Chapter 1 – Introduction to Statics

Engineering Mechanics: Statics and Structures

Chapter Outline

1.1 - Newton's Laws of Motion

- 1^{st} Law

  "*an object will remain at rest or in uniform motion in a straight line unless acted upon by an external force*."

- 2^{nd} Law

  A force will cause an object to accelerate in the direction of the net force, and the magnitude of the acceleration will be proportional to the net force but inversely proportional to the mass of the object

- 3^{rd} Law

  For every action, there is an equal and opposite reaction.

1.2 Units

Fundamental Units

Unit System	Force	Mass	Length	Time	g (Earth)
SI	N	kg	m	s	$9.81\ \mathrm{m/s^2}$
British Gravitational	lbf	slug	ft	s	$32.2\ \mathrm{ft/s^2}$
English Engineering	lbf	lbm	ft	s	$1\ \mathrm{lbf}/1\ \mathrm{lbm}$

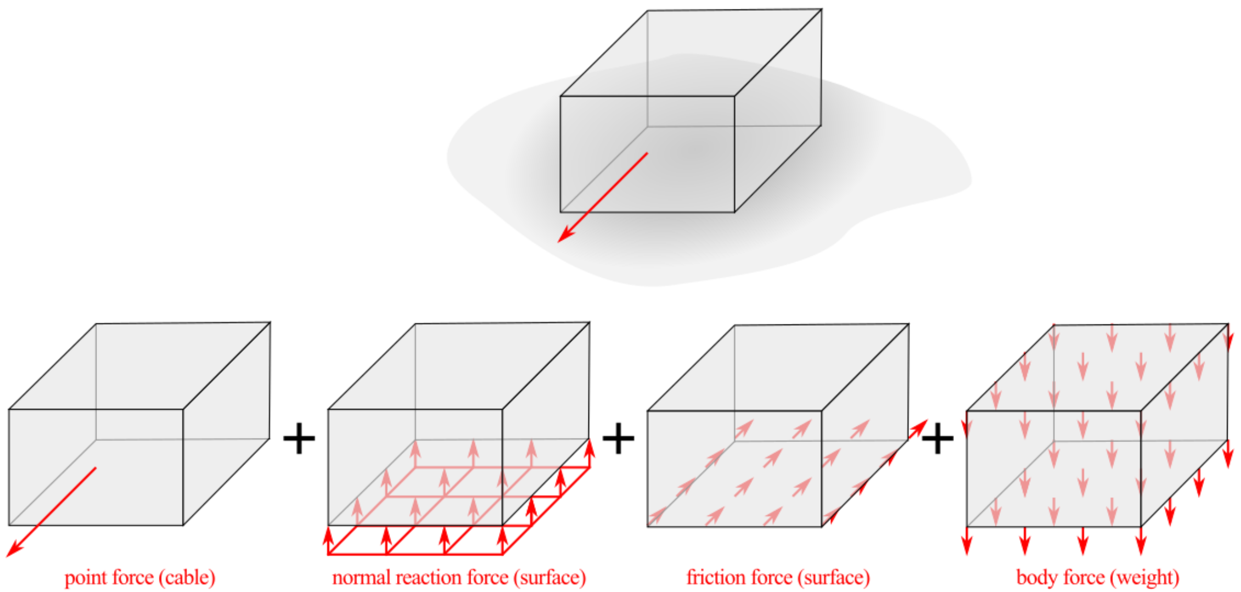
Base Units

- All unit systems are based around seven base units
- Important base units for Statics
  - Mass
  - Length
  - Time

- All other units of measurement are formed by combinations of the base units.

1.3 Forces

- At its simplest, a force is a “push or pull”



Types of Forces

- Point Force
- Body Force
- Reaction Force
- Distributed Force

1.4 Problem Solving

- Statics may be the first course you take where you are required to decide on your own how to approach a problem.
- Choosing a strategy gets easier with experience.
- For equilibrium problems, there are seven general problem-solving steps

Steps to Statics Problem Solving

1. Read and understand the problem.
2. Identify what you are asked to find and what is given.
3. Stop, think, and decide on an strategy.
4. Draw a free-body diagram and define variables.
5. Apply the strategy to solve for unknowns and check solutions.
6. Equations
   a. Write equations of equilibrium based on the free-body diagram.
   b. Check if the number of equations equals the number of unknowns. If it doesn't, you are missing something. You may need additional free-body diagrams or other relationships.
   c. Solve for unknowns.
7. Conceptually check solutions.

Textbook

Baker, Daniel W., and Haynes, William. "Chapter 1 – Introduction to Statics." *Engineering Statics: Open and Interactive*. May 21, 2023. https://engineeringstatics.org/

Chapter 2 – Forces and Other Vectors

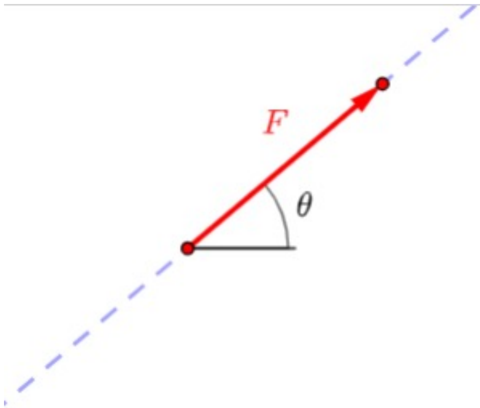
Engineering Mechanics: Statics and Structures

Chapter Outline

2.1 – Vectors

- Scalar Quantity
  - Physical quantities which have no associated direction
  - Described by a positive or negative number
  - Examples: Mass, time, temperature, and length
- Vector Quantity
  - Physical quantities which have magnitude and a direction
  - Examples: force, moment, and acceleration

Vector Convention



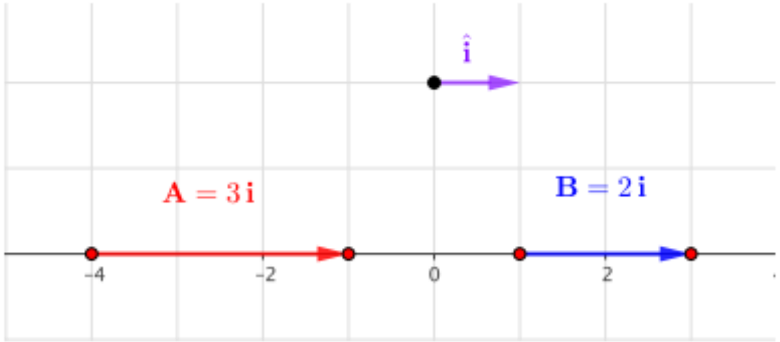
- Vector Visualization
  - An arrow pointing in a particular direction
  - The tip and tail of a vector define a line of action.
  - Standard notation for a vector

$$\vec{F} = \bar{F} = \mathbf{F} = \text{ a vector named } F$$

  - The vector's magnitude is a positive real number including units
  - Vector directions are described with respect to a coordinate system

2.2 – One-Dimensional Vectors

- Resultant Vector
  - Adding multiple vectors together finds the resultant vector
  - In the tip-to-tail technique, slide vector **B** until its tail is at the tip of **A** and the vector from the tail of **A** to the tip of **B** is the resultant **R**.

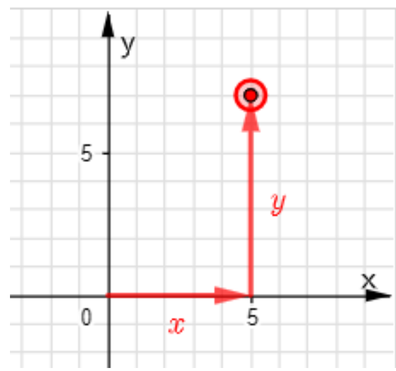


One-Dimensional Vectors

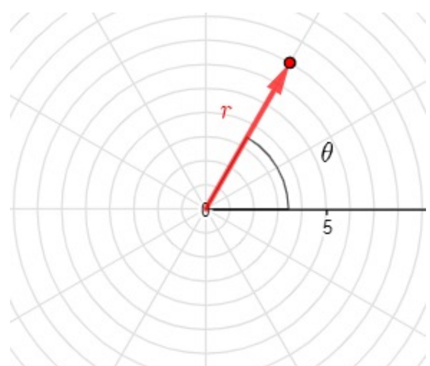
- Vector Subtraction
  - The easiest way to handle vector subtraction is to add the negative of the vector being subtracted to the other vector.
  - Can still use tip-to-tail technique
- Vector Multiplication
  - Multiplying or dividing a vector by a scalar changes the vector's magnitude but maintains its original line of action.

2.3 – Two Dimensional Coordinate Systems

- Rectangular Components (*x*-*y*)



- Polar Coordinates



Coordinate Transformation

- Rectangular To Polar for Points (Given: *x* and *y*)

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$P = (r\,;\theta)$$

- Polar to Rectangular for Points (Given: *r* and Θ)

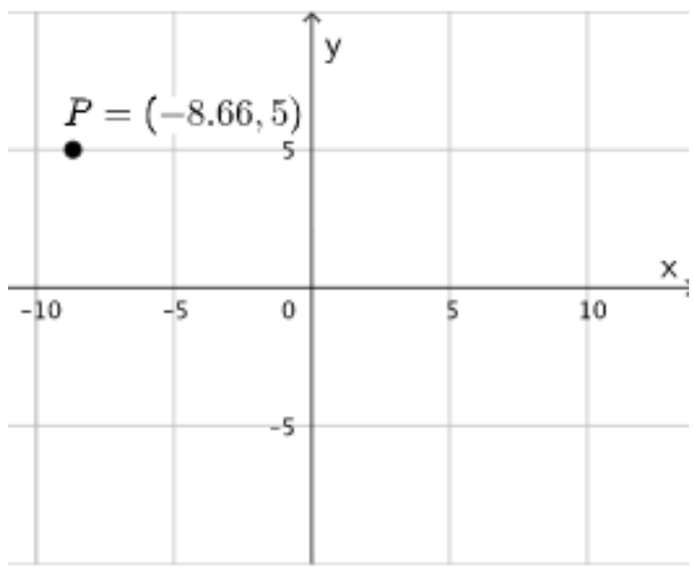
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$P = (x, y)$$

- Polar to Rectangular for forces (Given: magnitude and direction)

$$A_x = A\cos\theta$$
$$A_y = A\sin\theta$$
$$\mathbf{A} = \langle A_x, A_y\rangle = A\langle\cos\theta, \sin\theta\rangle$$

Example 2.1 (1/2)

Express point P = (−8.66, 5) in polar coordinates.



Example 2.1 (2/2)

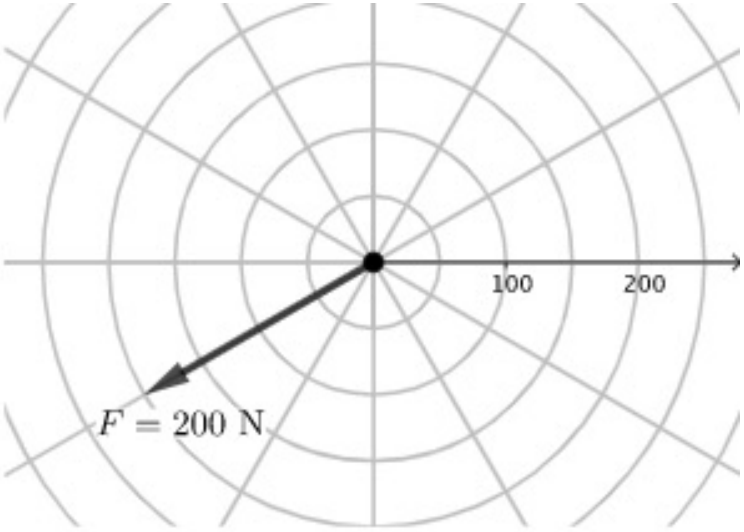
Solution 1. Given: $x = -8.66$, $y = 5$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-8.66)^2 + (5)^2} \\ &= 10 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{5}{-8.66}\right) \\ &= \tan^{-1}(-0.577) \\ &= -30^\circ \end{aligned}$$

Example 2.2 (1/2)

Express 200 N force **F** as a pair of scalar components.



Example 2.2 (2/2)

Solution 1. Given: The magnitude of force $\mathbf{F} = 200$ N, and from the diagram we see that the direction of $\mathbf{F}$ is $30°$ counter-clockwise from the negative x axis.

Letting $\theta = 30°$ we can find the components of $\mathbf{F}$ with right triangle trigonometry.

$$\begin{aligned} F_x &= F\cos\theta \\ &= 200 \text{ N} \cos 30° \\ &= 173.2 \text{ N} \end{aligned} \qquad \begin{aligned} F_y &= F\sin\theta \\ &= 200 \text{ N} \sin 30° \\ &= 100 \text{ N} \end{aligned}$$

Since the force points down and to the left into the third quadrant, these values are actually negative, and the signs must be applied manually.

After making this adjustment, the location of $\mathbf{F}$ expressed in rectangular coordinates is:

$$\mathbf{F} = \langle -173.2 \text{ N}, -100 \text{ N} \rangle$$

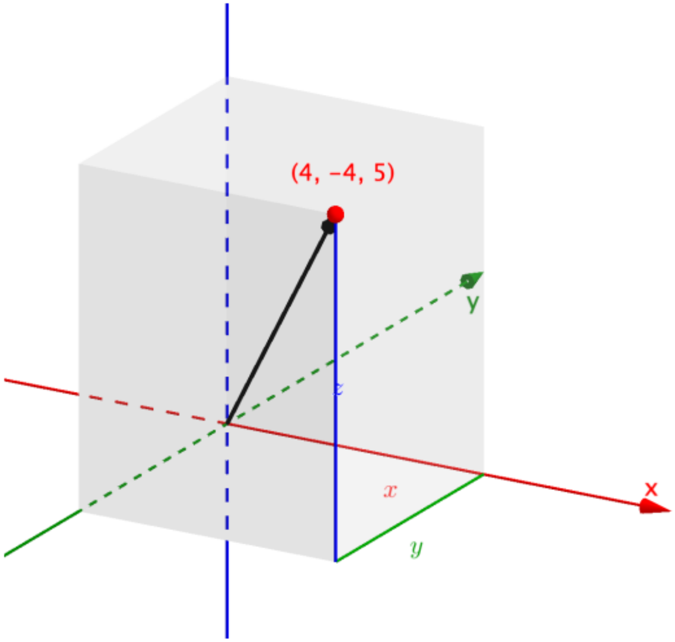
2.4 – Three Dimensional Coordinate Systems

- The most commonly use method is an extension of two-dimensional ***rectangular coordinates*** to three-dimensions. Alternately, points and vectors in three dimensions can be specified in terms of ***direction cosines***, or using ***spherical*** or cylindrical coordinate systems. These will be discussed in the following sections.

2.4.1 – Rectangular Coordinates

- Extension of the two-dimensional Cartesian coordinate system

$$P = (x, y, z) \qquad \mathbf{F} = \langle F_x, F_y, F_z \rangle$$



2.4.2 – Direction Cosine Angles

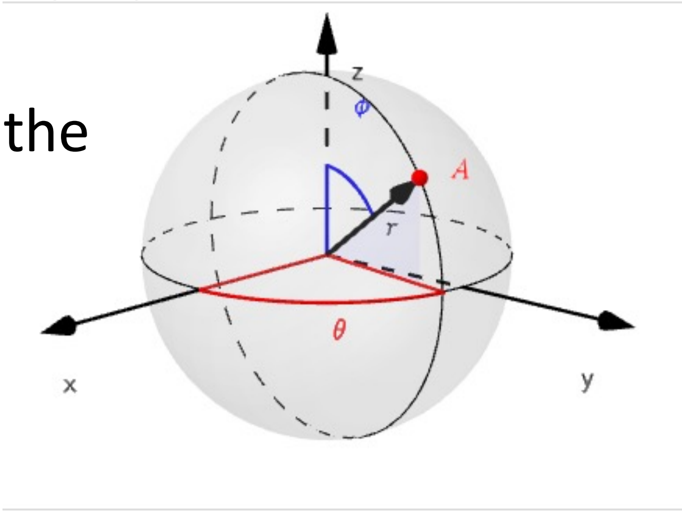
- The direction cosine angles are the angles between the positive *x, y*, and *z* axes to a given vector and are traditionally named Θ_x, Θ_y, and Θ_z.

$$\cos\theta_x = \frac{A_x}{|A|} \qquad \cos\theta_y = \frac{A_y}{|A|} \qquad \cos\theta_z = \frac{A_z}{|A|}$$

- The numerator of each direction cosine equation is positive or negative as defined by the coordinate system, and the vector magnitude in the denominator is always positive.
  - Direction cosines are signed value between -1 and 1.
  - Direction cosine angles must always be between 0° and 180°.

2.4.3 – Spherical Coordinates

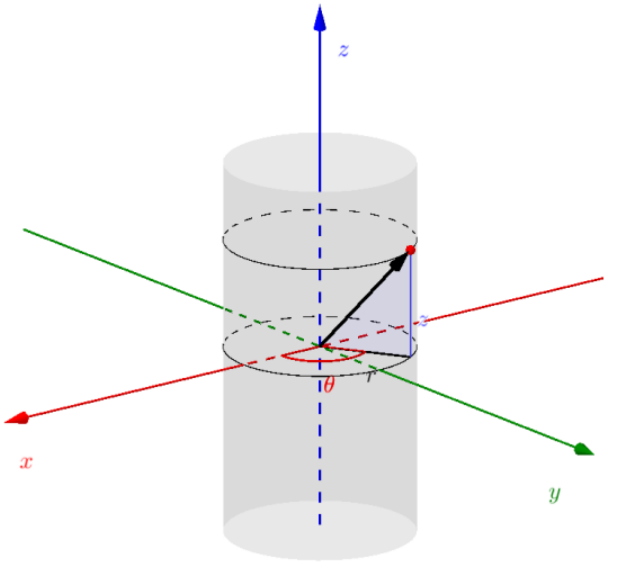
- In spherical coordinates, points are specified with these three coordinates
  - *r* - the distance from the origin to the tip of the vector
  - Θ - the angle, measured counter-clockwise from the positive *x* axis to the projection of the vector onto the *xy* plane, and
  - *Φ*, the polar angle from the z axis to the vector.



$A = \langle 0.6, 4, 3 \rangle \quad r = 5,\ \theta = 81.5^\circ,\ \phi = 53.4^\circ$

2.4.4 – Cylindrical Coordinates

- Seldom used in statics
- Cylindrical coordinates extend two-dimensional polar coordinates by adding a *z* coordinate indicating the distance above or below the *xy* plane.
- Points are specified with these three cylindrical coordinates.
  - *r* - the distance from the origin to the projection of the tip of the vector onto the xy plane
  - Θ - the angle, measured counter-clockwise from the positive x axis to the projection of the vector onto the xy plane
  - *z* - the vertical height of the vector tip



2.5 – Unit Vectors

A unit vector is a vector with a magnitude of one and no units.

- A unit vector represents a pure direction.
- By convention a unit vector is indicated by a *hat* over a vector symbol.

2.5.1 – Cartesian Unit Vectors

- A unit vector can point in any direction, but because they occur so frequently the unit vectors in each of the three Cartesian coordinate directions are given their own symbols, which are:
  - $\hat{\boldsymbol{\imath}}$ - for the unit vector pointing in the *x* direction
  - $\hat{\boldsymbol{\jmath}}$ - for the unit vector pointing in the *y* direction
  - $\widehat{\boldsymbol{k}}$ - for the unit vector pointing in the *z* direction.
- The *x* and *y* components of a point on the unit circle are also the scalar components of **F**

$$F_x = \cos\theta$$
$$F_y = \sin\theta$$
$$\begin{aligned}\hat{\mathbf{F}} &= \langle \cos\theta, \sin\theta \rangle \\ &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\end{aligned}$$

2.5.2 – Relation between Vectors and Unit Vectors

When a unit vector is multiplied by a scalar value it is *scaled* by that amount

In general,

$$\mathbf{F} = F\,\hat{\mathbf{F}}$$

where *F* is the magnitude of **F**, and $\widehat{\boldsymbol{F}}$ is the unit vector pointing in the direction of **F.**

2.5.2 – Relation between Vectors and Unit Vectors

To find the unit vector of known vector **F**

$$\hat{\mathbf{F}} = \frac{\mathbf{F}}{|\mathbf{F}|}$$

Example 2.3 (1/2)

Find the unit vector corresponding to a 100 N force at 60° from the *x*-axis.

Example 2.3 (2/2)

Solution. In polar coordinates, the unit vector is a vector of magnitude 1, pointing in the same direction as the force, so, by inspection

$$\mathbf{F} = (100\ \mathrm{N}\,;60°)$$
$$\hat{\mathbf{F}} = (1\,;60°)$$

In rectangular coordinates, first express $\mathbf{F}$ in terms of its x and y components.

$$F_x = F\cos 60°, F_y = F\sin 60°$$
$$\mathbf{F} = \langle F\cos 60°, F\sin 60°\rangle$$

Solve equation (2.5.2) for $\hat{\mathbf{F}}$

$$\begin{aligned}\hat{\mathbf{F}} &= \frac{\mathbf{F}}{F}\\ &= \frac{\langle F\cos 60°, F\sin 60°\rangle}{F}\\ &= \langle \cos 60°, \sin 60°\rangle\end{aligned}$$

2.5.3 – Force Vectors from Position Vectors

- Unit vectors are generally the best approach when working with forces and distances in three dimensions.
- When the location of two points on the line of action of a force are known, the unit vector of the line of action can be found and used to determine the components of a force acting along that line.

Components of a Force

Steps for determine the components of a force acting along the line given by points *A* and *B*.

1. Use the problem geometry to find **AB**, the displacement vector from point *A* to point *B.* Then subtract the coordinates of the starting point *A* from the coordinates of the destination point *B* to find the vector.

$$\begin{aligned} A &= (A_x, A_y, A_z) \\ B &= (B_x, B_y, B_z) \\ \mathbf{AB} &= (B_x - A_x)\mathbf{i} + (B_y - A_y)\mathbf{j} + (B_z - A_z)\mathbf{k}, \end{aligned}$$

Components of a Force

2. Find the direct distance between point *A* and point *B* using the Pythagorean Theorem. This distance is also the magnitude of AB

$$|\mathbf{AB}| = \sqrt{(AB_x)^2 + (AB_y)^2 + (AB_z)^2}$$

3. Find $\widehat{\mathbf{AB}}$, the unit vector from *A* to *B*, by dividing vector AB by its magnitude. This is a unitless vector with a magnitude of 1 which points from *A* to *B*.

$$\widehat{\mathbf{AB}} = \left\langle \frac{AB_x}{|AB|}, \frac{AB_y}{|AB|}, \frac{Ab_z}{|AB|} \right\rangle$$

Components of a Force

4. Multiply the magnitude of the force by the unit vector $\widehat{\mathbf{AB}}$ to get force $\mathbf{F}_{AB}$

$$\begin{aligned}\mathbf{F}_{AB} &= F_{AB}\,\widehat{\mathbf{AB}} \\ &= F_{AB}\left\langle \frac{AB_x}{|\mathbf{AB}|}, \frac{AB_y}{|\mathbf{AB}|}, \frac{AB_z}{|\mathbf{AB}|} \right\rangle\end{aligned}$$

Example 2.4 (1/3)

Determine the components of a 5 kN force **F** acting at point *A*, in the direction of a line from *A* to *B*.

Given: $A = (2, 3, -2.1)$ m and $B = (-2.5, 1.5, 2.2)$ m

Example 2.4 (2/3)

1. Find the displacement vector from *A* to *B*.

$$
\begin{aligned}
\mathbf{AB} &= (B_x - A_x)\,\mathbf{i} + (B_y - A_y)\,\mathbf{j} + (B_z - A_z)\,\mathbf{k} \\
&= [(-2.5 - 2)\,\mathbf{i} + (1.5 - 3)\,\mathbf{j} + (2.2 - (-2.1))\,\mathbf{k}]\ \ \mathrm{m} \\
&= (-4.5\mathbf{i} - 1.5\mathbf{j} + 4.3\mathbf{k})\ \ \mathrm{m} \\
&= \langle -4.5, -1.5, 4.3 \rangle\ \ \mathrm{m}
\end{aligned}
$$

2. Find the magnitude of the displacement vector

$$
\begin{aligned}
|\mathbf{AB}| &= \sqrt{(\Delta_x)^2 + (\Delta_y)^2 + (\Delta_z)^2} \\
&= \sqrt{(-4.5)^2 + (-1.5)^2 + 4.3^2\ \mathrm{m}^2} \\
&= \sqrt{40.99\ \mathrm{m}^2} \\
&= 6.402\ \mathrm{m}
\end{aligned}
$$

Example 2.4 (3/3)

3. Find the unit vector pointing from *A* to *B.*

$$
\begin{aligned}
\widehat{\mathbf{AB}} &= \left\langle \frac{\Delta_x}{|\mathbf{AB}|}, \frac{\Delta_y}{|\mathbf{AB}|}, \frac{\Delta_z}{|\mathbf{AB}|} \right\rangle \\
&= \left\langle \frac{-4.5}{6.402}, \frac{-1.5}{6.402}, \frac{4.3}{6.402} \right\rangle \\
\widehat{\mathbf{AB}} &= \langle -0.7, -0.23, 0.67 \rangle
\end{aligned}
$$

4. Find the force vector.

$$
\begin{aligned}
\mathbf{F}_{AB} &= F_{AB}\, \widehat{\mathbf{AB}} \\
&= 5\ \mathrm{kN} \langle -0.7, -0.23, 0.67 \rangle \\
&= \langle -3.51, -1.17, 3.36 \rangle\ \mathrm{kN}
\end{aligned}
$$

2.5.4 – Unit Vectors and Direction Cosines

- The cosine of each direction cosine angle collectively also computes the components of the unit vector

$$\hat{\mathbf{A}} = \cos\theta_x \mathbf{i} + \cos\theta_y \mathbf{j} + \cos\theta_z \mathbf{k}$$

- If two of the three direction cosine angles are know, the third can be found using the following equation

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

2.6 – Vector Addition

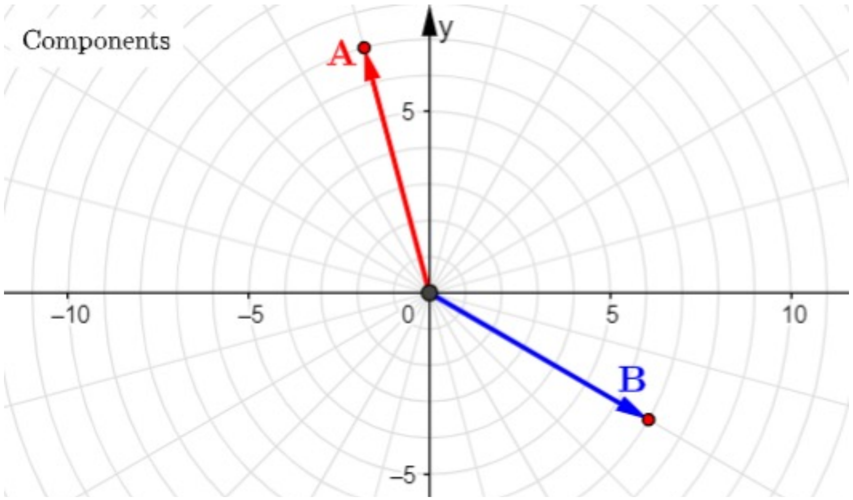
- Vectors being added together are called the ***components***, and the sum of the components is called the ***resultant vector***.
- There are five different methods for doing vector addition
  - Triangle Rule
  - Orthogonal Components
  - Graphical Addition
  - Trigonometric Addition
  - Algebraic Addition

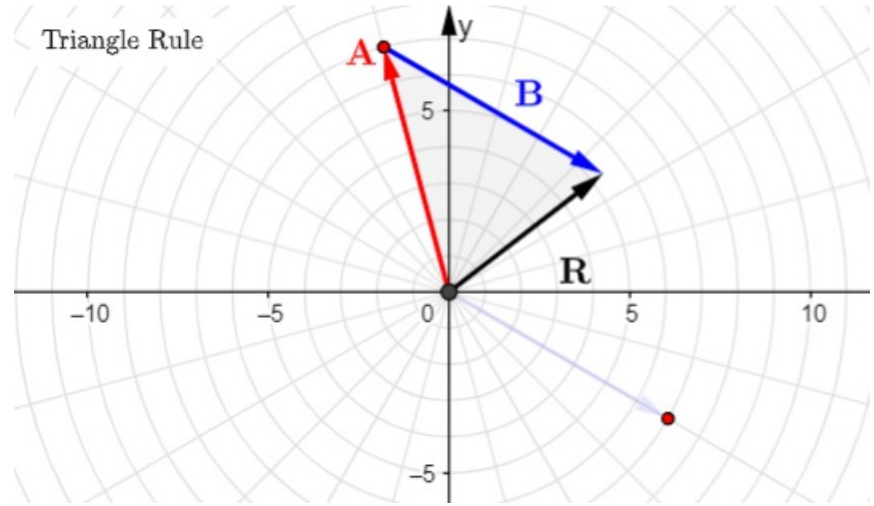
2.6.1 – Triangle Rule of Vector Addition

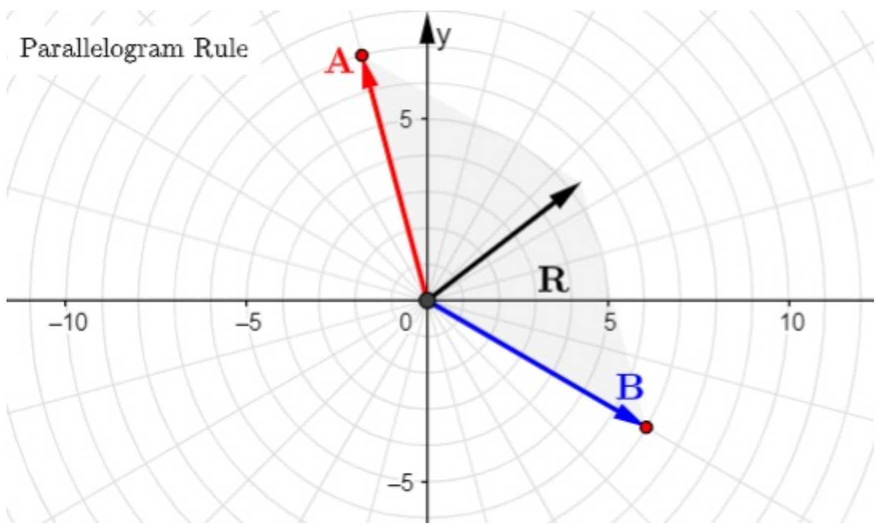
- All methods of vector addition are ultimately based on the tip-to-tail method
  - *Triangle Rule.*

    Place the tail of one vector at the tip of the other vector, then draw the resultant from the first vector's tail to the final vector's tip.
  - *Parallelogram Rule.*

    Place both vectors tails at the origin, then complete a parallelogram with lines parallel to each vector through the tip of the other. The resultant is equal to the diagonal from the tails to the opposite corner.


Components
A
B
y
5
−5
−10
−5
0
5
10


Triangle Rule
A
B
R
y
5
−5
−10
−5
0
5
10

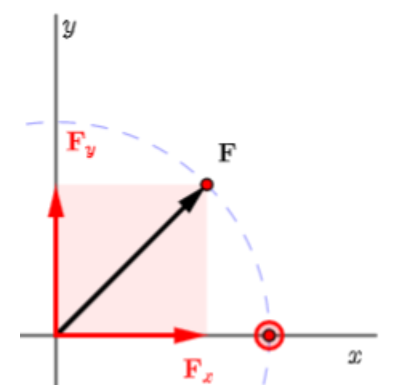

Parallelogram Rule
A
B
R
y
5
−5
−10
−5
0
5
10

2.6.2 – Orthogonal Components

- Any arbitrary vector **F** can broken into two component vectors which are the sides of a parallelogram having **F** as its diagonal.
- Rectangular Components
  - Vectors are resolved into components that align with the x and y axes.

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y = F_x\,\mathbf{i} + F_y\mathbf{j}$$

  - $\mathbf{F}_x$ and $\mathbf{F}_y$ are the scalar components of **F**



2.6.3 – Graphical Vector Addition

- Graphical vector addition involves drawing a scaled diagram using either the parallelogram or triangle rule, and then measuring the magnitudes and directions from the diagram.
  - Must carefully draw the triangle accurately to scale and use a protractor and ruler
  - Answer will only be as precise as the diagram and ability to read the tools

2.6.4 – Trigonometric Vector Addition

- Using triangle-based geometry to solve vector problems is a quick and powerful tool, but includes the following limitations:
  - There are only three sides in a triangle; thus vectors can only be added two at a time. If you need to add three or more vectors using this method, you must add the first two, then add the third to that sum and so on.
  - If you fail to draw the correct vector triangle, or identify the known sides and angles you will not find the correct answer.
  - The trigonometric functions are scalar functions. They are quick ways of solving for the magnitudes of vectors and the angle between vectors, but you may still need to find the vector components from a given datum.

2.6.5 – Algebraic Addition of Components

- The *algebraic method* uses the addition of *scalar* components.
- To find the sum of multiple vectors:
  - Find the scalar components of each component vector in the x and y directions
  - Algebraically sum the scalar components in each coordinate direction.
    - The scalar components will be positive if they point right or up, negative if they point left or down.
    - These sums are the scalar components of the resultant.
- Resolve the resultant's components to find the magnitude and direction of the resultant vector

$$\mathbf{F}_R = \sum F_x\, \mathbf{i} + \sum F_y\, \mathbf{j} + \sum F_z\, \mathbf{k}$$

Example 2.5 (1/3)

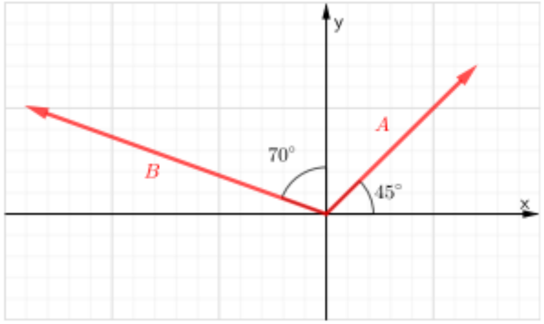
Vector **A** = 200 N∠ 45° counter-clockwise from the x axis, and vector **B** = 300 N ∠70∘ counter-clockwise from the *y* axis.

Find the resultant **R** = **A** + **B** by addition of scalar components.

Example 2.5 (2/3)

Solution.

Use the given information to draw a sketch of the situation. By imagining or sketching the parallelogram rule, it should be apparent that the resultant vector points up and to the left.



$$A_x = 200\ \text{N} \cos 45° = 141.4\ \text{N} \qquad B_x = -300\ \text{N} \sin 70° = -281.9\ \text{N}$$
$$A_y = 200\ \text{N} \sin 45° = 141.4\ \text{N} \qquad B_y = 300\ \text{N} \cos 70° = 102.6\ \text{N}$$

$$\begin{aligned} R_x &= A_x + B_x \\ &= 141.4\ \text{N} + -281.9\ \text{N} \\ &= -140.5\ \text{N} \end{aligned} \qquad \begin{aligned} R_y &= A_y + B_y \\ &= 141.4\ \text{N} + 102.6\ \text{N} \\ &= 244.0\ \text{N} \end{aligned}$$

$$\begin{aligned} R &= \sqrt{R_x^2 + R_y^2} \\ &= 281.6\ \text{N} \\ \theta &= \tan -1 \left(\frac{R_y}{R_x}\right) \\ &= -60.1° \end{aligned}$$

Example 2.5 (3/3)

This answer indicates that the resultant points down and to the left. This is because the calculator answers for the inverse trig function will always be in the first or fourth quadrant. To get the actual direction of the resultant, add 180° to the calculator result.

$$\theta = -60.1^\circ + 180^\circ = 119.9^\circ$$

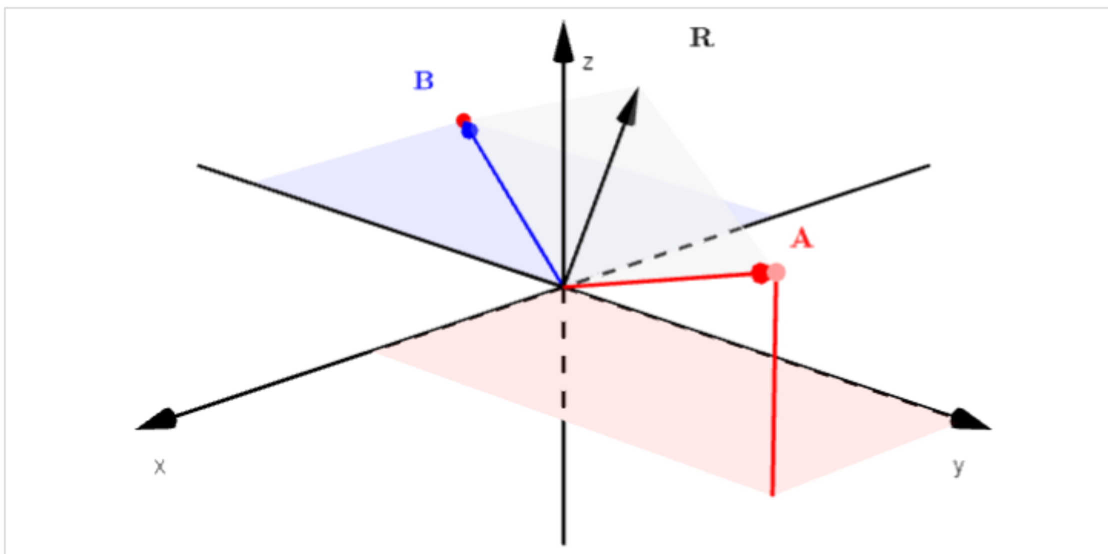
The final answer for the magnitude and direction of the resultant is

$$\mathbf{R} = 281.6 \text{ N}\angle 119.9^\circ$$

measured counter-clockwise from the x axis. □

2.6.5 – Algebraic Addition of Components

The process for adding vectors in space is exactly the same as in two dimensions, except that an additional z component is included.



	x	y	z
A	5	10	4
B	-6	-9	0
R	-1	1	4

2.6.6 – Vector Subtraction

- The easiest way to handle two dimensional vector subtraction is by taking the negative of a vector followed by vector addition.
- Multiplying a vector by -1 preserves its magnitude but flips its direction, which has the effect of changing the sign of the scalar components.

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

- Any of the vector addition techniques described may be used.

2.7 – Dot Products

- For two vectors **A**=⟨A_x, A_y, A_z⟩ and **B**=⟨B_x, B_y, B_z⟩, the dot product multiplication is computed by summing the products of the components.

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

or

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta = A\ B\ \cos\theta$$

- The dot product is a scalar value

2.7.1 – Magnitude of a Vector

- Dot products can be used to find vector magnitudes.

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

$$\mathbf{A} \cdot \mathbf{A} = A_x A_x + A_y A_y = A_x^2 + A_y^2$$

$$\sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_x^2 + A_y^2} = A = |\mathbf{A}|$$

Example 2.6 (1/2)

Find the magnitude of vector **F** with components F_x=30 N, F_y = −40 N and F_z = 50 N.

Example 2.6 (2/2)

Solution.

$$\mathbf{F} = \langle 30\ \mathrm{N}, -40\ \mathrm{N}, 50\ \mathrm{N} \rangle$$

$$\begin{aligned}\mathbf{F} \cdot \mathbf{F} &= F_x^2 + F_y^2 + F_z^2 \\ &= (30\ \mathrm{N})^2 + (-40\ \mathrm{N})^2 + (50\ \mathrm{N})^2 \\ &= 5000\ \mathrm{N}^2\end{aligned}$$

$$\begin{aligned}F = |\mathbf{F}| &= \sqrt{\mathbf{F} \cdot \mathbf{F}} \\ &= \sqrt{5000\ \mathrm{N}^2} \\ &= 70.7\ \mathrm{N}\end{aligned}$$

2.7.2 – Angle between Two Vectors

- The angle between two vectors can be found by rearranging the dot product equation

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$$
$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$$

Example 2.7 (1/2)

Find the angle between **F** = $\langle 100\ \text{N}, 200\ \text{N}, -50\ \text{N} \rangle$ and **G** = $\langle -75\ \text{N}, 150\ \text{N}, -40\ \text{N} \rangle$.

Example 2.7 (2/2)

Solution.

$$
\begin{aligned}
\cos\theta &= \frac{\mathbf{F}\cdot\mathbf{G}}{|\mathbf{F}||\mathbf{G}|} \\
&= \frac{F_xG_x + F_yG_y + F_zG_z}{\sqrt{F_x^2+F_y^2+F_z^2}\sqrt{G_x^2+G_y^2+G_z^2}} \\
&= \frac{(100)(-75) + (200)(150) + (-50)(-40)}{\sqrt{100^2+200^2+(-50)^2}\sqrt{(-75)^2+150^2+(-40)^2}} \\
&= \frac{24500}{(229.1)(172.4)} \\
&= 0.620
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \cos^{-1}(0.620) \\
&= 51.7^\circ
\end{aligned}
$$

2.8 – Cross Products

- The vector ***cross product*** is a mathematical operation applied to two vectors which produces a third mutually perpendicular vector as a result.
- Cross products are used in mechanics to find the moment of a force about a point.

$$\mathbf{A} \times \mathbf{B} = A\,B \sin\theta\;\hat{\mathbf{u}}$$

- If A and B are in the xy plane

$$\mathbf{A} \times \mathbf{B} = (A_y B_x - A_x B_y)\,\mathbf{k}$$

2.8.1 – Cross Product of Arbitrary Vectors

- The cross product of two three-dimensional vectors can be calculated by evaluating the *determinant* of this 3×3 matrix.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

2.8.1 – Cross Product of Arbitrary Vectors

$$\mathbf{A}\times\mathbf{B} = +\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}\mathbf{i} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix}\mathbf{j} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}\mathbf{k}$$

$$\mathbf{A}\times\mathbf{B} = \left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ A_x & A_y & A_z & A_x & A_y \\ B_x & B_y & B_z & B_x & B_y \end{array}\right.$$

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

Example 2.8 (1/2)

Find the cross product of **A** = $\langle 2, 4, -1\rangle$ and **B** = $\langle 10, 25, 20\rangle$. The components of **A** are in meters and **B** are in Newtons.

Example 2.8 (2/2)

Solution 1. To solve, set up the augmented determinant and evaluate it by adding the left-to-right diagonals and subtracting the right-to-left diagonals. (2.8.6).

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= \left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 2 & 4 & -1 & 2 & 4 \\ 10 & 25 & 20 & 10 & 25 \end{array}\right. \\
&= (4)(20)\,\mathbf{i} + (-1)(10)\,\mathbf{j} + (2)(25)\,\mathbf{k} - (4)(10)\,\mathbf{k} - (-1)(25)\,\mathbf{i} - (2)(20)\,\mathbf{j} \\
&= (80 + 25)\,\mathbf{i} + (-10 - 40)\,\mathbf{j} + (50 - 40)\,\mathbf{k} \\
&= \langle 105, -50, 10 \rangle \ \mathrm{N \cdot m}
\end{aligned}$$

Textbook

Baker, Daniel W., and Haynes, William. “Chapter 2 – Forces and Other Vectors.” *Engineering Statics: Open and Interactive*. May 21, 2023. https://engineeringstatics.org/

Chapter 3 – Equilibrium of Particles

Engineering Mechanics: Statics and Structures

Chapter Outline

3.1 Equilibrium

3.2 Particles

3.3 Particles in One Dimension

3.4 Particles in Two Dimensions

3.5 Particles in Three Dimensions

3.1 – Equilibrium

- Engineering statics is the study of rigid bodies in equilibrium so it’s appropriate to begin by defining what we mean by *rigid bodies* and what we mean by *equilibrium*.
  - A ***body*** is an object, possibly made up of many parts, which may be examined as a unit.
  - A ***rigid body*** is a body which doesn't deform under load.
  - A body in ***equilibrium*** is not accelerating.

$$\Sigma \mathbf{F} = 0$$

3.2 – Particles

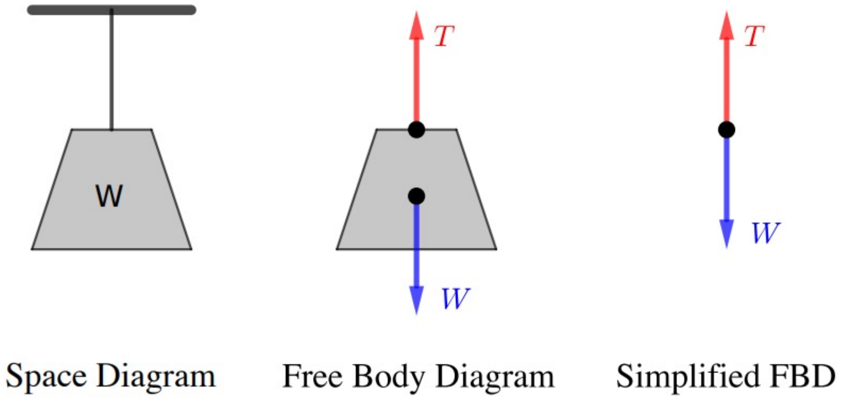
- The defining characteristic of a particle is that all forces that act on it are coincident or concurrent
  - Forces are coincident if they have the same line of action.
  - Forces are concurrent if they intersect at a point.
- All forces will be assumed to be ***concentrated***.
  - Concentrated forces act at a single point, have a well defined line of action, and can be represented with an arrow — in other words, they are vectors

3.3 – Particles in One Dimension

- In mechanics we are interested in studying the forces acting on objects

3.3.1 – A simple case

- The free-body diagram
  - A diagram which focuses on the forces acting on an object, not the mechanisms that hold it in place



- "Principle of Transmissibility"
  - A force can be moved along it's line of action and the net external effects remain the same.

3.3.1 – A simple case

- Drawing free-body diagrams can be surprisingly tricky.
  - Must identify all the forces acting on the object and correctly represent them on the free-body diagram.
  - If not all forces are accounted for or they are represented them incorrectly, the analysis will be incorrect.

3.3.2 – Scalar Components

- The scalar component of a vector is a *signed* number which indicates the vector's ***magnitude*** and ***sense***
- Scalar components can be added together algebraically, but only if they act "in the same direction."

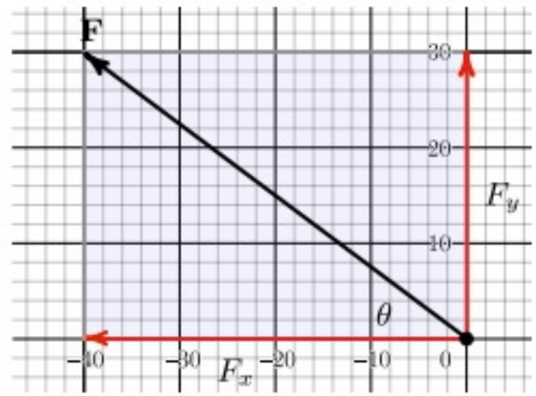
Example 3.1 (1/2)

If $F_x = -40$ N and $F_y = 30$ N, find the magnitude and direction of their resultant.

Example 3.1 (2/2)

Solution. In this example the scalar components have different subscripts indicating that they act along different lines of action, and this must be accounted for when they are added together.

Make a sketch of the two vectors and add them using the parallelogram rule to get



$$\begin{aligned}\theta &= \tan^{-1}\left|\frac{F_y}{F_x}\right| \\ &= \tan^{-1}\left|\frac{30\ \text{N}}{-40\ \text{N}}\right| \\ &= 36.9°\end{aligned}$$

$$\begin{aligned}F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{(-40\ \text{N})^2 + (30\ \text{N})^2} \\ &= 50\ \text{N}\end{aligned}$$

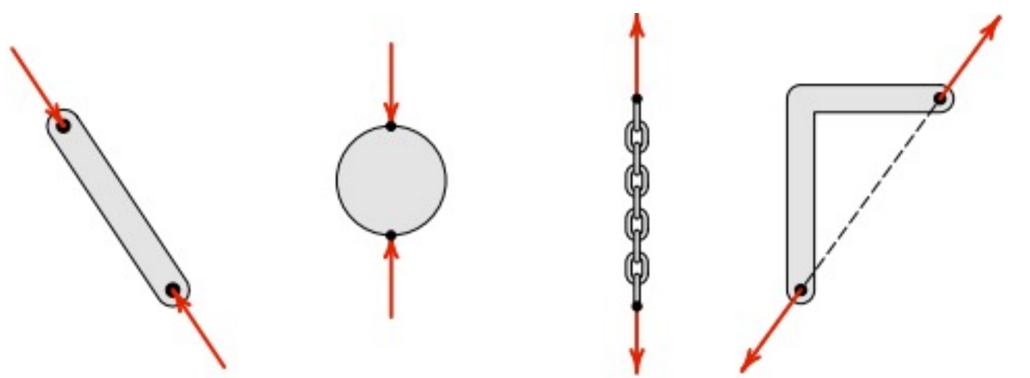
These are the magnitude and direction of vector **F**.

3.3.3 – Two-force Bodies

- A ***two-force body*** is a body with two forces acting on it
- The two forces must either
  - Share the same line of action, have the same magnitude, and point away from each other, or
  - Share the same line of action, have the same magnitude, and point towards each other, or
  - Both have zero magnitude.

3.3.3 – Two-force Bodies

- When two forces have the same magnitude but act in diametrically opposite directions, we say that they are ***equal-and-opposite***.
  - ***Compression -*** point towards each other
  - ***Tension -*** point away from each other



3.4 – Particles in Two Dimensions

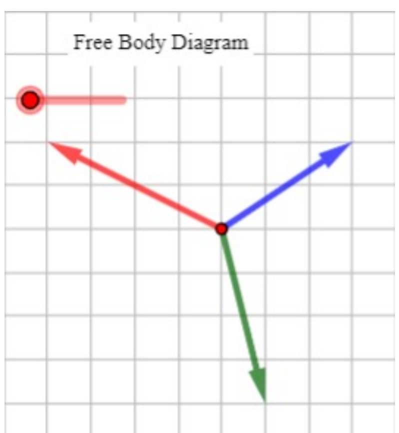
3.4.2 – General Procedure

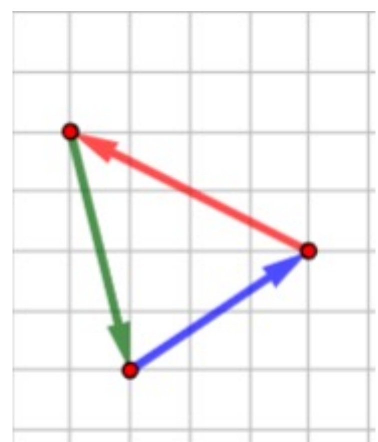
The general procedure for solving equilibrium of a particle problems in two dimensions is to:

1. Identify the particle.
2. Establish a coordinate system.
3. Draw a free-body diagram.
4. State any given values and identify the unknown values.
5. Find trivial angles.
6. Count knowns and unknowns.
7. Formulate equilibrium equations.
8. Simplify.
9. Substitute values for symbols.
10. Check your work.

3.4.3 – Force Triangle Method

- Applicable to situations where there are (exactly) three forces acting on a particle, and no more than two unknown magnitudes or directions.
  - If such a particle is in equilibrium then the three forces must add to zero.
- Graphically, force vectors are arranged tip-to-tail, and form a closed, three-sided polygon.





3.4.4 – Trigonometric Method

The general approach for solving particle equilibrium problems using the trigonometric method is to:

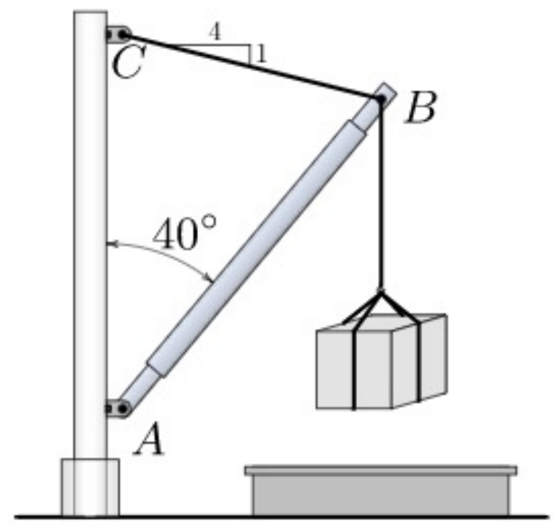
1. Draw and label a free-body diagram.
2. Rearrange the forces into a force triangle and label it.
3. Identify the knowns and unknowns.
4. Use trigonometry to find the unknown sides or angles of the triangle.

There must be no more than two unknowns

Example 3.2 (1/4)

A 24 kN crate is being lowered into the cargo hold of a ship.
Boom *AB* is 20 m long and acts at a 40° angle from kingpost *AC*. The boom is held in this position by topping lift *BC* which has a 1:4 slope.

Determine the forces in the boom and in the topping lift.



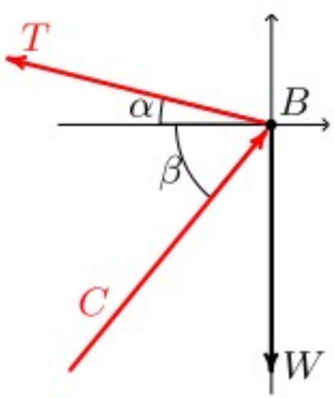
Example 3.2 (2/4)

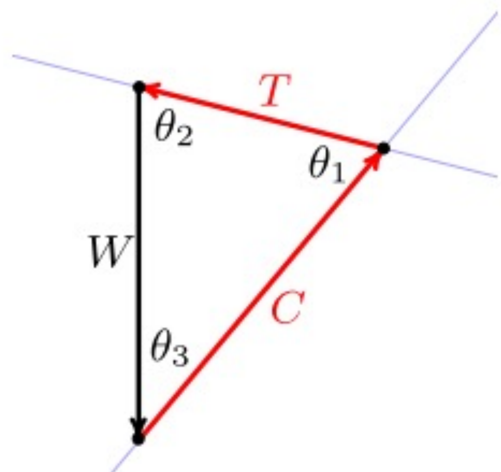
Solution.

1. *Draw diagrams.*

   Start by identifying the particle and drawing a free-body diagram. The particle in this case is point B at the end of the boom because it is the point where all three forces intersect. Let T be the tension of the topping lift, C be the force in the boom, and W be the weight of the load. Let α and β be the angles that forces T and C make with the horizontal.

   Rearrange the forces acting on point B to form a force triangle as was done in the previous example.





Example 3.2 (3/4)

2. *Find angles.*

   Angle α can be found from the slope of the topping lift.

$$\alpha = \tan^{-1}\left(\frac{1}{4}\right) = 14.0°$$

.

Angle β is the complement of the 40° angle the boom makes with the vertical kingpost.

$$\beta = 90° - 40° = 50°$$

Use these values to find the three angles in the force triangle.

$$\begin{aligned} \theta_1 &= \alpha + \beta = 64.0° \\ \theta_2 &= 90° - \alpha = 76.0° \\ \theta_3 &= 90° - \beta = 40.0° \end{aligned}$$

Example 3.2 (4/4)

3. *Solve force triangle.*

   With the angles and one side of the force triangle known, apply the Law of Sines to find the two unknown sides.

$$\frac{\sin\theta_1}{W} = \frac{\sin\theta_2}{C} = \frac{\sin\theta_3}{T}$$

$$T = W\left(\frac{\sin\theta_3}{\sin\theta_1}\right)$$
$$T = 24\text{ kN}\left(\frac{\sin 40.0°}{\sin 64.0°}\right)$$
$$T = 17.16\text{ kN}$$

$$C = W\left(\frac{\sin\theta_2}{\sin\theta_1}\right)$$
$$C = 24\text{ kN}\left(\frac{\sin 76.0°}{\sin 64.0°}\right)$$
$$C = 25.9\text{ kN}$$

3.4.5 – Scalar Components Method

- The general statement of equilibrium of forces, can be expressed as the sum of forces in the $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\widehat{\boldsymbol{k}}$ directions.

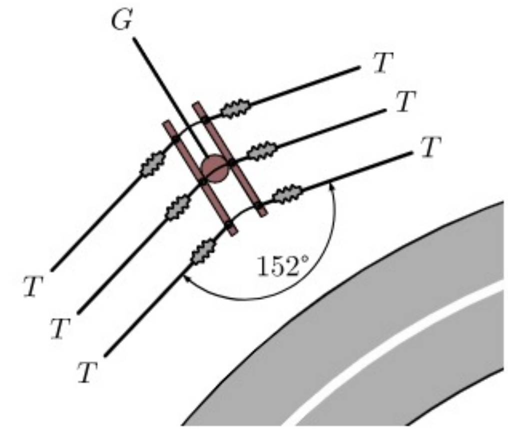
$$\Sigma \mathbf{F} = \Sigma F_x\,\mathbf{i} + \Sigma F_y\,\mathbf{j} + \Sigma F_z\,\mathbf{k} = 0$$

- This statement will only be true if all three coefficients of the unit vectors are themselves equal to zero,

$$\Sigma \mathbf{F} = 0 \implies \begin{cases} \Sigma F_x = 0 \\ \Sigma F_y = 0 \\ \Sigma F_z = 0 \end{cases} \qquad \text{(three dimensions)}$$

Example 3.3 (1/4)

Consider the utility pole next to the road shown below. A top view is shown in the right hand diagram. If each of the six power lines pulls with a force of 10.0 kN, determine the magnitude of the tension in the guy wire.



Example 3.3 (2/4)

1. *Assumptions.*

   A utility pole isn't two-dimensional, but we can solve this problem as if it was by first considering the force components acting in a horizontal plane, and then considering the components in a vertical plane.

   It also isn't a concurrent force problem because the lines of action of the forces don't all intersect at a single point. However, we can make it into one by replacing the forces of the three power lines in each direction with a single force three times larger. This is an example of an *equivalent transformation*, a trick engineers use frequently to turn complex situations into simpler ones. It works here because all the tensions are equal, and the outside wires are equidistant from the center wire. You must be careful to justify all equivalent transformations, because they will lead to errors if they are not applied correctly. Equivalent transformations will be discussed in greater detail in Section 4.7 later.

Example 3.3 (3/4)

2. *Givens.*

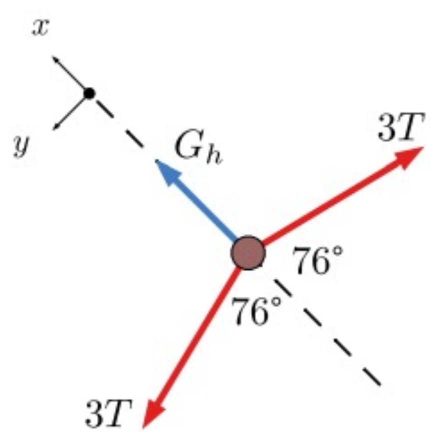
   $T = 10.0$ kN and 38° and 152° angles.

3. *Free Body Diagram.*

   Begin by drawing a neat, labeled, free-body diagram of the top view of the pole, establishing a coordinate system and indicating the directions of the forces.

   Call the tension in one power line T and the tension in the guy wire G. Resolve the the tension of the guy wire into a horizontal component G_h, and a vertical component G_v. Only the horizontal component of G is visible in the top view.

   Although it is not necessary, it simplifies this problem considerably to note the symmetry and establish the x axis along the axis of symmetry.



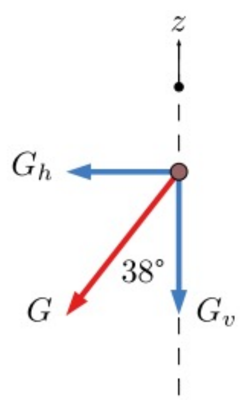
Example 3.3 (4/4)

4. *Solution.*

Solve for G_h by applying the equations of equilibrium. The symmetry of this problem means that the ΣF_x equation is sufficient.

$$\begin{aligned}
\Sigma F_x &= 0 \\
G_h - 6\,T_x &= 0 \\
G_h &= 6\,(T\cos 76°) \\
&= 14.5 \text{ kN}
\end{aligned}$$

Once G_h is determined, the tension of the guy wire G is easily found by considering the components of G in the side view. Note that the vertical component G_v tends to compress the pole.

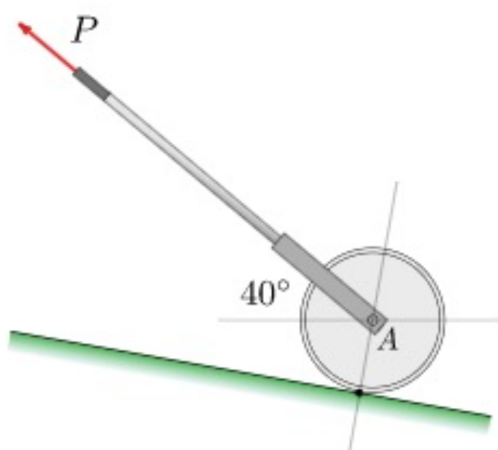


$$\begin{aligned}
G_h/G &= \sin 38° \\
G &= G_h/\sin 38° \\
G &= 23.6 \text{ kN}
\end{aligned}$$

Example 3.4 (1/3)

A lawn roller which weighs 160 lb is being pulled up a 10° slope at a constant velocity.

Determine the required pulling force .



Example 3.4 (2/3)

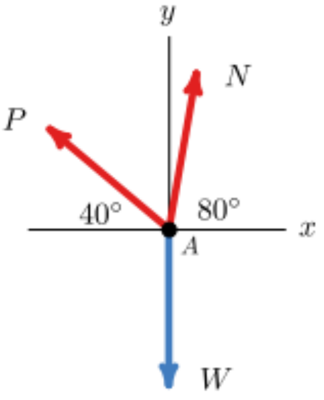
Solution 1.

1. *Strategy.*

   (a) Select a coordinate system, in this case horizontal and vertical.

   (b) Draw a free-body diagram

   (c) Solve the equations of equilibrium using the scalar approach.



Example 3.4 (3/3)

2. *Procedure.*

$$\Sigma F_x = 0 \qquad\qquad \Sigma F_y = 0$$

$$\begin{aligned} -P_x + N_x &= 0 \\ N\cos 80^\circ &= P\cos 40^\circ \\ N &= P\left(\frac{0.766}{0.174}\right) \end{aligned} \qquad\qquad \begin{aligned} P_y + N_y &= 0 \\ P\sin 40^\circ + N\sin 80^\circ &= W \\ 0.643P + 0.985N &= 160 \text{ lb} \end{aligned}$$

Solving simultaneously for P

$$\begin{aligned} 0.643P + 0.985(4.40P) &= 160 \text{ lb} \\ 4.98P &= 160 \text{ lb} \\ P &= 32.1 \text{ lb} \end{aligned}$$

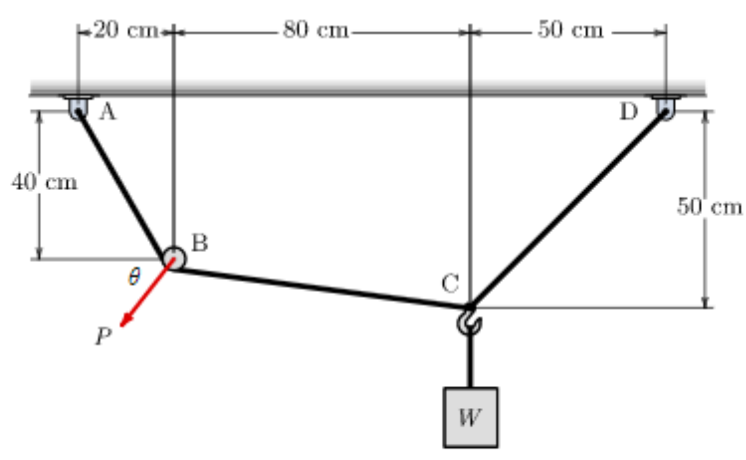
3.4.6 – Multi-Particle Equilibrium

- When two or more particles interact with each other there will always be common forces between them as a result of Newton's Third Law, the action-reaction principle.

Example 3.5 (1/4)

A 100 N weight *W* is supported by cable *ABCD*. There is a frictionless pulley at *B* and the hook is firmly attached to the cable at point *C*.

What is the magnitude and direction of force **P** required to hold the system in the position shown?

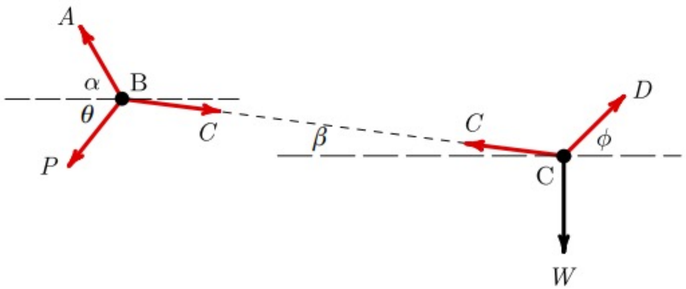


Example 3.5 (2/4)

Solution.

1. *Strategy.*

   Following the General Procedure we identify the particles as points A and B, and draw free-body diagrams of each. We label the rope tensions A, C, and D for the endpoints of the rope segments, and label the angles of the forces α, β, and ϕ. We will use the standard Cartesian coordinate system and use the scalar components method.



Weight W was given, and we can easily find angles α, β, and ϕ so the knowns are:

$$W = 100 \text{ N}$$
$$\alpha = \tan^{-1}\left(\frac{40}{20}\right) = 63.4^\circ$$
$$\beta = \tan^{-1}\left(\frac{10}{80}\right) = 7.13^\circ$$
$$\phi = \tan^{-1}\left(\frac{50}{50}\right) = 45^\circ$$

Counting unknowns we find that there are two on the free-body diagram of particle C (C and D), but four on particle B, (A C, P and θ).

Two unknowns on particle C means it is solvable since there are two equilibrium equations available, so we begin there.

Example 3.5 (3/4)

2. *Solve Particle C.*

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0$$

$$-C_x + D_x = 0 \qquad C_y + D_y - W = 0$$

$$C\cos\beta = D\cos\phi \qquad C\sin\beta + D\sin\phi = W$$

$$C = D\left(\frac{\cos 45^\circ}{\cos 7.13^\circ}\right) \qquad C\sin 7.13^\circ + D\sin 45^\circ = 100\text{ N}$$

$$C = 0.713D \qquad 0.124C + 0.707D = 100\text{ N}$$

Solving these two equations simultaneously gives

$$C = 89.6\text{ N} \qquad D = 125.7\text{ N}.$$

Example 3.5 (4/4)

3. *Solve Particle B.*

   Referring to the FBD for particle B we can write these equations.

$$\begin{aligned} \Sigma F_x &= 0 \\ -A_x - P_x + C_x &= 0 \\ P\cos\theta &= C\cos\beta - A\cos\alpha \end{aligned} \qquad \begin{aligned} \Sigma F_y &= 0 \\ A_y - P_y - C_y &= 0 \\ P\sin\theta &= A\sin\alpha - C\sin\beta \end{aligned}$$

   Since $A = C = 89.6$ N, substituting and solving simultaneously gives

$$\begin{aligned} P\cos\theta &= 48.8 \text{ N} \\ P &= 84.5 \text{ N} \end{aligned} \qquad \begin{aligned} P\sin\theta &= 69.0 \text{ N} \\ \theta &= 54.7°. \end{aligned}$$

   These are the magnitude and direction of vector **P**. If you wish, you can express **P** in terms of its scalar components. The negative signs on the components have been applied by hand since **P** points down and to the left.

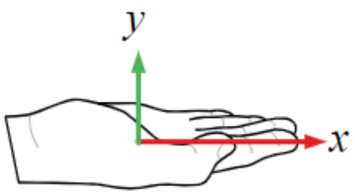
$$\begin{aligned} \mathbf{P} &= \langle -P\cos\theta, -P\sin\theta \rangle \\ &= \langle -48.8 \text{ N}, -69.0 \text{ N} \rangle \end{aligned}$$

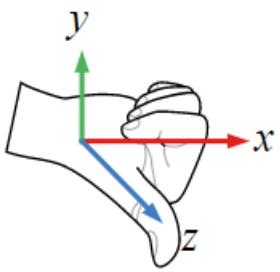
3.5 – Particles in Three Dimensions

- To model real-world problems we will have to consider all three dimensions.
- All the principles learned thus far still apply
- Especially important to have good diagrams and keep work neat and organized.

3.5.1 – Three-Dimensional Coordinate Frame

- For equilibrium of a particle, usually the origin of the coordinate frame is at the particle, the x axis is horizontal, the y axis is vertical just as in a two-dimensional situation, and the z axis is determined by the right hand rule.





3.5.2 – Free Body Diagrams

- Begin the analysis by drawing a free-body diagram which shows all forces and moment acting on the object of interest.

3.5.3 – Angles

- When working in three dimensions you actually need three angles to determine the direction of the vector.
- As with two dimensions, angles can be determined from geometry — a distance vector going in the same direction as the force vector.

3.5.3 – Angles

- The line of action goes through two points *A* and *B*, and the direction of the force is from *A* towards *B*.
- To determine the three angles, write the distance vector (r_{AB}) from *A* to *B.*
  - Starting at point *A*, you need to determine how to get to point *B* by moving in each of the three directions.
  - When writing these scalar components pay attention to which direction moved along the axes.
  - Determine the total distance from point *A* and *B*

$$r_{AB} = \sqrt{(r_{AB_x})^2 + (r_{AB_y})^2 + (r_{AB_z})^2}$$

3.5.3 – Angles

- The angles are determined by the direction cosines

$$r_{AB} = \sqrt{(r_{AB_x})^2 + (r_{AB_y})^2 + (r_{AB_z})^2}$$

- Since the force vector has the same line of action as the distance vector, by the three-dimensional version of similar triangles

$$\frac{r_{AB_x}}{r_{AB}} = \frac{F_x}{F} \qquad \frac{r_{AB_y}}{r_{AB}} = \frac{F_y}{F} \qquad \frac{r_{AB_z}}{r_{AB}} = \frac{F_z}{F}$$

$$F_x = \left(\frac{r_{AB_x}}{r_{AB}}\right)F \qquad F_y = \left(\frac{r_{AB_y}}{r_{AB}}\right)F \qquad F_z = \left(\frac{r_{AB_z}}{r_{AB}}\right)F$$

3.5.4 – General Procedure

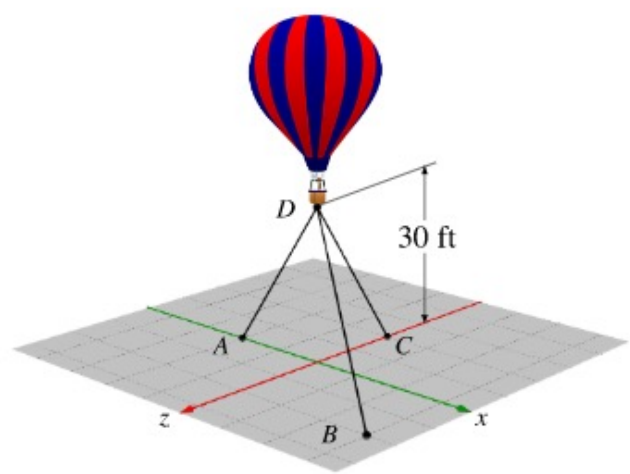
The general procedure for solving equilibrium of a particle (or concurrent force) problems in three dimensions

1. Identify the particle.
2. Establish a coordinate system.
3. Draw a free-body diagram.
4. State any given values and identify the unknown values.
5. Determine the direction of each of the force vectors
6. Count knowns and unknowns.
7. Formulate equilibrium equations.
8. Simplify.
9. Substitute values for symbols.
10. Check your work.

Example 3.6 (1/5)

A hot air balloon 30 ft above the ground is tethered by three cables as shown in the diagram. If the balloon is pulling upwards with a force of 900 lb, what is the tension in each of the three cables?

The grid lines on the ground plane are spaced 10 ft apart.

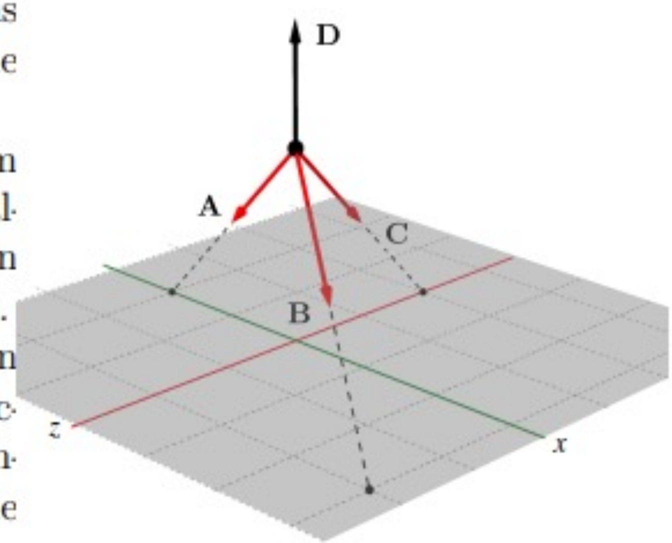


Example 3.6 (2/5)

Solution.

1. *Strategy.*

   The three tensions are the unknowns which we can find by applying the three equilibrium equations.
   We'll establish a coordinate system with the origin directly below the balloon and the y axis vertical, then draw and label a free-body diagram.
   Next we'll use the given information to find two points on each line of action, and use them to find the components of each force in terms of the unknowns.
   When the x, y and z components of all forces can be expressed in terms of known values, the equilibrium equations can be solved.



Example 3.6 (3/5)

2. *Geometry.*

From the diagram, the coordinates of the points are

$$\mathrm{A} = (-20, 0, 0) \qquad \mathrm{B} = (30, 0, 20) \qquad \mathrm{C} = (0, 0, -20) \qquad \mathrm{D} = (0, 30, 0)$$

Use the point coordinates to find the x, y and z components of the forces.

$$A_x = \frac{-20}{L_A}A \qquad A_y = \frac{-30}{L_A}A \qquad A_z = \frac{0}{L_A}A$$

$$B_x = \frac{30}{L_B}B \qquad B_y = \frac{-30}{L_B}B \qquad B_z = \frac{20}{L_B}B$$

$$C_x = \frac{0}{L_C}C \qquad C_y = \frac{-30}{L_C}C \qquad C_z = \frac{-20}{L_C}C$$

Where L_A, L_B and L_C are the lengths of the three cables found with the distance formula.

$$\begin{aligned} L_A &= \sqrt{(-20)^2 + (-30)^2 + 0^2} && = 36.1 \text{ ft} \\ L_B &= \sqrt{30^2 + (-30)^2 + 20^2} && = 46.9 \text{ ft} \\ L_C &= \sqrt{0^2 + (-30)^2 + (-20)^2} && = 36.1 \text{ ft} \end{aligned}$$

Example 3.6 (4/5)

3. *Equilibrium Equations.*

   Applying the three equations of equilibrium yields three equations in terms of the three unknown tensions.

$$\Sigma F_x = 0$$
$$A_x + B_x + C_x = 0$$
$$-\frac{20}{36.1}A + \frac{30}{46.9}B + 0\,C = 0$$
$$A = 1.153\,B \tag{1}$$

$$\Sigma F_z = 0$$
$$A_z + B_z + C_z = 0$$
$$0\,A + \frac{20}{46.9}B - \frac{20}{36.1}C = 0$$
$$C = 0.769\,B \tag{2}$$

$$\Sigma F_y = 0$$
$$A_y + B_y + C_y + D = 0$$
$$-\frac{30}{36.1}A - \frac{30}{46.9}B - \frac{30}{36.1}C + 900 = 0$$
$$0.832\,A + 0.640\,B + 0.832\,C = 900\text{ lb} \tag{3}$$

Example 3.6 (5/5)

Solving these equations simultaneously yields the answers we are seeking. One way to do this is to substitute equations (1) and (2) into (3) to eliminate A and C and solve the resulting equation for B.

$$0.832\,(1.153\,B) + 0.640\,B + 0.832\,(0.769\,B) = 900 \text{ lb}$$

$$\begin{aligned} 2.24B &= 900 \text{ lb} \\ B &= 402 \text{ lb} \end{aligned}$$

With B known, substitute it into equations (1) and (2) to find A and C.

$$\begin{aligned} A &= 1.153\,B \\ &= 464 \text{ lb} \end{aligned} \qquad \begin{aligned} C &= 0.769\,B \\ &= 309 \text{ lb} \end{aligned}$$

Textbook

Baker, Daniel W., and Haynes, William. “Chapter 3 – Equilibrium of Particles.” *Engineering Statics: Open and Interactive*. May 21, 2023. https://engineeringstatics.org/

Chapter 4 – Moments and Static Equivalence

Engineering Mechanics: Statics and Structures

Chapter Outline

4.1 – Direction of a Moment

- In a two-dimensional problem the direction of a moment can be determined easily by inspection as either clockwise or counter-clockwise.
- In three-dimensions a moment vector may point in any direction in space and is more difficult to visualize. The direction is established by the *right hand rule*.

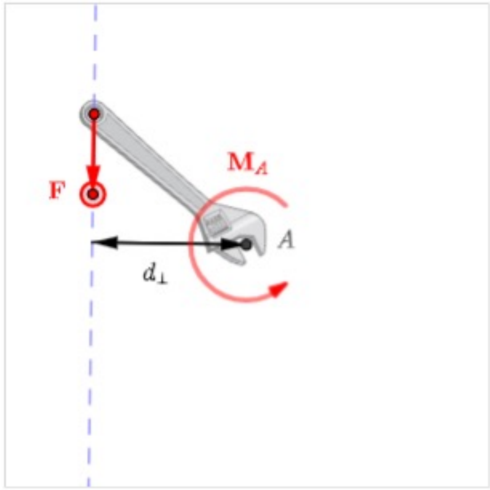
4.2 – Magnitude of a Moment

- The turning effect produced by a wrench depends on where and how much force applied to the wrench, and the optimum direction to apply the force is at right angles to the wrench's handle.

4.2.1 – Definition of a Moment

- The magnitude of a moment is found by multiplying the magnitude of force **F** times the ***moment arm***
  - The moment arm is defined as the ***perpendicular distance,*** $d_{\perp}$, from the center of rotation to the line of action of the force

$$M = Fd_{\perp}$$



4.3 – Scalar Components

- Vectors can be expressed as the product of a ***scalar component*** and a unit vector.
- Moments in two dimensions are either clockwise or counter-clockwise.
  - Counter-clockwise moments are positive
  - Clockwise moments are negative
- In three dimensions, moments, like forces, can be resolved into components in the *x*, *y*, and *z* directions.

$$\mathbf{M} = M_x\,\mathbf{i} + M_y\,\mathbf{j} + M_x\,\mathbf{k}$$

4.4 – Varignon’s Theorem

- Varignon’s Theorem is a method to calculate moments
  - It states that sum of the moments of several concurrent forces about a point is equal to the moment of the resultant of those forces, or alternately, the moment of a force about a point equals the sum of the moments of its components.
- The moment of a force can be calculated by first breaking it into components, evaluating the scalar moments of the individual components, and finally summing them to find the net moment about the point

4.4.1 – Rectangular Components

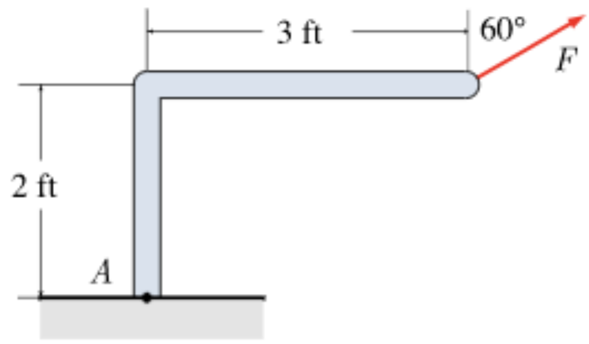
- The moment of a force is the sum of the moments of the components

$$M = \pm F_x d_y \pm F_y d_x$$

- A positive moment rotates the object counter-clockwise and a negative moment tend to rotate it clockwise

Example 4.1 (1/2)

A 750 lb force is applied to the frame as shown. Determine the moment this force makes about point *A*.



Example 4.1 (2/2)

Solution. Force **F** acts 60° from the vertical with a 750 lb magnitude, so its horizontal and vertical components are

$$F_x = F\sin 60^\circ = 649.5 \text{ lb}$$
$$F_y = F\cos 60^\circ = 375.0 \text{ lb}$$

For component F_x, the perpendicular distance from point A is 2 ft so the moment of this component is

$$M_1 = 2F_x = 1299 \text{ ft}\cdot\text{lb Clockwise}$$

.

For component F_x, the perpendicular distance from point A is 3 ft so the moment of this component is

$$M_2 = 3F_y = 1125 \text{ ft}\cdot\text{lb Counter-clockwise}$$

.

Assigning a negative sign to M_1 and a positive sign to M_2 to account for their directions and summing, gives the moment of **F** about A.

$$\begin{aligned} M_A &= -M_1 + M_2 \\ &= -1299 + 1125 \\ &= -174 \text{ ft}\cdot\text{lb} \end{aligned}$$

4.5 – Moments in Three Dimensions

- Moments are vectors and they will typically have components in the *x*, *y*, and *z* directions in three-dimensional situations
- The vector cross product will be used to calculate the moment in three dimensions

4.5.1 – Moment Cross Products

- The general method to find the moment of a force is to use the vector cross product

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

  - **F** is the force creating the moment in vector form
  - **r** is a position vector from the moment center to the line of action of the force

$$\begin{aligned} \mathbf{M} &= \mathbf{r} \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \\ &= (r_y F_z - r_z F_y)\,\mathbf{i} - (r_x F_z - r_z F_x)\,\mathbf{j} + (r_x F_y - r_y F_x)\,\mathbf{k} \end{aligned}$$

4.5.1 – Moment Cross Products

- The magnitude of the resultant moment can be calculated using the three-dimensional Pythagorean Theorem

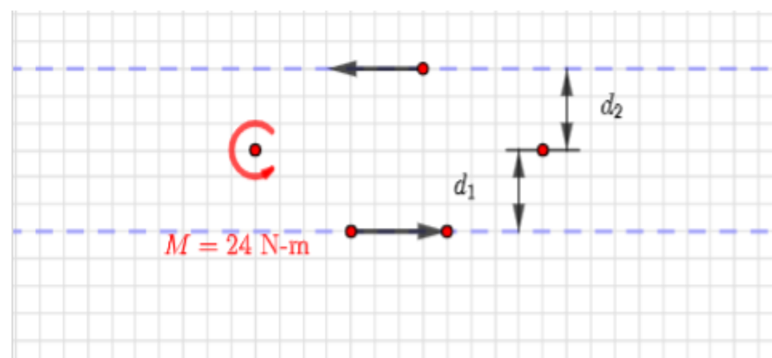
$$M = |\mathbf{M}| = \sqrt{{M_x}^2 + {M_y}^2 + {M_z}^2}$$

4.5.1 – Moment Cross Products

- Important notes when calculating the moment using the cross product
  - The order must always be **r** x **F**, never **F** x **r.**
  - The moment arm **r** must always be measured from moment center to the line of action of the force.
  - The signs of the components of **r** and **F** must follow those of a right-hand coordinate system.

4.6 – Couples

- A couple consists of two parallel forces, equal in magnitude, opposite in direction, and non-coincident.
- Couples are special because the pair of forces always cancel each other, which means that a couple produces a rotational effect but never translation.
  - “Pure Moment”



$$F_1 = 4\,\mathrm{N} \qquad d_1 = 3\,\mathrm{m}$$
$$F_2 = 4\,\mathrm{N} \qquad d_2 = 3\,\mathrm{m}$$
$$M = (12) + (12) \qquad = 24\,\mathrm{N\text{-}m}$$

4.6 – Couples

- When adding moments to find the total or resultant moment, you must include couple-moments as well the **r** × **F** moments.

$$\Sigma M_P = \Sigma(\mathbf{r} \times \mathbf{F}) + \Sigma(\mathbf{M}_{\text{couple}})$$

4.8 Statically Equivalent Systems

- A loading system is a combination of load forces and moments which act on an object.
- Any loading system may be replaced with a simpler *statically equivalent system* consisting of one *resultant force* at a specific point and one *resultant moment*

4.8 – Statically Equivalent Systems

- The resultant force acting on a system, **R**, can be found from adding the individual forces, $\mathbf{F}_i$

$$\mathbf{R} = \sum \mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \ldots$$

- The resultant moment, $\mathbf{M}_O$, about a point *O*, can be found from adding all of the moments **M**, about that point, including both **r** × **F** moments and concentrated moments.

$$\mathbf{M_O} = \sum \mathbf{M}_i = \mathbf{M_1} + \mathbf{M_2} + \mathbf{M_3} + \ldots$$

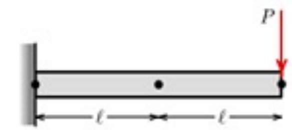
4.8 – Statically Equivalent Systems

- Force-Couple Systems
  - One transformation includes moving a force to another location. While *sliding* a force along its line of action is fine, *moving* a force to another point changes its line of action and thus its rotational effect on the object, so moving a force to a new line of action is *not* an equivalent transformation.
  - You *can* move a force to a new line of action in an equivalent fashion if you add a “compensatory couple” to undo the effect of changing the line of action.

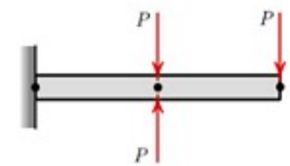
4.8 – Statically Equivalent Systems

Force-Couple System General Procedure

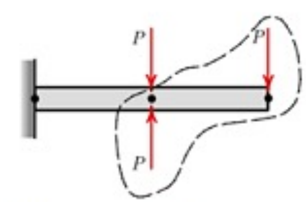
a. Draw the original system
b. Add two equal and opposite forces to desired location
c. Recognize couple formed
d. Replace couple with an equivalent couple-moment



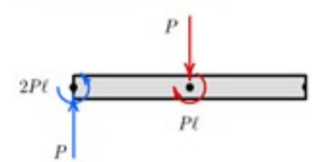
(a) Original situation.



(b) Add two equal and opposite forces at midpoint.



(c) Recognize couple.



(d) Replace couple to produce equivalent force-couple system,

Example 4.2 (1/2)

Replace the system of forces in diagram (a) with an equivalent force-couple system at *A.*

Replace the force-couple system at *A* with a single equivalent force and specify its location.

(a) (b) (c)

Example 4.2 (2/2)

Solution. The original system is shown in (a).

Since the F_1 and F_2 are parallel, the magnitude of the resultant force is just the sum of the two magnitudes and it points down.

$$R = F_1 + F_2$$

The resultant moment about point A is

$$M_A = F_1 d_1 + F_2(d_1 + d_2)$$

.

To create the equivalent system (b), the resultant force and resultant moment are placed at point A.

The system in (b) can be further simplified to eliminate the moment at M_A, by performing the process in reverse.

In (c) we place the resultant force R a distance d away from point A such that the resultant moment around point A remains the same. This distance can be found using $M = Fd$.

$$d = M_A/R$$

The systems in (a), (b), and (c) are all statically equivalent □

4.8 – Statically Equivalent Systems

There are four common special cases which are worth highlighting individually.

- **Concurrent forces**
- **Parallel forces**
- **Coplanar forces**
- **Wrench resultant**

4.8 – Statically Equivalent Systems

Concurrent forces

When all forces in a system are concurrent, the resultant moment about that their common intersection point will always be zero. We then need only find the resultant force and place it at the point of intersection. The resultant moment about any other point is the moment of the resultant force **R** about that point.

4.8 – Statically Equivalent Systems

Parallel forces

When all forces in a system are parallel, the resultant force will act in this direction with a magnitude equal to the sum of the individual magnitudes. There will be no moment created about this axis, but we need to find the resultant moment about the other two rectangular axes. That is, if all forces act in the x direction, we need only find the resultant force in the x direction and the resultant moment about the y and z axes

4.8 – Statically Equivalent Systems

Coplanar forces

When all forces in a system are coplanar we need only find the resultant force in this plane and the resultant moment about the axis perpendicular to this plane. That is, if all forces exist in the *x*-*y* plane, we need only to sum components in the *x* and *y* directions to find resultant force **R**, and use these to determine the resultant moment about the *z* axis. All two-dimensional problems fall into this category.

4.8 – Statically Equivalent Systems

Wrench resultant

A wrench resultant is a special case where the resultant moment acts around the axis of the resultant force. The directions of the resultant force vector and the resultant moment vector are the same.

Textbook

Baker, Daniel W., and Haynes, William. “Chapter 4 – Moments and Static Equivalence.” *Engineering Statics: Open and Interactive*. May 21, 2023. https://engineeringstatics.org/

Chapter 5 – Rigid Body Equilibrium

Engineering Mechanics: Statics and Structures

Chapter Outline

5.1 – Degree of Freedom

- *Degrees of freedom* refers to the number of independent parameters or values required to specify the *state* of an object.
  - Two-dimensional rigid bodies in the *xy* plane have three degrees of freedom.
  - Three-dimensional rigid bodies have six degrees of freedom
- For a body to be in static equilibrium, all possible movements of the body need to be adequately restrained.

5.2 – Free Body Diagrams

- Free body diagrams are used to identify the forces and moments that influence an object.
- Drawing a correct free-body diagram is the first and <u>most important</u> step in the process of solving an equilibrium problem.
  - A good free-body diagram is neat and clearly drawn and contains all the information necessary to solve the equilibrium.

5.2 – Free Body Diagrams

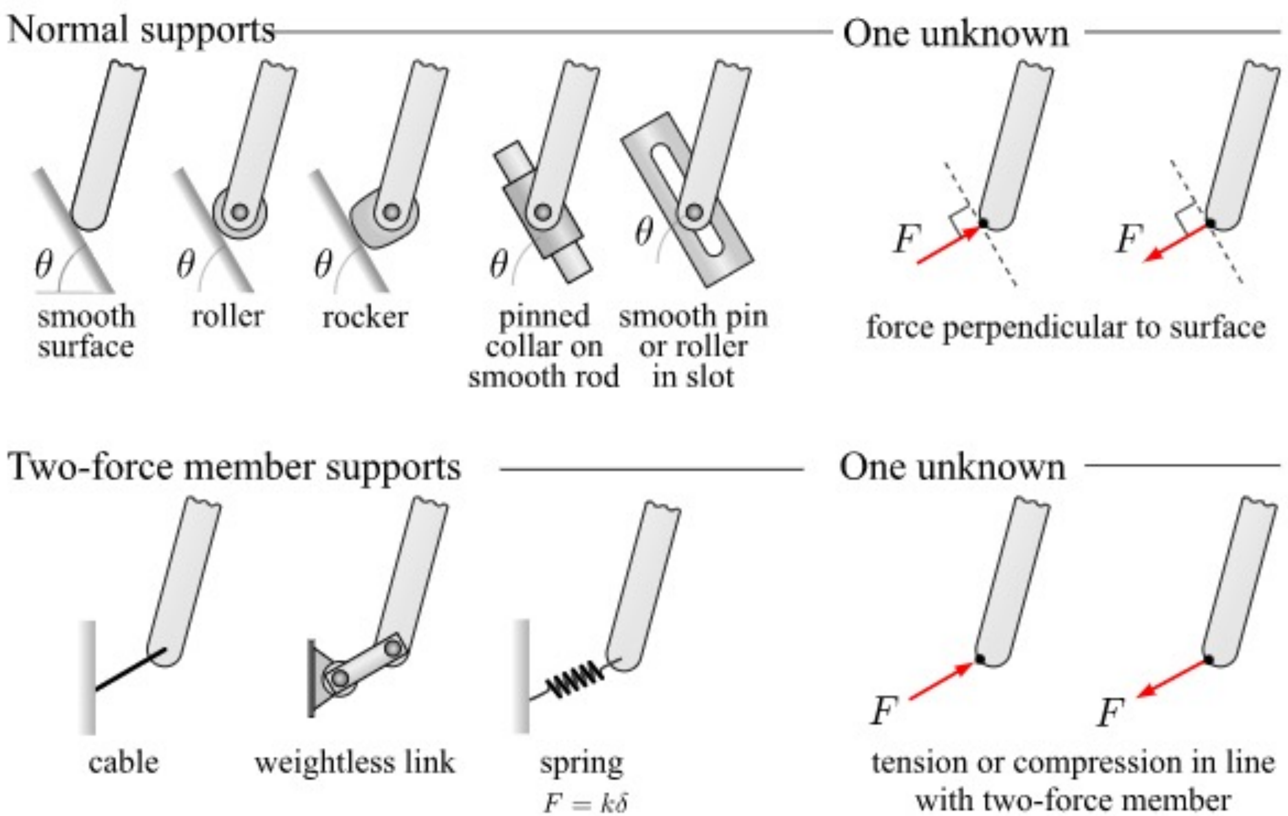
Every equilibrium problem begins by drawing and labeling a free-body diagram!

The basic steps to creating a FBD

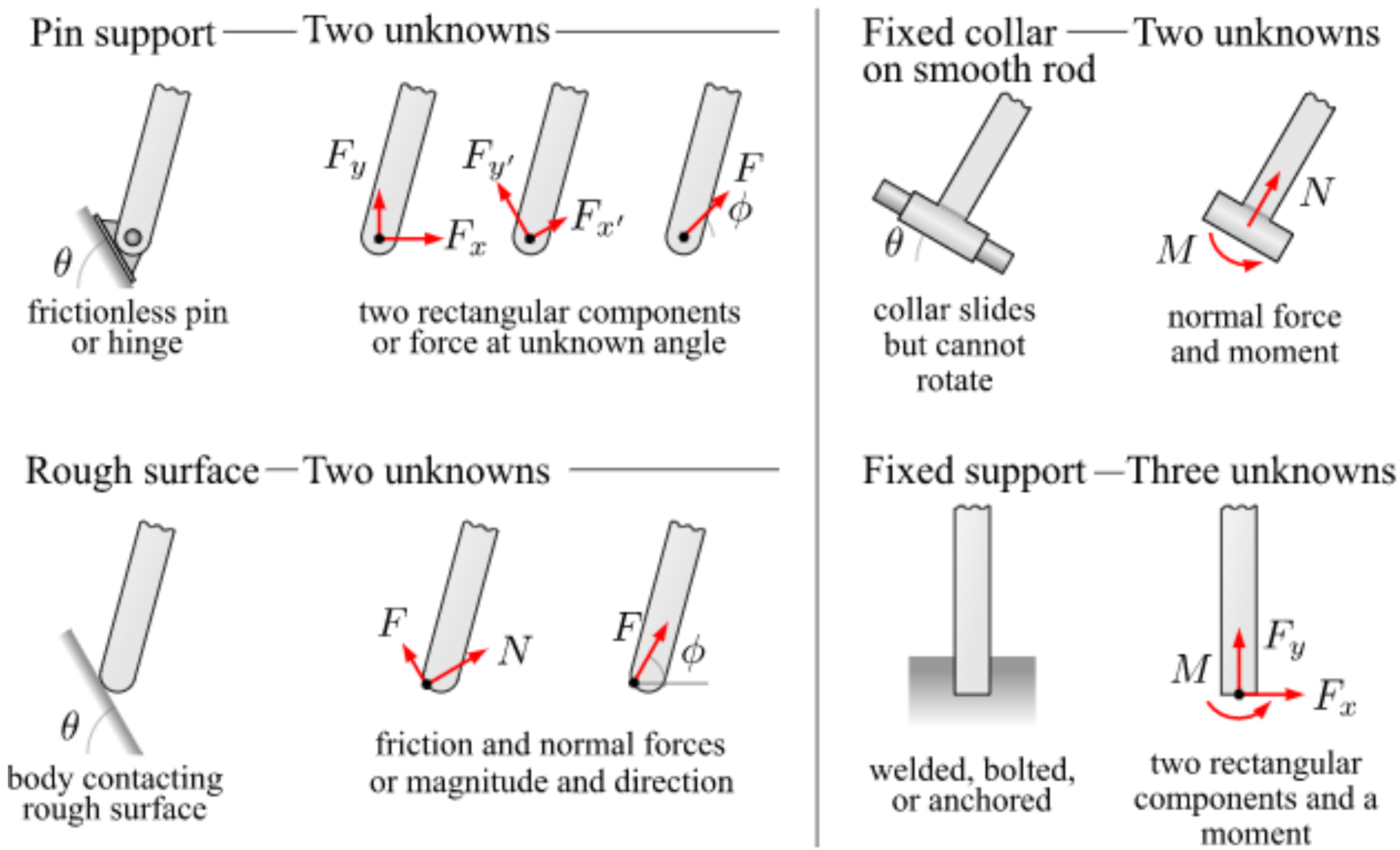
1. Select and isolate an object.
2. Select a reference frame.
3. Identify all loads.
4. Identify all reactions.
5. Label the diagram.

5.2 – Free Body Diagrams

Two-dimensional Reactions

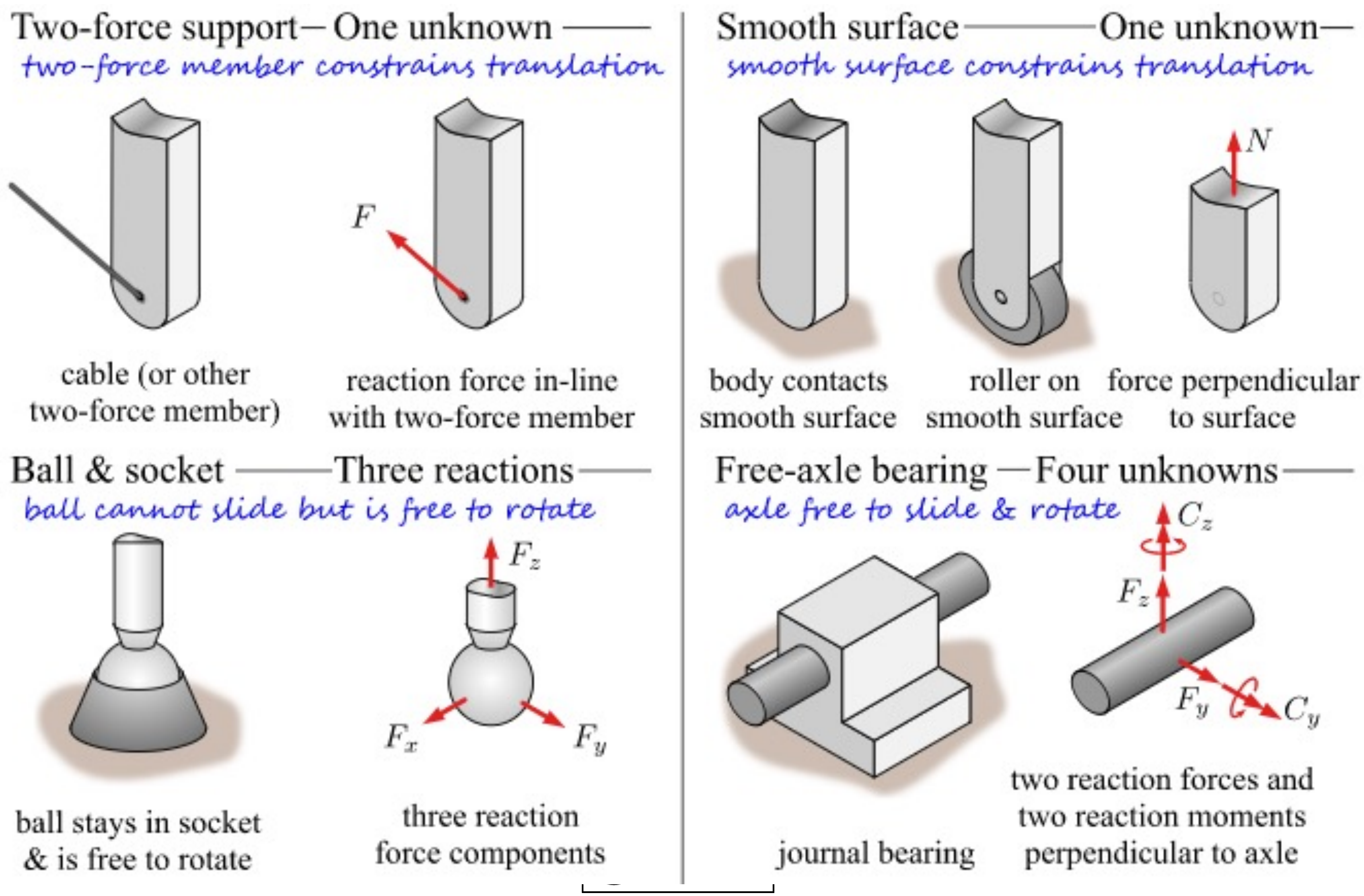


5.2 – Free Body Diagrams

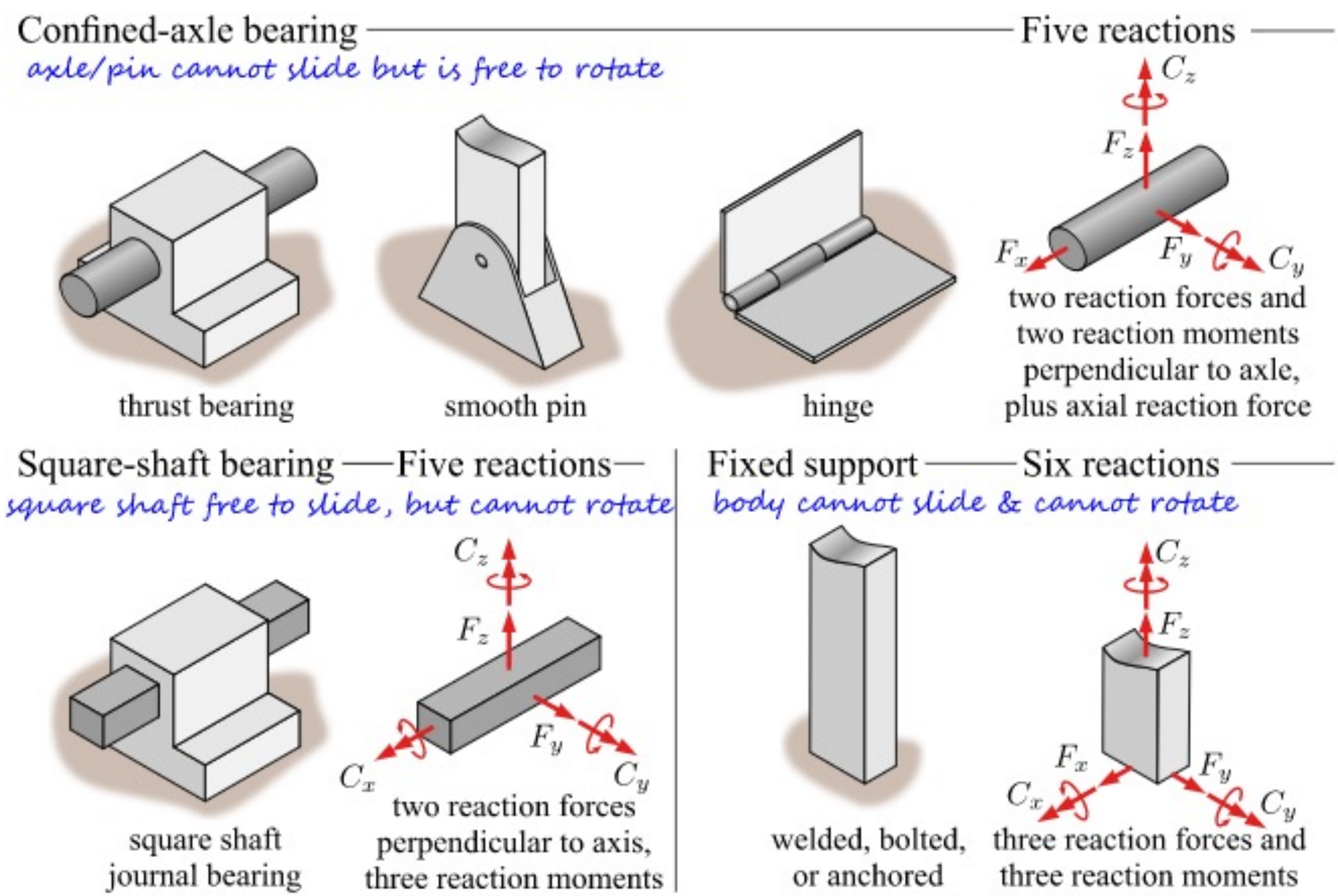


5.2 – Free Body Diagrams

Three-dimensional Reactions



5.2 – Free Body Diagrams



5.3 – Equations of Equilibrium

- The focus in statics is on systems where both linear acceleration **a** and angular acceleration α are zero. These systems are frequently stationary, but could be moving with constant velocity.
- Newton’s Second Law

$$\sum \mathbf{F} = 0$$

$$\sum \mathbf{M} = 0$$

5.3 – Equations of Equilibrium

- The scalar equations become

$$\sum \mathbf{F} = 0 \implies \begin{cases} \Sigma F_x & = 0 \\ \Sigma F_y & = 0 \\ \Sigma F_z & = 0 \end{cases} \qquad \sum \mathbf{M} = 0 \implies \begin{cases} \Sigma M_x & = 0 \\ \Sigma M_y & = 0 \\ \Sigma M_z & = 0 \end{cases}$$

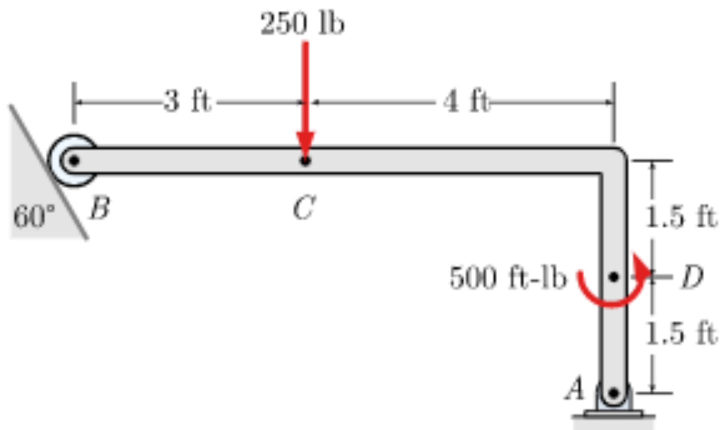
5.4 – 2D Rigid Body Equilibrium

- Two-dimensional rigid bodies have three degrees of freedom, so they only require three independent equilibrium equations to solve.

$$\begin{aligned} \sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum M_A &= 0 \end{aligned}$$

Example 5.1 (1/4)

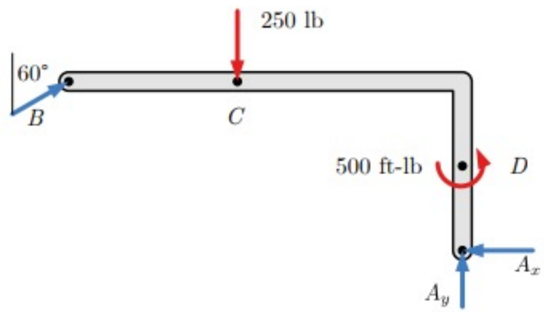
The L-shaped body is supported by a roller at *B* and a frictionless pin at *A*. The body supports a 250 lb vertical force at *C* and a 500 ft·lb couple-moment at *D*. Determine the reactions at *A* and *B*.



Example 5.1 (2/4)

Solution 1.

Solutions always start with a free-body diagram, showing all forces and moments acting on the object. Here, the known loads $C = 250$ lb (down) and $D = 500$ ft·lb (CCW) are red, and the unknown reactions A_x, A_y and B are blue.



The force at B is drawn along its known line-of-action perpendicular to the roller surface, and drawn pointing up and right because that will oppose the rotation of the frame about A caused by load C and moment D. The force at A is represented by unknown components A_x and A_y. The sense of these components is unknown, so we have arbitrarily assigned the arrowheads pointing left and up.

We have chosen the standard coordinate system with positive x to the right and positive y pointing up, and resolved force A into components in the x and y directions.

The magnitude of force B is unknown but its direction is known, so the x and y components of B can be expressed as

$$B_x = B \sin 60^\circ \qquad B_y = B \cos 60^\circ$$

Example 5.1 (3/4)

We choose to solve equation set $\{A\}$, and choose to take moments about point A, because unknowns A_x and A_y intersect there. Substituting the variables into the equation and solving for the unknowns gives

$$\begin{aligned} \sum F_x &= 0 \\ B_x - A_x &= 0 \\ A_x &= B \ \sin 60^\circ \end{aligned} \tag{1}$$

$$\begin{aligned} \sum F_y &= 0 \\ B_y - C + A_y &= 0 \\ A_y &= C - B \ \cos 60^\circ \end{aligned} \tag{2}$$

$$\begin{aligned} \sum M_A &= 0 \\ -B_x(3) - B_y(7) + C(4) + D &= 0 \\ 3B\cos 60^\circ + 7B\sin 60^\circ &= 4C + D \\ B(3\sin 60^\circ + 7\cos 60^\circ) &= 4C + D \\ B &= \frac{4C + D}{6.098} \end{aligned}$$

Example 5.1 (4/4)

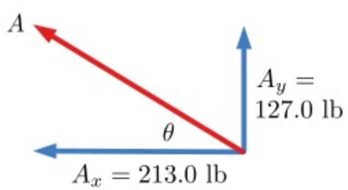
Now with the magnitude of B known, A_x and A_y can be found with (1) and (2).

$$\begin{aligned} A_x &= B \sin 60^\circ \\ &= 246.0 \sin 60^\circ \\ &= 213.0 \text{ lb} \end{aligned}$$

$$\begin{aligned} A_y &= C - B \cos 60^\circ \\ &= 250 - 246.0 \cos 60^\circ \\ &= 127.0 \text{ lb} \end{aligned}$$

The positive signs on these values indicate that the directions assumed on the free-body diagram were correct.

The magnitude and direction of force **A** can be found from the scalar components A_x and A_y using a rectangular to polar conversion.



$$A = \sqrt{A_x^2 + A_y^2} = 248.0 \text{ lb}$$

$$\theta = \tan^{-1} \left| \frac{A_y}{A_x} \right| = 30.8^\circ$$

The final values for **A** and **B**, with angles measured counter-clockwise from the positive x axis are

$$\mathbf{A} = 248.0 \text{ lb} \measuredangle 149.2^\circ,$$

$$\mathbf{B} = 246.0 \text{ lb} \measuredangle 30^\circ$$

5.5 – 3D Rigid Body Equilibrium

- Three-dimensional systems are closer to reality than two-dimensional systems and the basic principles to solving both are the same.
- Three-dimensional problems are usually solved using vector algebra rather than the scalar approach.

5.5 – 3D Rigid Body Equilibrium

Resolving Forces and Moments into Components

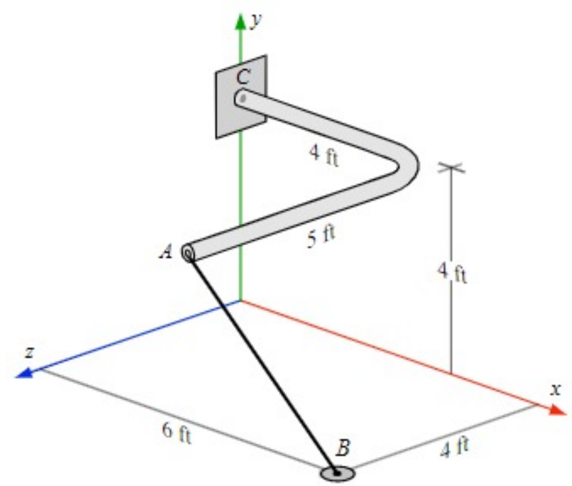
- Three-dimensional forces will need to be broken into components
- When summing moments, make sure to consider both the **r** × **F** moments and the couple-moments

Example 5.2 (1/4)

The bent bar shown is held in a horizontal plane by a fixed connection at *C* while cable *AB* exerts a 500 lb force on point *A*.

Given *A* = (4,4,5) *B* = (6,0,4) and *C* = (0, 4, 0).

Find the reaction force F and concentrated moment M required to hold the bar in this position under this condition.

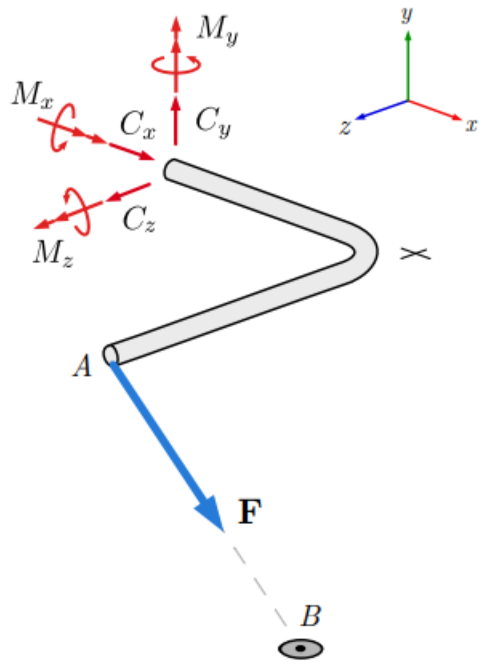


Example 5.2 (2/4)

Solution.

1. *Draw a free-body diagram.*

   As always, begin by drawing a free-body diagram.



Example 5.2 (3/4)

2. *Determine the force acting at point A in Cartesian form.*

   The force of the cable acts from A to B. This direction is described by the displacement vector from A to B

$$\mathbf{r}_{AB} = (2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k})\ \text{ft}$$

or the corresponding unit vector

$$\begin{aligned}\lambda_{AB} &= \frac{\mathbf{r}_{AB}}{|\mathbf{r}_{AB}|}\\ &= \frac{2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k}}{\sqrt{(2)^2 + (-4)^2 + (-1)^2}}\\ &= \frac{2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k}}{\sqrt{21}}\end{aligned}$$

.

Multiplying the unit vector by the cable tension gives the force acting on A as a three-dimensional Cartesian force vector

$$\begin{aligned}\mathbf{F} &= \lambda_{AB} T\\ &= \left(\frac{2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k}}{\sqrt{21}}\right) 500\ \text{lb}\\ &= (2\mathbf{i} - 4\mathbf{j} - 1\mathbf{k}) \left(\frac{500}{\sqrt{21}}\right)\ \text{lb}\\ \mathbf{F} &= (218\mathbf{i} - 436\mathbf{j} - 109\mathbf{k})\ \text{lb}\end{aligned}$$

Example 5.2 (4/4)

4. *Apply the equations of equilibrium.*

$$\Sigma\mathbf{F} = 0 \quad \begin{cases} \Sigma F_x = 0: & C_x + F_x = 0 \\ & C_x = -218 \text{ lb} \\ \Sigma T_y = 0: & C_y - F_y = 0 \\ & C_y = +436 \text{ lb} \\ \Sigma T_z = 0: & C_z - F_z = 0 \\ & C_z = +109 \text{ lb} \end{cases}$$

$$\Sigma\mathbf{M} = 0 \quad \begin{cases} \Sigma M_x = 0: & M_x + M_{Cx} = 0 \\ & M_x = -2180 \text{ ft}\cdot\text{lb} \\ \Sigma M_y = 0: & M_y + M_{Cy} = 0 \\ & M_y = -1530 \text{ ft}\cdot\text{lb} \\ \Sigma M_z = 0: & M_z + M_{Cz} = 0 \\ & M_z = +1750 \text{ ft}\cdot\text{lb} \end{cases}$$

$$\mathbf{C} = (-218\mathbf{i} + 436\mathbf{j} + 109\mathbf{k}) \text{ lb}$$
$$\mathbf{M} = (-2180\mathbf{i} - 1530\mathbf{j} + 1750\mathbf{k}) \text{ ft}\cdot\text{lb}$$

5.6 – Stability and Determinacy

- **Determinate vs. Indeterminate.** A static system is ***determinate*** if it is possible to find the unknown reactions using the methods of statics, that is, by using equilibrium equations, otherwise it is ***indeterminate***.
- **Stable vs. Unstable.** When there are sufficient supports to restrain a body from moving, we say that the body is ***stable***. A stable body is prevented from translating and rotating in all directions. A body which *can* move is ***unstable*** even if it is not currently moving.

5.6 – Stability and Determinacy

Rules to Validate a Stable and Determinate System.

1. *Rule 1:* Are there exactly three reaction components on a two-dimensional body?

   If YES, the system is determinate.

   If NO, the system is indeterminate or not stable

2. *Rule 2:* Are all the reaction force components parallel to one another?

   If YES, the system is unstable for translation.
   If NO, the system is stable for translation.

3. *Rule 3:* Do the lines of action of the reaction forces intersect at a single point?

   If YES, the system is unstable for rotation about the single intersection point.
   If NO, the system is stable for rotation.

Textbook

Baker, Daniel W., and Haynes, William. “Chapter 5 – Chapter 5 Rigid Body Equilibrium.” *Engineering Statics: Open and Interactive*. May 21, 2023. https://engineeringstatics.org/

Chapter 6 – Equilibrium of Structures

Engineering Mechanics: Statics and Structures

Chapter Outline

6.1 – Structures

Structures fall into three broad categories: trusses, frames, and machines.

- ***Truss***: a multi-body structure made up of long slender members connected at their ends in triangular subunits. Truss members carry axial forces only.
- ***Frame***: a multi-part, rigid, stationary structure primarily designed to support some type of load. A frame contains at least one multi-force member.
- ***Machine***: very similar to a frame, except that it includes some moving parts. The purpose of a machine is usually to provide a mechanical advantage and multiply forces.

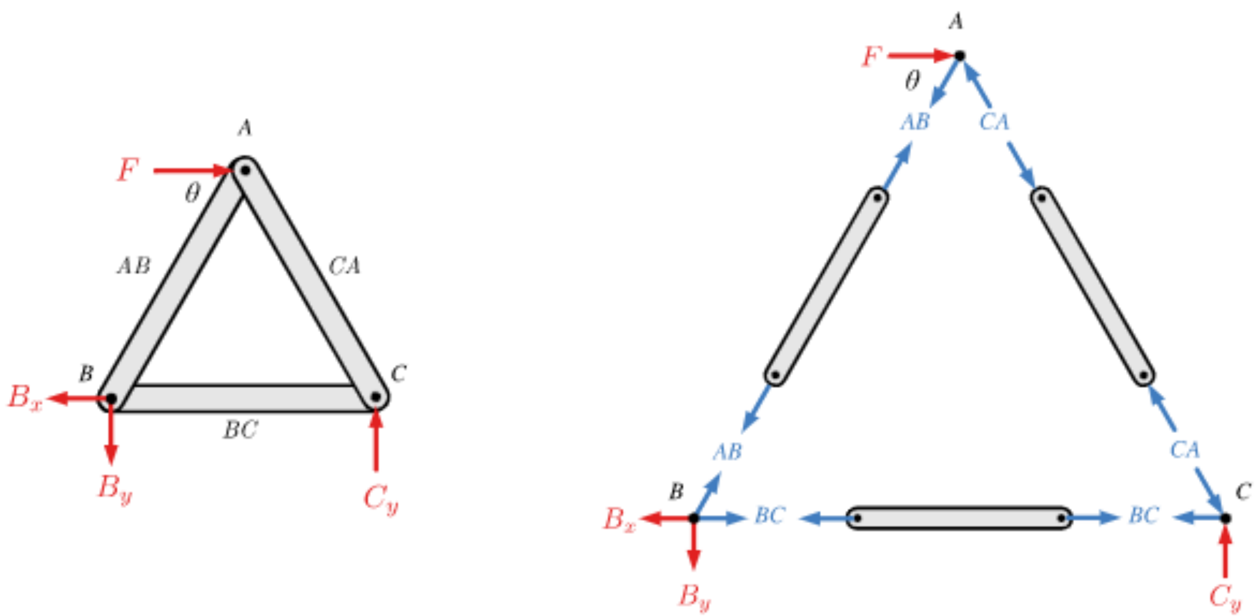
6.1 – Structures

Two-force Members

- Many structures contain at least one two-force member, and trusses consist of two-force members exclusively.
- Identifying two-force members is helpful when solving structures because they automatically establish the line of action of the two forces.
- The common way to express the force of a two-force member is with a magnitude and a sense, where the sense is either tension or compression.

6.2 – Interactions between members

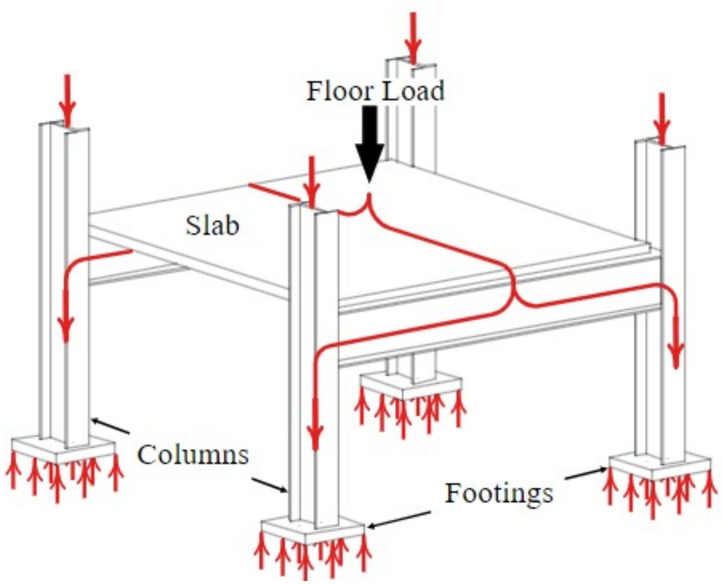
- Newton's 3rd Law, "For every action, there is an equal and opposite reaction."



(a) Whole Truss **(b)** Exploded

6.2.1 – Load Paths

- Load paths show how applied forces pass through the interconnected members of the structure until they end up at the fixed support reactions.



6.3 – Trusses

- A ***truss*** is a rigid engineering structure made up of long, slender members connected at their ends.
- Simple trusses are made of all two-force members and all joints are modeled as frictionless pins.
- All applied and reaction forces of a simple truss are applied only to these joints.

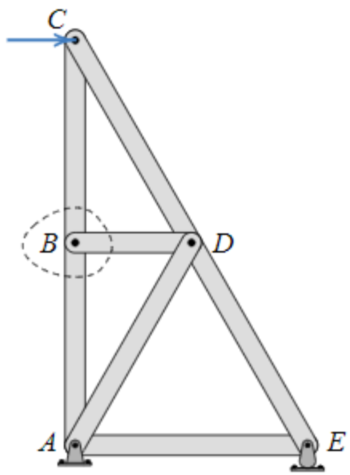
6.3.3 – Solving Trusses

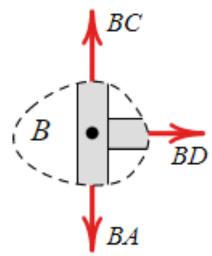
- "Solving" a truss means identifying and determining the unknown forces carried by the members of the truss when supporting the assumed load.
- There are two strategies to solve trusses:
  - Method of Joints
  - Method of Sections

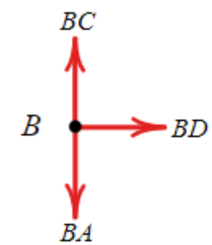
6.3.4 Zero-Force Members

- Zero-force members carry no force and thus support no load.
- zero-force members can be identified by inspection:
  - Rule 1: If *two* non-collinear members meet at an unloaded joint, then both are zero-force members.
  - Rule 2: If *three* forces (interaction, reaction, or applied forces) meet at a joint and two are collinear, then the third is a zero-force member.

6.3.4 – Zero-Force Members



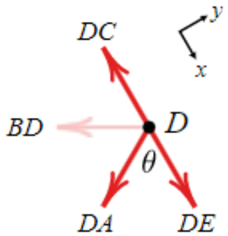




Vertically, forces *BC* and *BA* must be equal, and horizontally, force *BD* must be zero to satisfy $\Sigma F_x=0$. We learn that member *BD* is a zero-force member.

6.3.4 – Zero-Force Members

Finding zero-force members is an iterative process.



$$\Sigma F_y = 0$$
$$DA \sin\theta = 0$$

Therefore, force *DA* must be zero, and we can conclude that member *DA* is a zero-force member as well.

6.4 – Method of Joints

- The ***method of joints*** is a process used to solve for the unknown forces acting on members of a truss.
- The method centers on the joints or connection points between the members.

6.4.1 – Procedure (MOJ)

1. Determine if the structure is a truss and if it is determinate.
2. Identify and remove all zero-force members.
3. Determine if you need to find the external reactions.
   a. A solvable joint includes one or more known forces and no more than two unknown forces
4. Identify a solvable joint and solve it using the methods of Chapter 3. When drawing free-body diagrams of joints you should
   a. Represent the joint as a dot.
   b. Draw all known forces in their known directions with arrowheads indicating their sense. Known forces are the given loads, and forces determined from previously solved joints.
   c. Assume the sense of unknown forces. A common practice is to assume that all unknown forces are in tension, i.e. pulling away from the free-body diagram of the pin, and label them based on the member they represent.

6.4.1 – Procedure (MOJ)

5. Finally, write out and solve the force equilibrium equations for the joint
6. Once the unknown forces acting on a joint are determined, carry these values to the adjacent joints and repeat step four until all the joints have been solved.
7. If you solved for the reactions in step two, you will have more equations available than unknown forces when you reach the last joint. The extra equations can be used to check your work.

6.5 – Method of Sections

- The ***method of sections*** is used to solve for the unknown forces within specific members of a truss without solving for them all.
- The method involves dividing the truss into sections by cutting through the selected members and analyzing the section as a rigid body.

6.5.1 – Procedure (MOS)

1. Determine if a truss can be modeled as a simple truss.
2. Identify and eliminate all zero-force members.
3. Solve for the external reactions, if necessary.
4. Use your imaginary chain saw to cut the truss into two pieces by cutting through some or all of the members you are interested in. The cut does not need to be a straight line.
    a. Every cut member exposes an unknown internal force, so if you cut three members you'll expose three unknowns. Exposing more than three members is not advised because you create more unknowns than available equilibrium equations.

6.5.1 – Procedure (MOS)

5. Select the easier of the two halves of the truss and draw its free-body diagram.
    a. Include all applied and reaction forces acting on the section, and show known forces acting in their known directions.
    b. Draw unknown forces in assumed directions and label them. A common practice is to assume that all unknown forces are in tension and label them based on the endpoints of the member they represent.
6. Write out and solve the equilibrium equations for your chosen section. If you assumed that unknown forces were tensile, negative answers indicate compression.
7. If you have not found all the required forces with one section cut, repeat the process using another imaginary cut or proceed with the method of joints if it is more convenient.

6.6 – Frames and Machines

Frame and machines are engineering structures that contain at least one multi-force member.

- ***Frames*** are rigid, stationary structures designed to support loads and must include at least one multi-force member.
- ***Machines*** are non-rigid structures where the parts can move relative to one another. Generally they have an input and an output force and are designed produce a mechanical advantage.

6.6.1 – Analyzing Frames and Machines

Analyzing a frame or machine means determining all applied, reaction, and internal forces and couples acting on the structure and all of its parts.

- Multi-part structures are analyzed by mentally taking them apart and analyzing the entire structure and each part separately. Each component is analyzed as an separate rigid body

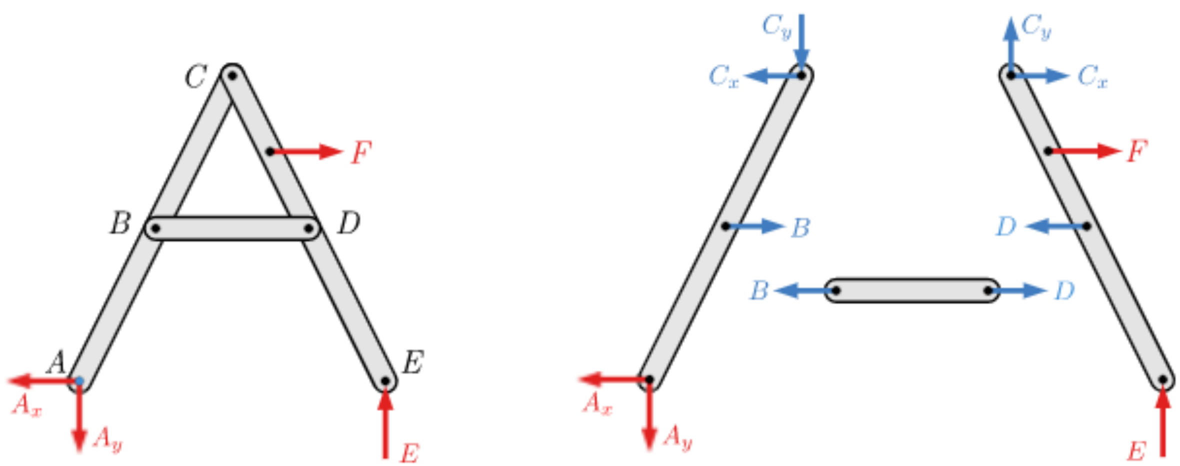
6.6.1 – Analyzing Frames and Machines

Procedure

1. Determine if the entire structure is independently rigid.
2. Draw a free-body diagram for each of the members in the structure.
   a. Applied forces and couples and the weights of the components if non-negligible.
   b. Interaction forces due to two-force members.
   c. All reaction forces and moments at the connection points between members.

6.6.1 – Analyzing Frames and Machines

All interaction forces and moments between connected bodies *must* be shown as equal-and-opposite action-reaction pairs.



6.6.1 – Analyzing Frames and Machines

5. Write out the equilibrium equations for each free-body diagram.
6. Solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once.

   Negative magnitudes indicate that the assumed direction of that term was incorrect, and the actual force/moment is opposite the assumed direction.

6.6.1 – Analyzing Frames and Machines

Free-body diagram of structures

The toggle clamp shown is used to quickly secure wooden furniture parts to the bedplate of a CNC router in order to cut mortise and tenon joints. The machine will be used to show how to create a FBD.

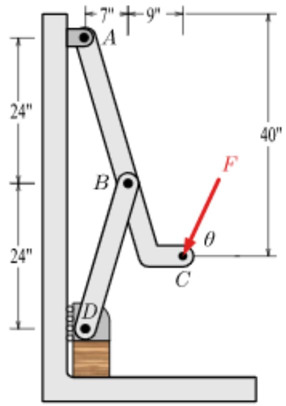


Figure 6.6.5. Original diagram

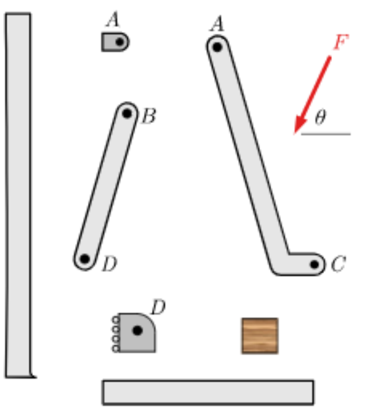


Figure 6.6.6. Component parts.

6.6.1 – Analyzing Frames and Machines

Exclude the floor

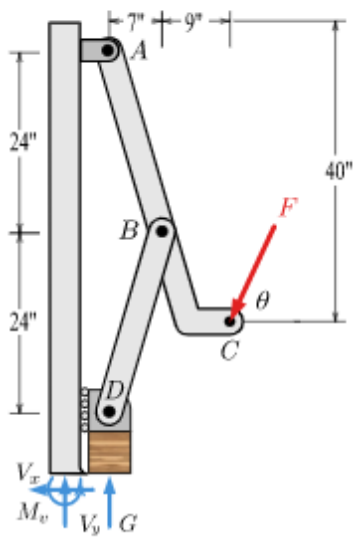


Figure 6.6.7. Free-body Diagram 1

Included	Excluded
Lever ABC, Short Link BD, Wooden Block, Roller D, Wall, Bearing A	Floor

Tips.

- Include friction if it's given or obvious.
- Internal forces in rigid bodies should be modeled as a fixed support.
- If you need info which you don't have, select a variable to act as its name.

6.6.1 – Analyzing Frames and Machines

Exclude the wall

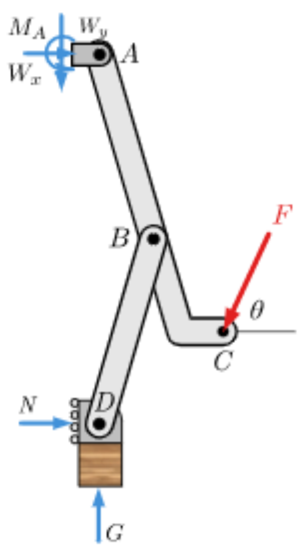


Figure 6.6.8. Free-body diagram 2

Included	Excluded
Lever ABC, Short Link BD, Wooden Block, Roller D, Bearing A	Floor, Wall

Tips.

- Every force needs a point of application and a clear arrowhead.
- Indicate any distances and angles needed and not available on the original diagram.
- Define the direction of forces which are not vertical or horizontal with an angle from a reference direction.
- Define a coordinate system unless you are using the standard x-y axes.
- Do not add forces that don't act on your body.

6.6.1 – Analyzing Frames and Machines

Exclude the bearing at *A*

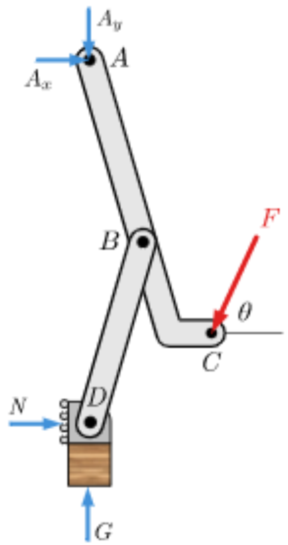


Figure 6.6.9. Free-body diagram 3

Included	Excluded
Lever ABC, Short Link BD, Wooden Block, Roller D,	Floor, Wall, Bearing A

Tips.

- Look for free-body diagrams which include only three unknowns in two dimensions or six unknowns in three.
- Don't include internal loads on your free-body diagrams.

6.6.1 – Analyzing Frames and Machines

Examine the wooden block

A free-body diagram of the block shows the clamping force Q, which we are seeking.

Note that $Q \neq G$. These forces are given different names since they may have different magnitudes. If the weight of the block is small[1] in comparison to the other forces acting on the object it may be neglected, in which case $Q = G$ and they could be given the same name.

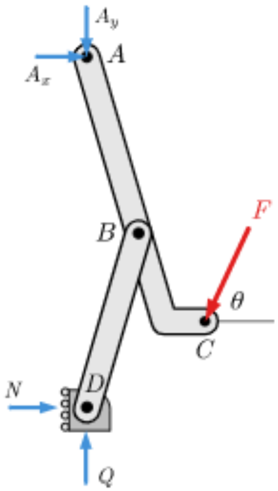
Figure 6.6.10. Free-body diagram 4 (block)

Tips.

- If the two forces are not the same don't identify them by the same name.
- Make as few assumptions as you possibly can. Make a note of any assumptions you make.
- In textbook problems, if the weight of an object is not mentioned it may be neglected.

6.6.1 – Analyzing Frames and Machines

Exclude the wooden block



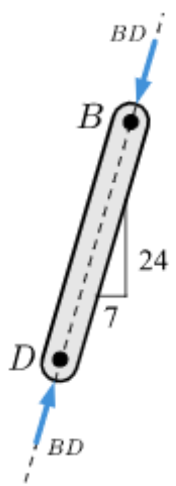
Included	Excluded
Lever ABC, Short Link BD, Roller D	Floor, Wall, Bearing A, Wooden Block

Figure 6.6.11. Free-body diagram 5 (lever and link)

6.6.1 – Analyzing Frames and Machines

Examine the short link *BD*

BD is a two-force member.



When drawing free-body diagrams, forces with known directions should be drawn pointing in that direction rather than breaking them into components, otherwise you may lose track of the fact that the x and y components are not independent but are actually related by the direction of the force.

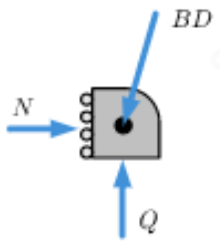
Figure 6.6.12. Free-body diagram 6

Tips.

- A short-link is a two-force body.
- Recognize two-force bodies because they give you information about direction.
- Represent the force of a two-force bodies as a force with unknown magnitude acting along a known line of action, not as x and y components.
- If you don't know the sense of a force along its line of action, assume one. If you guess wrong, the analysis will give you a negative value.

6.6.1 – Analyzing Frames and Machines

Examine the roller at *D*



Note that the force BD acting on the roller is shown pointing down and to the left. This is the opposite to the force acting on the link at D, which acts up and to the right. These two must act in opposite directions because they are an action-reaction pair.

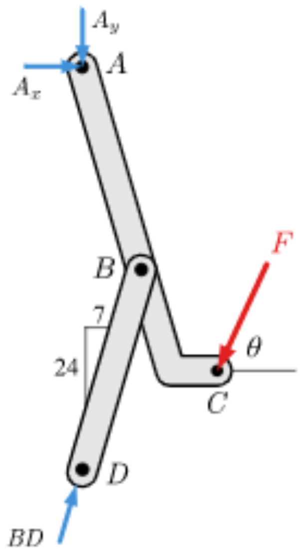
Figure 6.6.13. Free-body diagram 7 (roller)

Tips.

- Recognize three-force bodies and use their special properties to your advantage.
- Use the same name for the exposed forces on interacting bodies since they are equal-and-opposite halves of an action-reaction pair.

6.6.1 – Analyzing Frames and Machines

Exclude the roller

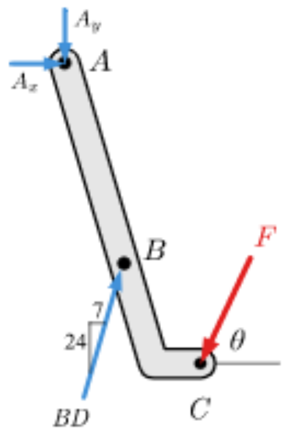


Included	Excluded
Lever ABC, Short Link BD	Floor, Wall, Bearing A, Wooden Block, Roller D

Figure 6.6.14. Free-body diagram 8

6.6.1 – Analyzing Frames and Machines

Exclude the short link

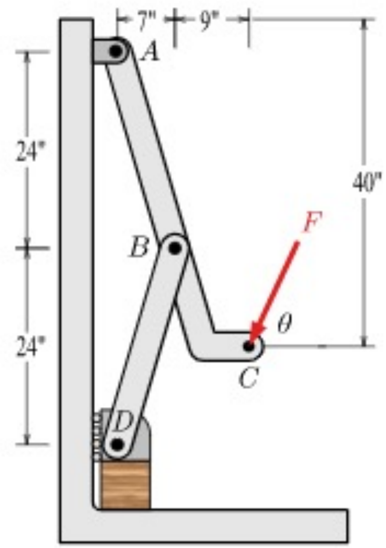


Included	Excluded
Lever ABC	Floor, Wall, Bearing A, Wooden Block, Roller D, Short Link BD

Figure 6.6.15. Free-body diagram 9 (lever)

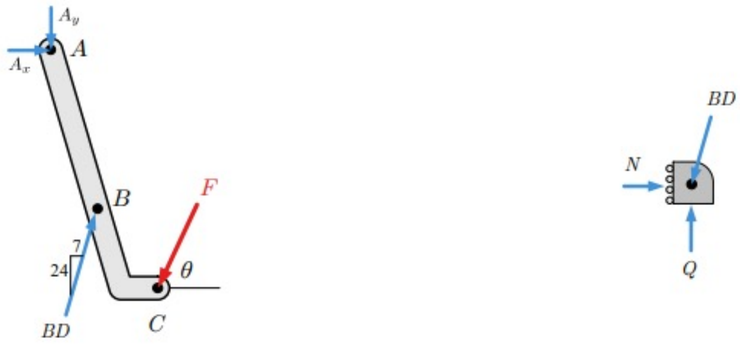
Example 6.1 (1/3)

Knowing that angle Θ = 60°, find the vertical clamping force acting on the piece at *D* and the magnitude of the force exerted on member *ABC* at pin *B* in terms of force *F* applied to the clamp arm at *C*.



Example 6.1 (2/3)

Solution. For this problem, we need two free-body diagrams. The first links the input force F to the link force BD, and the second links BD to the clamping force Q.



(a) FBD I **(b)** FBD II

Figure 6.6.18

We will assume the two-force member BD is in compression based on the physical situation. The forces acting on the link, lever and roller are all directed along a line-of-action defined by a 7-24-25 triangle. Similar triangles gives

$$BD_x = \left(\frac{7}{25}\right) BD$$

$$BD_y = \left(\frac{24}{25}\right) BD$$

Example 6.1 (3/3)

Applying $\sum M = 0$ at A to the free-body diagram of the lever gives BD in terms of F.

$$\text{FBD I:} \quad \Sigma M_A = 0$$
$$BD_x(24) + BD_y(7) - F_x(40) - F_y(16) = 0$$
$$\left(\frac{7}{25}BD\right)(24) + \left(\frac{24}{25}BD\right)(7) = (F\cos 60^\circ)(40) + (F\sin 60^\circ)(16)$$
$$13.44BD = 33.86F$$
$$BD = 2.52F$$

The positive sign on the answer reveals that our assumption that member BD was in compression was correct.

Applying $\sum F_y = 0$ to the free-body diagram of the roller will give Q in terms of F.

$$\text{FBD II:} \quad \Sigma F_y = 0$$
$$Q - BD_y = 0$$
$$Q = \frac{24}{25}BD$$
$$= \frac{24}{25}(2.52F)$$
$$= 2.42F$$

6.7 – Summary

Particle Equilibrium.

An object may be treated as a particle when the forces acting on it are coincident, that is, all of their lines of action intersect at a common point. In this case, they produce no moment to rotate the object, and $\Sigma\mathbf{M}=0$ is not helpful. The applicable equation is

$$\Sigma\mathbf{F}=0$$

which produces two scalar equations in two dimensions and three scalar equations in three dimensions.

6.7 – Summary

Rigid Body Equilibrium.

A rigid body can rotate and translate so both force and moment equilibrium must be considered.

$$\Sigma \mathbf{F}=0 \quad \Sigma \mathbf{M}=0$$

In two dimensions, these equations produce in two scalar force equations and one scalar moment equation. Up to three unknowns can be determined.

In three dimension, they produce three scalar force equations and scalar three moment equations. Up to six unknowns can be determined.

6.7 – Summary

Trusses.

A truss is a structure which consists entirely of two-force members and only carries forces at the joints connecting members. Two-force members and loading at joints allows free-body diagram of the joints to expose the axial loads in members.

In addition to the equations provided by treating the entire truss as a rigid body, each joint provides two additional equations for two-dimensional trusses, and three for non-planar trusses.

6.7 – Summary

Frames and Machines.

Frames and machines are structures which contain multiple rigid body systems. Frames don't move and are designed to support loads. Machines are generally designed to multiply forces, and usually have moving parts. Both frames and machines can be solved using the same methods.

All interactions between bodies are equal and opposite action-reaction pairs.

When solving frames and machines:

> Two-force members provide one useful equilibrium equation, and can determine one unknown.
>
> In two dimensions, rigid bodies result in two scalar force equations and one scalar moment equation. Up to three unknowns can be determined.
>
> In three dimensions, rigid bodies produce three scalar force equations and scalar three moment equations. Up to six unknowns can be determined.

Textbook

Baker, Daniel W., and Haynes, William. "Chapter 6 – Equilibrium of Structures." *Engineering Statics: Open and Interactive*. May 21, 2023. https://engineeringstatics.org/

Chapter 7 – Centroids and Centers of Gravity

Engineering Mechanics: Statics and Structures

Chapter Outline

7.1 – Weighted Averages

In situations where some values are more important than others, we use a ***weighted average***.

In general terms a weighted average is

$$\bar{a} = \frac{\sum a_i w_i}{\sum w_i}$$

Where a_i are the values we are averaging and w_i are the corresponding weighting factors.

Example 7.1 (1/3)

The mechanics syllabus says that there are two exams worth 25% each, homework is 10%, and the final is worth 40%. You have a 40 on the first exam, a 80 on the second exam, and your homework grade is 90.

What do you have to earn on the final exam to get a 70 in the class?

Example 7.1 (2/3)

Solution. Your known grades and the weighting factors are

$$G_i = [40, 80, 90, FE]$$
$$w_i = [25\%, 25\%, 10\%, 40\%]$$

Find final exam score FE so that your average grade $\bar{G}$ is 70%.

$$\bar{G} = \frac{\sum G_i w_i}{\sum w_i}$$
$$70 = \frac{(40 \times 0.25) + (80 \times 0.25) + (90 \times 0.1) + (FE \times 0.4)}{(0.25 + 0.25 + 0.1 + 0.4)}$$
$$FE = \frac{70(1) - (10 + 20 + 9)}{0.4} = 77.5$$

7.2 – Center of Gravity

The ***center of gravity*** is the point where all of an object's weight may be concentrated and still have the same *external* effect on the body.

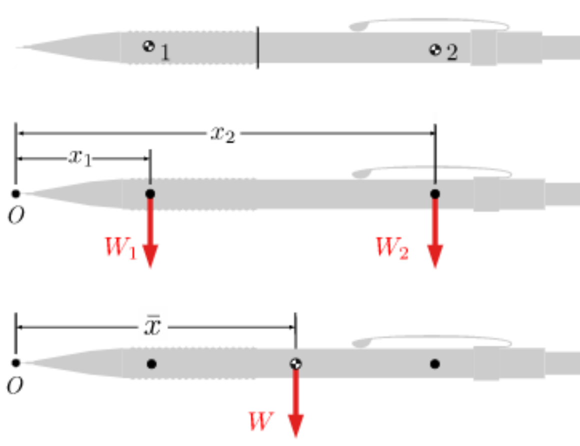
The center of gravity of a body is fixed with respect to the body, but the coordinates depend on the choice of coordinate system.

7.2 – Center of Gravity

Let's look at the center of gravity for a pencil

Assume that the two halves of the pencil have known weights acting at points 1 and 2.

To be equivalent, the total weight must equal the total weight of the parts. $W=W_1+W_2$. Common sense also tells us that W will act somewhere between W_1 and W_2.



7.2 – Center of Gravity

- Pick the tip of the pencil to be point *O* and calculate the total moment about point *O* due to the two weights.

$$\sum M_O = -x_1 W_1 - x_2 W_2$$

- Need a single equivalent force acting at the (unknown) center of gravity. Call the distance from the origin to the center of gravity $\bar{x}$.
- $\bar{x}$ represents the mean distance of the weight, mass, or area depending on the context of the problem. We are evaluating weights in this problem, so $\bar{x}$ represents the distance from *O* to the center of gravity.
- The sum of moments around point *O* for the equivalent system can be written as:

$$\sum M_O = -\bar{x} W$$

7.2 – Center of Gravity

- Set the two equations equal to each other

$$-\bar{x}W = -x_1W_1 - x_2W_2$$
$$\bar{x} = \frac{x_1W_1 + x_2W_2}{W_1 + W_2}$$

- Location of the centroid is

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i}$$

7.2 – Center of Gravity

- There are similar formula for the other dimensions as well

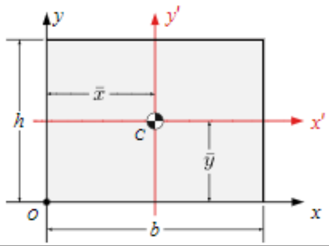
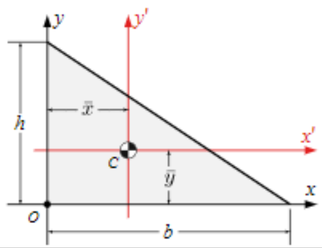
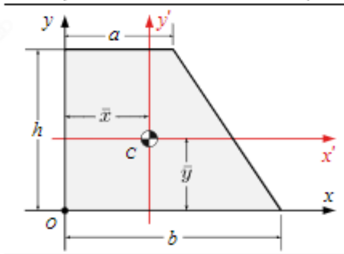
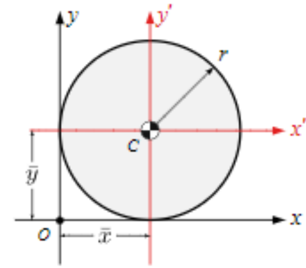
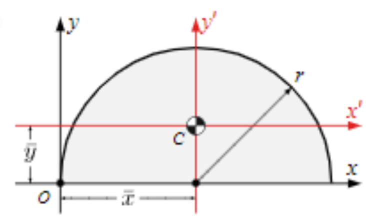
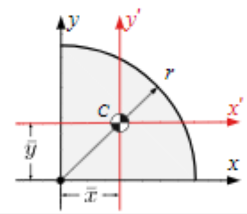
$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \quad \bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \quad \bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i}$$

7.4 – Centroids

- A centroid is a weighted average like the center of gravity, but weighted with a geometric property like area or volume, and not a physical property like weight or mass.
- A centroid is a weighted average like the center of gravity, but weighted with a geometric property like area or volume, and not a physical property like weight or mass.

$$\bar{x} = \frac{\sum \bar{x}_i V_i}{\sum V_i} \quad \bar{y} = \frac{\sum \bar{y}_i V_i}{\sum V_i} \quad \bar{z} = \frac{\sum \bar{z}_i V_i}{\sum V_i}$$

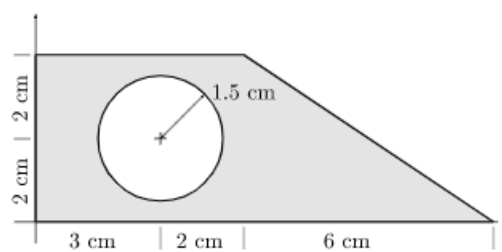
7.4.1 – Properties of Common Shapes

Shape	Area	$\bar{x}$	$\bar{y}$
	$A = bh$	$b/2$	$h/2$
	$\dfrac{bh}{2}$	$b/3$	$h/3$
	$\dfrac{(a+b)h}{2}$	$\dfrac{a^2+ab+b^2}{3(a+b)}$	$\dfrac{h(2a+b)}{3(a+b)}$
	πr^2	r	r
	$\dfrac{\pi r^2}{2}$	r	$\dfrac{4r}{3\pi}$
	$\dfrac{\pi r^2}{4}$	$\dfrac{4r}{3\pi}$	$\dfrac{4r}{3\pi}$

7.5.1 – Composite Parts Method

The steps to finding a centroid using the composite parts method are:

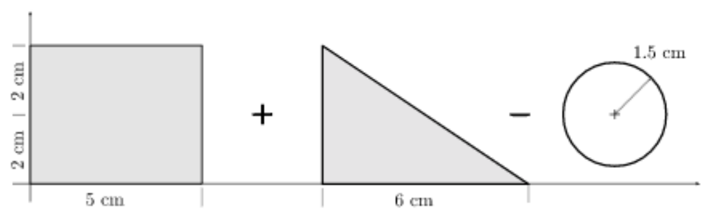
1. Break the overall shape into simpler parts.
2. Collect the areas and centroid coordinates, and
3. Combine the pieces to find the overall centroid.



7.5.1 – Composite Parts Method

1. Break the overall shape into simpler parts.

- Begin with a sketch of the shape and establish a coordinate system.
- Divide the shape into several simpler shapes.



7.5.1 – Composite Parts Method

2. Collect the areas and centroid coordinates.

- Once the complex shape has been divided into parts, determine the area and centroidal coordinates for each part.
- Record the information in a table.

Part	A_i [cm^2]	$\bar{x}_i$ [cm]	$\bar{y}_i$ [cm]	$A_i\bar{x}_i$ [cm^3]	$A_i\bar{y}_i$ [cm^3]
1	20	2.5	2	50	40
2	12	7	4/3	84	16
3	$-2.25\,\pi$	3	2	$-6.75\,\pi$	-4.5π
Σ	24.93	—	—	112.8	41.86

7.5.1 – Composite Parts Method

3. Combine the pieces to find the overall centroid.

- After completing the table, calculate the coordinates of the centroid.

$$\bar{x} = \frac{Q_y}{A} \qquad = \frac{112.8}{24.93} = 4.52 \text{ cm}$$
$$\bar{y} = \frac{Q_x}{A} \qquad = \frac{41.86}{24.93} = 1.692 \text{ cm}$$

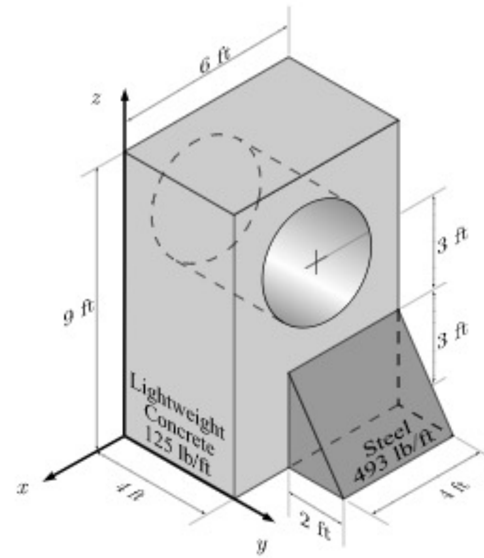
7.5.2 – Centroids of 3D objects

- The centroid of a three-dimensional volume is found similarly to two-dimensional centroids, but with volume used instead of area for the weighting factor.

$$\bar{x} = \frac{\sum \bar{x}_i V_i}{\sum V_i} \qquad \bar{y} = \frac{\sum \bar{y}_i V_i}{\sum V_i} \qquad \bar{z} = \frac{\sum \bar{z}_i V_i}{\sum V_i}$$

Example 7.2 (1/2)

A composite solid consists of a rectangular block of lightweight concrete and a triangular wedge of steel with dimensions as shown. The rectangular block has a 2 ft radius circular hole, centered and drilled through its full depth, perpendicular to the front and back faces.



Assume γ_c=125 lb/ft³, and γ_s =493 lb/ft³

Find the center of mass of this composite solid.

Example 7.2 (2/2)

Solution.

Table 7.5.4

Part	V_i [ft^3]	γ [lb/ft^3]	W_i [lb]	$\bar{x}_i$ [ft]	$\bar{y}_i$ [ft]	$\bar{y}_i$ [ft]	$W_i\ \bar{x}_i$ [ft·lb]	$W_i\ \bar{y}_i$ [ft·lb]	$W_i\ \bar{z}_i$ [ft·lb]
block	216	125	27000	-3	2	4.5	-81000	54000	121500
hole	-50.27	125	-6283	-3	2	6	18850	-12566	-37699
wedge	12	493	5916	-4	4.67	1	-23664	27608	5916
			26633				-85814	69042	89717

$$\bar{x} = \frac{\sum W_i\ \bar{x}_i}{\sum W_i} = \frac{-85814 \text{ ft}\cdot\text{lb}}{26633 \text{ lb}} = -3.22 \text{ ft}$$

$$\bar{y} = \frac{\sum W_i\ \bar{y}_i}{\sum W_i} = \frac{69042 \text{ ft}\cdot\text{lb}}{26633 \text{ lb}} = 2.59 \text{ ft}$$

$$\bar{z} = \frac{\sum W_i\ \bar{z}_i}{\sum W_i} = \frac{89717 \text{ ft}\cdot\text{lb}}{26633 \text{ lb}} = 3.37 \text{ ft}$$

7.6 – Average Value of a Function

- To find the average of an infinite number of values or values which change continuously, use a definite integral

$$\int_a^b f(x)\, dx$$

- In other words, to transform a discrete summation to an equivalent continuous integral form you:

  - Replace the summation with integration, .Σ⇒∫.
  - Replace the discrete weighting factor with the corresponding differential element,
  - Rename the value being averaged to eliminate the index i. We often use el as a subscript when referring to a differential element.

$$\begin{cases} A_i & \Rightarrow dA \\ V_i & \Rightarrow dV \\ W_i & \Rightarrow dW \end{cases} \qquad \text{etc.}$$

7.7 – Centroids using Integration

- Integrals can be used to find the centroids of non-homogenous objects or shapes with curved boundaries.

$$\bar{x} = \frac{\int \bar{x}_{\text{el}}\, dA}{\int dA} \qquad \bar{y} = \frac{\int \bar{y}_{\text{el}}\, dA}{\int dA} \qquad \bar{z} = \frac{\int \bar{z}_{\text{el}}\, dA}{\int dA}$$

7.7.1 – Integration Process

- Determining the centroid of a area using integration involves finding weighted average values $\bar{x}$ and $\bar{y}$, by evaluating the three integrals

$$A = \int dA, \qquad Q_x = \int \bar{y}_{\text{el}}\, dA \qquad Q_y = \int \bar{x}_{\text{el}}\, dA,$$

- where
- *dA* is a differential bit of area called the *element*.
- *A* is the *total area* enclosed by the shape, and is found by evaluating the first integral.
- $\bar{x}$ and $\bar{y}$ are the coordinates of the *centroid of the element*. These are frequently functions of *x* or *y*, not constant values.
- Q_x and Q_y are the *First moments of Area* with respect to the *x* and *y* axis.

7.7.1 – Integration Process

Procedure for finding the centroid using integrals

1. *Set up the integrals.*

    Draw a sketch

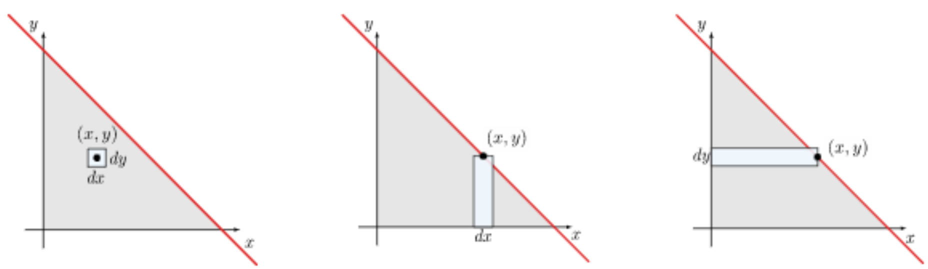
    Choose and element of area *dA*

7.7.1 – Integration Process

The two most common choices for differential elements are:

Square elements and double integrals.

Rectangular elements and single integrals.



(a) Square element   **(b)** Vertical strip   **(c)** Horizontal strip

7.7.1 – Integration Process

2. *Solve the integrals*

    Be neat, work carefully, and check your work as you go along. Use proper mathematics notation

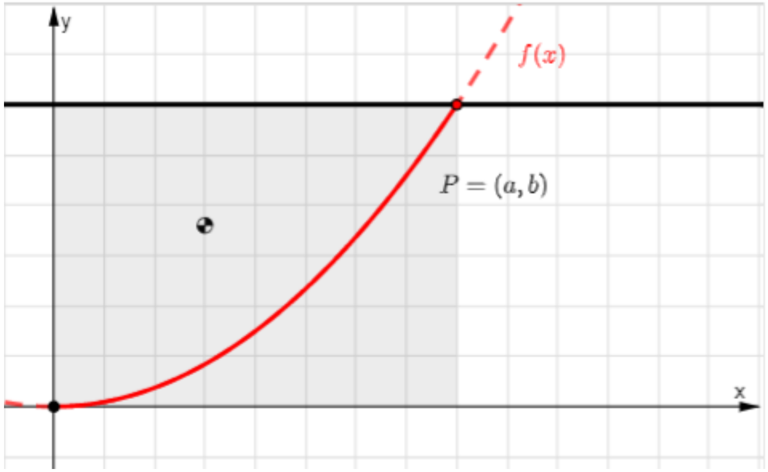
3. *Evaluate the centroid.*

    Solve for $\bar{x}$ and $\bar{y}$

    Plot the centroid and determine if the location make sense

Example 7.3 (1/6)

Find the coordinates of the centroid of a parabolic spandrel bounded by the *y* axis, a horizontal line passing through the point (*a, b*), and a parabola with a vertex at the origin and passing through the same point. *a* and *b* are positive integers.



Example 7.3 (2/6)

1. *Set up the integrals.*

   Determining the bounding functions and setting up the integrals is usually the most difficult part of problems like this. Begin by drawing and labeling a sketch of the situation.

   (a) Place a point in the first quadrant and label it $P = (a, b)$. This point is in the first quadrant and fixed since we are told that a and b are positive integers

   (b) Place a horizontal line through P to make the upper bound.

   (c) Sketch in a parabola with a vertex at the origin and passing through P and shade in the enclosed area.

   (d) Decide which differential element you intend to use. For this example we choose to use vertical strips, which you can see if you tick `show`

Example 7.3 (3/6)

`strips` in the interactive above. Horizontal strips $dA = x\ dy$ would give the same result, but you would need to define the equation for the parabola in terms of y.

Determining the equation of the parabola and expressing it in terms of x and any known constants is a critical step. You should remember from algebra that the general equation of parabola with a vertex at the origin is $y = kx^2$, where k is a constant which determines the shape of the parabola. If $k > 0$, the parabola opens upward and if $k < 0$, the parabola opens downward.

To find the value of k, substitute the coordinates of P into the general equation, then solve for k.

$$\begin{aligned} y &= kx^2, \text{ so at } P \\ (b) &= k(a)^2 \\ k &= \frac{b}{a^2} \end{aligned}$$

The resulting function of the parabola is

$$y = y(x) = \frac{b}{a^2}x^2$$

Example 7.3 (4/6)

To perform the integrations, express the area and centroidal coordinates of the element in terms of the points at the top and bottom of the strip. The area of the strip is its height times its base, so

$$dA = (b - y)\ dx$$

. If you incorrectly used $dA = y\ dx$, you would find the centroid of the spandrel below the curve.

For vertical strips, the bottom is at (x, y) on the parabola, and the top is directly above at (x, b). The strip has a differential width dx. The centroid of the strip is located at its midpoint and the coordinates are found by averaging the x and y coordinates of the points at the top and bottom.

$$\bar{x}_{\text{el}} = (x + x)/2 = x$$
$$\bar{y}_{\text{el}} = (y + b)/2$$

For vertical strips, the integrations are with respect to x, and the limits on the integrals are $x = 0$ on the left to $x = a$ on the right.

Example 7.3 (5/6)

2. *Solve the integrals.*

We have already established that $y(x) = kx^2$ where $k = b/a^2$. To simplify the algebra, it is best not to prematurely substitute y(x) and k, but you must substitute in any functions of x before you do the integration step.

$$
\begin{aligned}
A &= \int dA \\
&= \int_0^a (b - y)\ dx \\
&= \int_0^a (b - kx^2)\ dx \\
&= bx - k\frac{x^3}{3}\bigg|_0^a \\
&= ba - k\frac{a^3}{3} \\
&= ba - \left(\frac{b}{a^2}\right)\frac{a^3}{3} \\
&= \frac{3ba}{3} - \frac{ba}{3} \\
&= \frac{2}{3}ba \\
A &= \frac{2}{3}ba
\end{aligned}
$$

$$
\begin{aligned}
Q_x &= \int \bar{y}_{\text{el}} dA \\
&= \int_0^a \frac{(b+y)}{2}(b - y)dx \\
&= \frac{1}{2}\int_0^a (b^2 - y^2)\ dx \\
&= \frac{1}{2}\int_0^a (b^2 - (kx^2)^2)\ dx \\
&= \frac{1}{2}\int_0^a (b^2 - k^2x^4)\ dx \\
&= \frac{1}{2}\left[b^2x - k^2\frac{x^5}{5}\right]_0^a \\
&= \frac{1}{2}\left[b^2a - \left(\frac{b}{a^2}\right)^2\frac{a^5}{5}\right] \\
&= \frac{1}{2}b^2a\left[1 - \frac{1}{5}\right] \\
Q_x &= \frac{2}{5}b^2a
\end{aligned}
$$

$$
\begin{aligned}
Q_y &= \int \bar{x}_{\text{el}} dA \\
&= \int_0^a x(b - y)\ dx \\
&= \int_o^a x(b - y)\ dx \\
&= \int_o^a x(b - kx^2)\ dx \\
&= \int_o^a (bx - kx^3)\ dx \\
&= \left[\frac{bx^2}{2} - k\frac{x^4}{4}\right]_0^a \\
&= \left[\frac{ba^2}{2} - \left(\frac{b}{a^2}\right)\frac{4^4}{4}\right] \\
&= ba^2\left[\frac{1}{2} - \frac{1}{4}\right] \\
Q_y &= \frac{1}{4}ba^2
\end{aligned}
$$

The area of the spandrel is 2/3 of the area of the enclosing rectangle and the moments of area have units of [length]^3.

Example 7.3 (6/6)

3. *Find the centroid.*

   Substituting the results into the definitions gives

$$\bar{x} = \frac{Q_y}{A} = \frac{ba^2}{4}\bigg/\frac{2ba}{3} = \frac{3}{8}a \qquad \bar{y} = Q_x A = \frac{2b^2a}{5}\bigg/\frac{2ba}{3} = \frac{2}{5}b$$

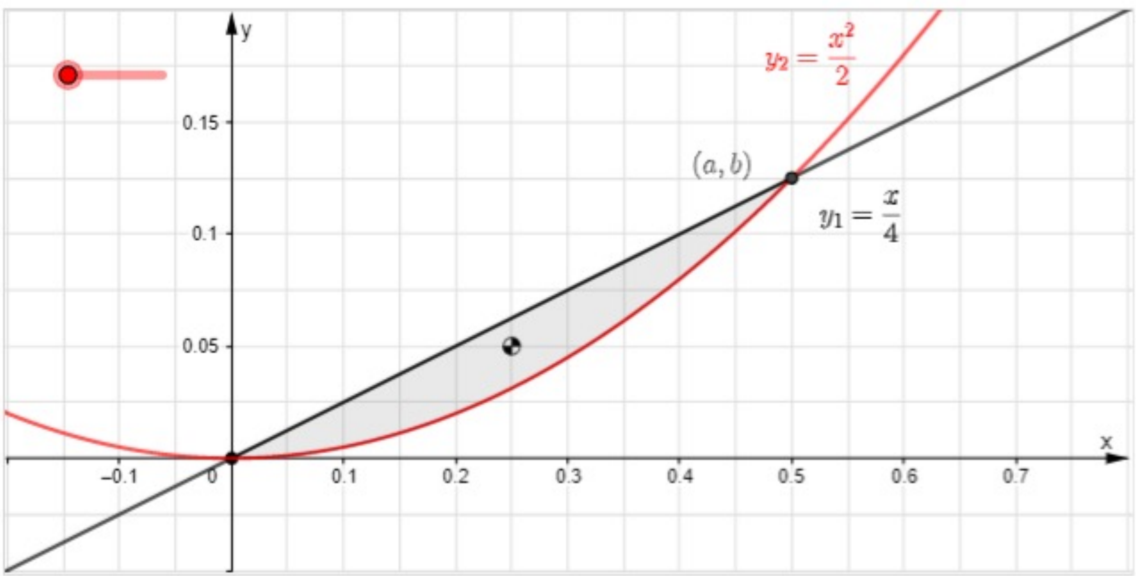
.

$\bar{x}$ is 3/8 of the width and $\bar{y}$ is 2/5 of the height of the enclosing rectangle.

Example 7.4 (1/4)

Use integration to locate the centroid of the area bounde

$$y_1 = \frac{x}{4} \text{ and } y_2 = \frac{x^2}{2}$$



Example 7.4 (2/4)

1. *Set up the integrals.*

   The bounding functions in this example are the x axis, the vertical line $x = b$, and the straight line through the origin with a slope of $\frac{h}{b}$. Using the slope-intercept form of the equation of a line, the upper bounding function is

   $$y = f(x) = \frac{h}{b}x$$

   and any point on this line is designated (x, y).

   The strip extends from $(x, 0)$ on the x axis to (x, y) on the function, has a height of y, and a differential width dx. The area of this strip is

   $$dA = y dx$$

   .

   The centroid of the strip is located at its midpoint so, by inspection

   $$\bar{x}_{\text{el}} = x$$
   $$\bar{y}_{\text{el}} = y/2$$

   With vertical strips the variable of integration is x, and the limits are $x = 0$ to $x = b$.

Example 7.4 (3/4)

2. *Solve the integrals.*

   Substitute dA, $\bar{x}_{\text{el}}$, and $\bar{y}_{\text{el}}$ into (7.7.2) and integrate. In contrast to the rectangle example both dA and $\bar{y}_{\text{el}}$ are functions of x, and will have to be integrated accordingly.

$$
\begin{aligned}
A &= \int dA \\
&= \int_0^{1/2} (y_1 - y_2)\, dx \\
&= \int_0^{1/2} \left(\frac{x}{4} - \frac{x^2}{2}\right) dx \\
&= \left[\frac{x^2}{8} - \frac{x^3}{6}\right]_0^{1/2} \\
&= \left[\frac{1}{32} - \frac{1}{48}\right] \\
A &= \frac{1}{96}
\end{aligned}
$$

$$
\begin{aligned}
Q_x &= \int \bar{y}_{\text{el}}\, dA \\
&= \int_0^{1/2} \left(\frac{y_1 + y_2}{2}\right)(y_1 - y_2)\, dx \\
&= \frac{1}{2}\int_0^{1/2} \left(y_1^2 - y_2^2\right) dx \\
&= \frac{1}{2}\int_0^{1/2} \left(\frac{x^2}{16} - \frac{x^4}{4}\right) dx \\
&= \frac{1}{2}\left[\frac{x^3}{48} - \frac{x^5}{20}\right]_0^{1/2} \\
&= \frac{1}{2}\left[\frac{1}{384} - \frac{1}{640}\right] \\
Q_x &= \frac{1}{1920}
\end{aligned}
$$

$$
\begin{aligned}
Q_y &= \int \bar{x}_{\text{el}}\, dA \\
&= \int_0^{1/2} x(y_1 - y_2)\, dx \\
&= \int_0^{1/2} x\left(\frac{x}{4} - \frac{x^2}{2}\right) dx \\
&= \int_0^{1/2} \left(\frac{x^2}{4} - \frac{x^3}{2}\right) dx \\
&= \left[\frac{x^3}{12} - \frac{x^4}{8}\right]_0^{1/2} \\
&= \left[\frac{1}{96} - \frac{1}{128}\right] \\
Q_y &= \frac{1}{384}
\end{aligned}
$$

Example 7.4 (4/4)

3. *Find the Centroid.*

   Substituting the results into the definitions gives

$$\bar{x} = \frac{Q_y}{A} = \frac{1}{384}\Big/\frac{1}{96} \qquad \bar{y} = \frac{Q_x}{A} = \frac{1}{1920}\Big/\frac{1}{96}$$

$$\bar{x} = \frac{1}{4} \qquad \bar{y} = \frac{1}{20}$$

7.8 – Distributed Loads

- ***Distributed loads*** are forces which are spread out over a length, area, or volume.
- Distributed load is a force per unit length or force per unit area.
- Computational tools discussed in the previous chapters can be used to handle distributed loads if first converted to equivalent point force.
  - To be equivalent, the point force must have a:
  - Magnitude equal to the area or volume under the distributed load function.
  - Line of action that passes through the centroid of the distributed load distribution.

7.8.1 – Equivalent Magnitude

- The magnitude of the distributed load is equal to the value of the distributed load times the length it acts over.

$$W = w(x)\ell$$

$$\text{total weight} = \frac{\text{weight}}{\text{length}} \times \text{ length of shelf}$$

- This total load is simply the area under the curve *w*(*x*), and has units of force. If the loading function is not uniform, integration may be necessary to find the area.

7.8.2 – Equivalent Location

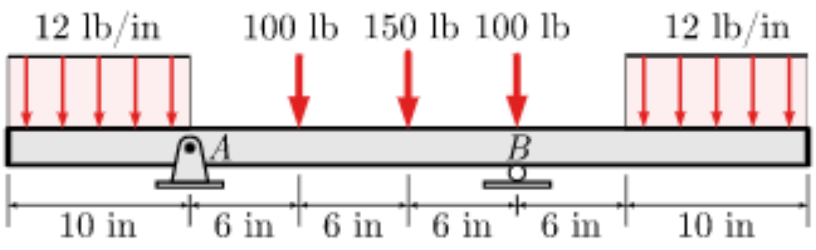
- The line of action of the equivalent force acts through the centroid of area under the load intensity curve.
  - For a rectangular loading, the centroid is in the center.
  - For a triangular distributed load — also called a ***uniformly varying load*** — the magnitude of the equivalent force is the area of the triangle, $bh/2$ and the line of action passes through the centroid of the triangle.

7.8.3 – Distributed Load Applications

- Once the distributed loads are converted to the resultant point force, the problem can be solved like a general equilibrium problem.

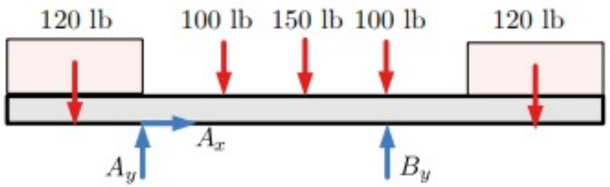
Example 7.5 (1/2)

Find the reactions at the supports for the beam shown.



Example 7.5 (2/2)

Solution. Start by drawing a free-body diagram of the beam with the two distributed loads replaced with equivalent concentrated loads. The two distributed loads are (10 in)(12 lb/in) = 120 lb each.



Then apply the equations of equilibrium.

$$\sum M_A = 0$$

$$\begin{aligned}
&+(12\text{ lb/in})(10\text{ in})(5\text{ in}) - (100\text{ lb})(6\text{ in}) \\
&\quad -(150\text{ lb})(12\text{ in}) - (100\text{ lb})(18\text{ in}) \\
&\quad +(B_y)(18\text{ in}) - (12\text{ lb/in})(10\text{ in})(29\text{ in}) = 0 \rightarrow \qquad B_y = 393.3\text{ lb}
\end{aligned}$$

$$\sum F_y = 0$$

$$\begin{aligned}
&-(12\text{ lb/in})(10\text{ in}) + B_y - 100\text{ lb} - 150\text{ lb} \\
&\quad -100\text{ lb} + B_y - (12\text{ lb/in})(10\text{ in}) = 0 \rightarrow \qquad B_y = 196.7\text{ lb}
\end{aligned}$$

$$\sum F_x = 0 \rightarrow \qquad A_x = 0$$

7.9 – Fluid Statics

Pressure is the term used for a force distributed over an area

$$P = \frac{F}{A}$$

Pressure can be measured in two different ways

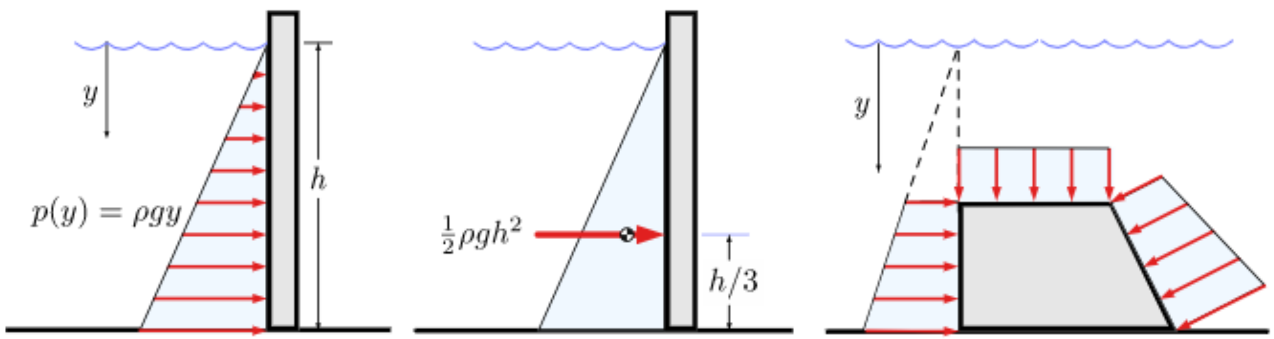
- ***Absolute pressure***
- ***Gage pressure***

7.9.1 – Principles of Fluid Statics

- A fluid, like water or air, exerts a pressure on its surroundings. This pressure applies a distributed load on surfaces surrounding the fluid.
- At the surface, the gage pressure is zero.
- The fluid pressure *P* increases with depth according to the equation

$$P = \rho g h$$

- Fluid pressure increases linearly with depth. It behaves as a distributed load which increases linearly from 0 at the surface to $\rho g h$ at depth h, acting normal to the surface.



(a) Distributed pressure.

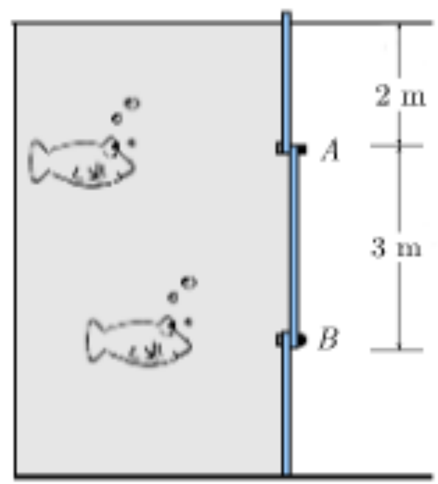
(b) Equivalent force.

(c) Pressure is perpendicular to the surface.

Example 7.6 (1/3)

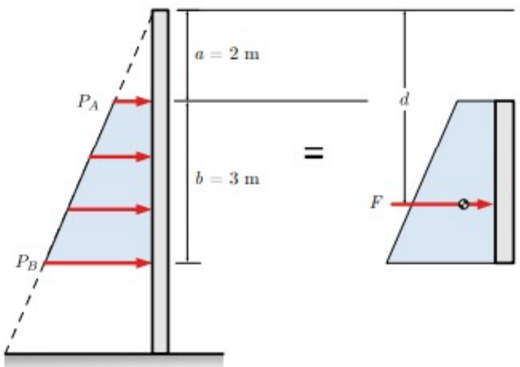
An aquarium tank has a 3 m × 1.5 m window AB for viewing the inhabitants. The tank contains water with density ρ=1000 kg/m³.

Find the force of the water on the window, and the location of the equivalent point load.



Example 7.6 (2/3)

Solution 1. Begin by drawing a diagram of the window showing the load intensity and the equivalent concentrated force.



The pressure at the top and the bottom of the window are

$$P_A = \rho\, g(2 \text{ m}) = 19620 \text{ N/m}^2$$
$$P_B = \rho\, g(5 \text{ m}) = 49050 \text{ N/m}^2$$

Since the loading is linear, the average pressure acting on the window is

$$\begin{aligned} P_{ave} &= (P_A + P_B)/2 \\ &= 34300 \text{ N/m}^2 \end{aligned}$$

The total force acting on the window is the average pressure times the area of the window

$$\begin{aligned} F &= (P_{ave})(3 \text{ m} \times 1.5 \text{ m}) \\ &= 155 \text{ kN} \end{aligned}$$

Example 7.6 (3/3)

This force may also be visualized as the volume of a trapezoidal prism with a 1.5 m depth into the page.

The line of action of the equivalent force passes through the centroid of the trapezoid, which may be calculated using composite areas, see Section 7.5.

Dividing the trapezoid into a triangle and a rectangle and measuring down from the surface of the tank, the distance to the equivalent force is

$$d = \frac{\sum A_i \bar{y}_i}{\sum A_i}$$

$$d = \frac{\left[P_A(3\text{ m})\right](3.5\text{ m}) + \left[\frac{1}{2}(P_B - P_A)(3\text{ m})\right](4\text{ m})}{\left[P_A(3\text{ m})\right] + \left[\frac{1}{2}(P_B - P_A)(3\text{ m})\right]}$$

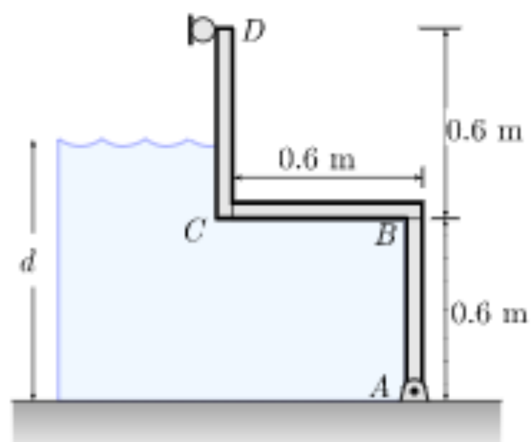
$$d = 3.71\text{ m}$$

If you prefer, you may use the formula from the Centroid table to locate the centroid of the trapezoid instead.

Example 7.7 (1/3)

A gate at the end of a freshwater channel is fabricated from three 125 kg, 0.6 m × 1 m rectangular steel plates. The gate is hinged at *A* and rests against a frictionless support at *D*. The depth of the water d=0.75 m.

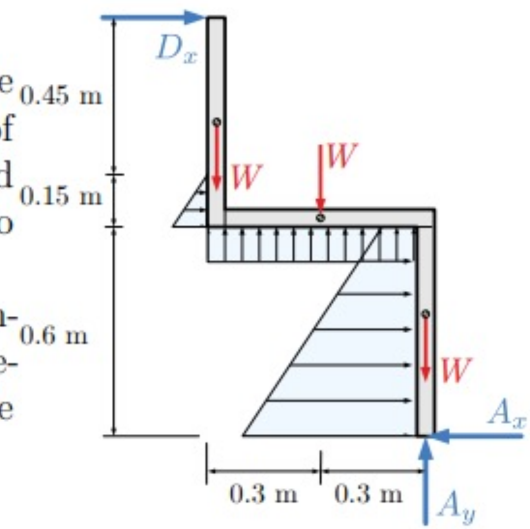
Draw the free-body diagram and determine the reactions at *A* and *D*.



Example 7.7 (2/3)

Solution.

A free-body diagram of a cross section of the gate is shown. For simplicity the thickness of the steel plates has been ignored. You should ensure that sufficient distances are provided to locate the loads.
The easiest way to solve this is to apply the principle of transmissibility: slide the lower trapezoid left until it aligns with the upper triangle and makes a triangular loading.



Example 7.7 (3/3)

The total horizontal force from the water will be

$$
\begin{aligned}
F_x &= P_{ave}\ A \\
&= \left[\frac{1}{2}\rho\ g\ 0.75\ \mathrm{m}\right](0.75\ \mathrm{m} \times 1\ \mathrm{m}) \\
&= \left[\frac{1}{2}(1000\ \mathrm{kg/m^3})(9.81\ \mathrm{m/s^2})\ 0.75\ \mathrm{m}\right](0.75\ \mathrm{m} \times 1\ \mathrm{m}) \\
&= 2760\ \mathrm{N}
\end{aligned}
$$

acting to the right 0.25 m above point A.

The total vertical load from the water is

$$
\begin{aligned}
F_y &= P_{ave}\ A \\
&= [\rho\ g\ (0.15\ \mathrm{m})](0.6\ \mathrm{m} \times 1\ \mathrm{m}) \\
&= [(1000\ \mathrm{kg/m^3})(9.81\ \mathrm{m/s^2})\ (0.15\ \mathrm{m})](0.6\ \mathrm{m} \times 1\ \mathrm{m}) \\
&= 882.9\ \mathrm{N}
\end{aligned}
$$

acting upward 0.3 m to the left of A.

Each plate weighs

$$
\begin{aligned}
W &= mg \\
&= (125\ \mathrm{kg})(9.81\ \mathrm{m/s^2}) \\
&= 1226\ \mathrm{N}
\end{aligned}
$$

.

From here solve the equilibrium equations to find the reactions. You should complete this for practice.

$$
\begin{aligned}
\Sigma M_A &= 0 & \rightarrow && D_x &= 239\ \mathrm{N\ right} \\
\Sigma F_x &= 0 & \rightarrow && A_x &= 2998\ \mathrm{N\ left} \\
\Sigma F_y &= 0 & \rightarrow && A_y &= 2795\ \mathrm{N\ up}
\end{aligned}
$$

Textbook

Baker, Daniel W., and Haynes, William. "Chapter 7 – Centroids and Centers of Gravity." *Engineering Statics: Open and Interactive*. May 21, 2023. https://engineeringstatics.org/

Chapter 8 – Internal Forces

Engineering Mechanics: Statics and Structures

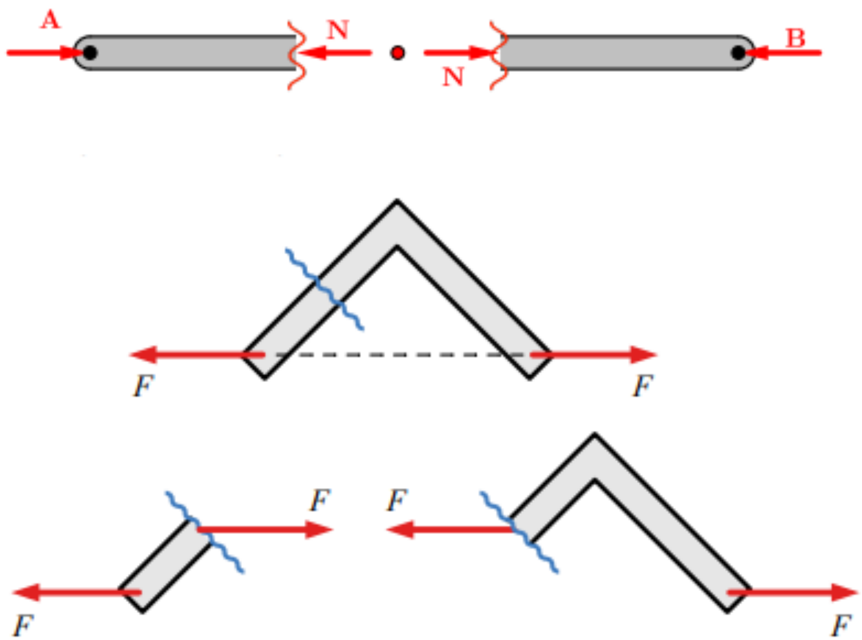
Chapter Outline

8.1 – Internal Forces

- One of the fundamental assumptions we make in statics is that bodies are rigid, that is, they do not deform, bend, or change shape.
- Internal forces are present at every point within a rigid body, but they always occur in equal-and-opposite pairs which cancel each other out.
- Two internal forces found in two-dimensional systems, the ***internal shear*** and ***internal bending moment*** will be examined.

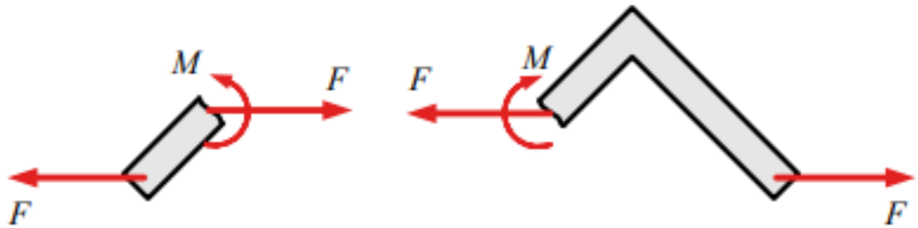
8.1 – Internal Forces

- Internal forces of a two-force member

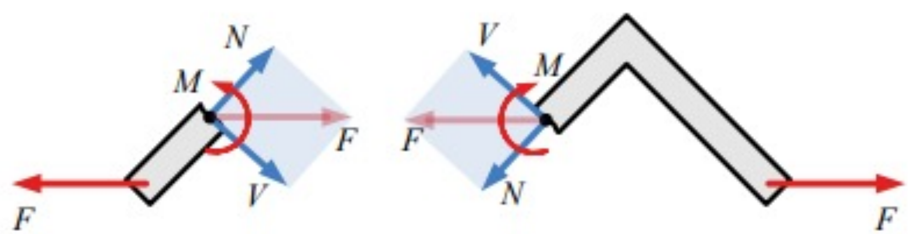


8.1 – Internal Forces

- Internal bending moment

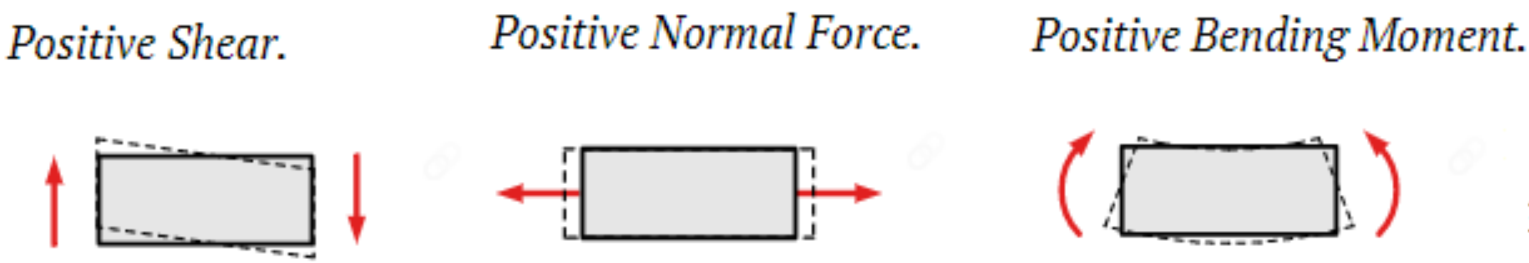


- Internal normal force



8.2 – Sign Conventions

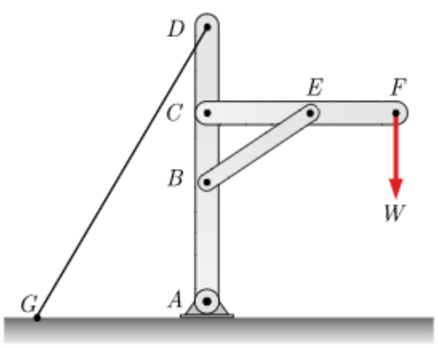
- Standard sign convention used for shear force, normal force, and bending moment



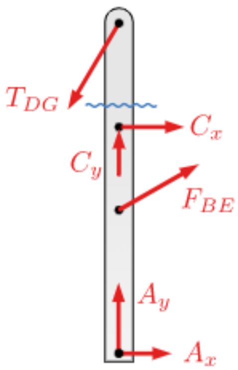
- This new sign convention applies to internal forces, it doesn't change the sign convention for the equations of equilibrium at all.

8.3 – Internal Forces at a Point

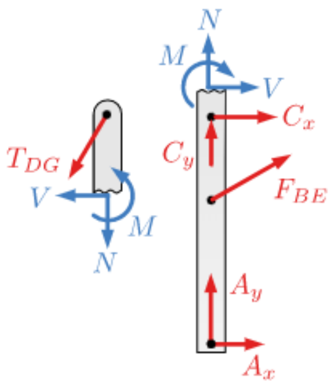
- To find the internal forces at a point, imagine making a cut at that point.



(a) A frame supporting a load at F.



(b) Wavy line indicates the location of the imaginary cut.

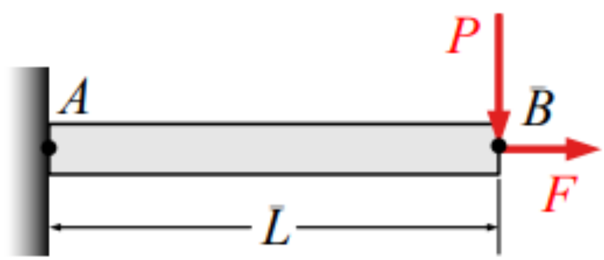


(c) Internal forces are exposed by the cut.

Example 8.1 (1/4)

Consider a cantilever beam which is supported by a fixed connection at *A*, and loaded by a vertical force *P* and horizontal force *F* at the free end *B*.

Determine the internal forces at a point a distance a from the left end.

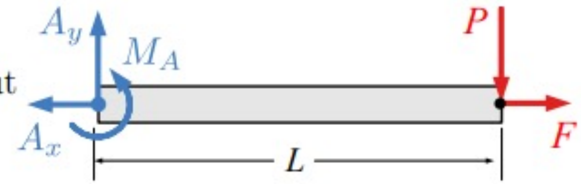


Example 8.1 (2/4)

Solution.

1. *Determine the reactions.*

Draw an FBD of the entire, uncut beam and determine the reactions.



Notice that only the applied loads and support reactions are included on this uncut beam FBD. The internal forces are only exposed and shown on a FBD after the beam is cut.

Use this free-body diagram and the equations of equilibrium to determine the external reaction forces.

$$\Sigma F_x = 0 \quad \Longrightarrow \quad A_x = F$$
$$\Sigma F_y = 0 \quad \Longrightarrow \quad A_y = P$$
$$\Sigma M_A = 0 \quad \Longrightarrow \quad M_A = PL$$

Example 8.1 (3/4)

2. *Section the beam.*

   Take a cut at the point of interest and draw a FBD of either or both parts. Try to choose the simpler free-body diagram. If one side has no external reactions, then you can skip the previous step if you choose that side.

   The free-body diagrams of both portions have been drawn with the internal forces and moments drawn in the positive direction defined by the standard sign convention.

   The axial force is shown in tension on both parts. This force has been named N so its name doesn't conflict with the forces at point A.

   The shear force V is positive when the shear is down on the right face of the cut and up on the left face.

   The bending moment M is positive if the bending direction would tend to bend the beam into a concave upward curve.

   Always assume that the unknown internal forces act in the positive direction as defined by the standard sign convention.

Example 8.1 (4/4)

3. *Solve for the internal forces.*

   Selecting the right hand diagram and solving for the unknown internal forces gives:

$$\begin{aligned} \Sigma F_x = 0 &\implies N = F && \text{from before } F = A_x \\ \Sigma F_y = 0 &\implies V = P && \text{from before } P = A_y \\ \Sigma M_{\text{cut}} = 0 &\implies M = -Pb = -P(L-a) && \text{since } a + b = L \end{aligned}$$

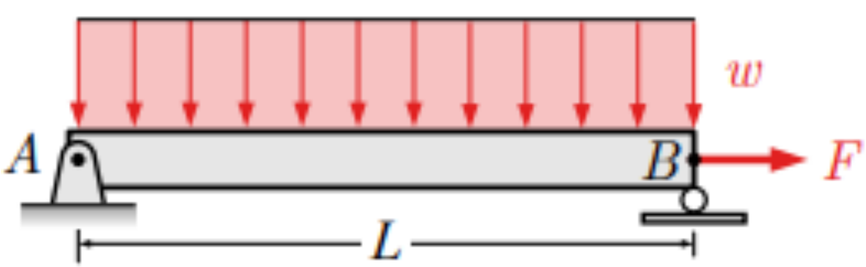
8.3 – Internal Forces at a Point

- General Procedure
  - Establishing a horizontal *x* and vertical *y* coordinate system.
  - Taking a cut at the point of interest.
  - Assuming that the internal forces act in the positive direction and drawing a free-body diagram accordingly
  - Using $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M_z = 0$ to solve for the three unknown internal forces.
- The shear force **V**, normal force N**,** and bending moment **M** are scalar components and they may be positive, zero, or negative depending on the applied The signs of the scalar components together with the sign convention for internal forces establish the actual directions of the shear force, normal force and bending moment vectors.

Example 8.2 (1/6)

A beam of length L is supported by a pin at A and a roller at B and is subjected to a horizontal force F applied to point B and a uniformly distributed load over its entire length. The intensity of the distributed load is w with units of [force/length].

Find the internal forces at the midpoint of the beam.

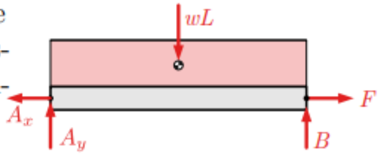


Example 8.2 (2/6)

Solution.

1. *Find the external reactions.*

   Begin by drawing a free-body diagram of the entire beam, simplified by replacing the distributed load w with an equivalent concentrated load at the centroid of the rectangle.



The magnitude of the equivalent load W is equal to the "area" under the rectangular loading curve.

$$W = w(L)$$

Then apply and simplify the equations of equilibrium to find the external reactions at A and B.

$$\begin{aligned}
\Sigma M_A &= 0 \\
-(wL)(\cancel{L}/2) + (B)\cancel{L} &= 0 \\
B &= wL/2
\end{aligned}$$

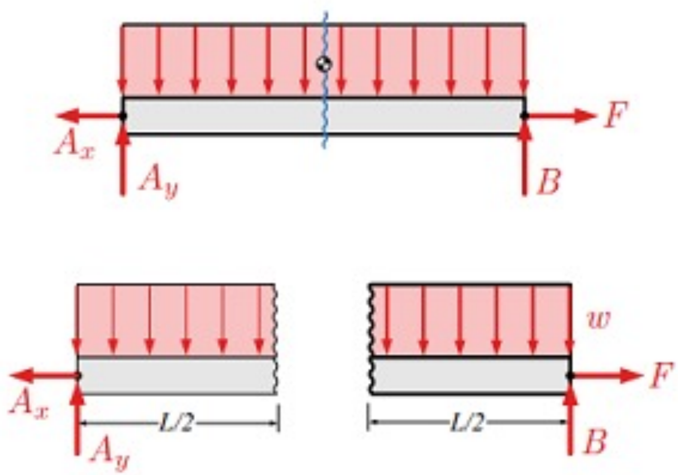
$$\begin{aligned}
\Sigma F_x &= 0 \\
-A_x + F &= 0 \\
A_x &= F
\end{aligned}$$

$$\begin{aligned}
\Sigma F_y &= 0 \\
A_y - wL + B_y &= 0 \\
A_y &= wL - wL/2 \\
&= wL/2
\end{aligned}$$

Example 8.2 (3/6)

2. *Cut the beam.*

Cut the beam at the point of interest and separate the beam into two sections. Notice that as the beam is cut in two, the distributed load w is cut as well. Each of these distributed load halves will support equivalent point loads of $wL/2$ acting through the centroid of each cut half.



Example 8.2 (4/6)

3. *Add the internal forces.*

   At each cut, a shear force, a normal force, and a bending moment will be exposed, and these need to be included on the free-body diagram.

   At this point, we don't know the actual directions of the internal forces, but we do know that they act in opposite directions. We will assume that they act in the positive sense as defined by the standard sign convention.

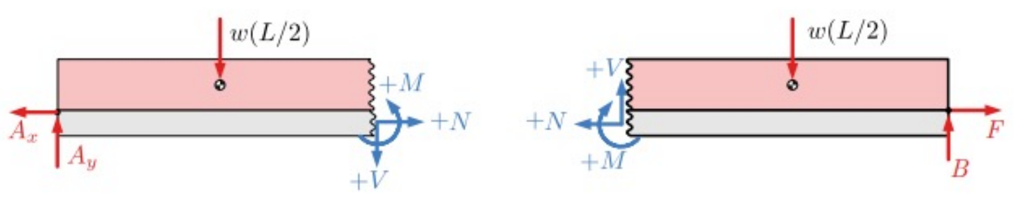
   Axial forces are positive in tension and act in opposite directions on the two halves of the cut beam.

   Positive shear forces act down when looking at the cut from the right, and up when looking at the cut from the left. An alternate definition of positive shears is that the positive shears cause clockwise rotation. This definition is useful if you are dealing with a vertical column instead of a horizontal beam.

   Bending moments are positive when the moment tends to bend the beam into a smiling U-shape. Negative moments bend the beam into a frowning shape.

   For vertical columns, positive bending moments bend a beam into a C shape and negative into a backward C-shape.

   The final free-body diagrams look like this.



Example 8.2 (5/6)

4. *Solve for the internal forces.*

   You may uses either FBD to find the internal forces using the techniques you have already learned. So, with a standard xy coordinate system, forces to the right or up are positive when summing forces and counter-clockwise moments are positive when summing moments.

   Using the left free-body diagram and substituting in the reactions, we get:

$$\begin{aligned} \Sigma F_x &= 0 \\ -A_x + N &= 0 \\ N &= A_x \end{aligned}$$

$$\begin{aligned} \Sigma F_y &= 0 \\ A_y - wL/2 - V &= 0 \\ V &= wL/2 - wL/2 \\ V &= 0 \end{aligned}$$

$$\begin{aligned} \Sigma M_{\text{cut}} &= 0 \\ (wL/2)(L/4) - (A_y)(L/2) + M &= 0 \\ M &= -wL^2/8 + wL^2/4 \\ M &= wL^2/8 \end{aligned}$$

Example 8.2 (6/6)

Using the right side free-body diagram we get:

$$\begin{aligned}
\Sigma F_x &= 0 \\
-N + F &= 0 \\
N &= F
\end{aligned}$$

$$\begin{aligned}
\Sigma F_y &= 0 \\
V - wL/2 + B_y &= 0 \\
V &= wL/2 - B_y \\
V &= wL/2 - wL/2 \\
V &= 0
\end{aligned}$$

$$\begin{aligned}
\Sigma M_{\text{cut}} &= 0 \\
-M - (L/4)(wL/2) + (L/2)(B_y) &= 0 \\
M &= -wL^2/8 + wL^2/4 \\
M &= wL^2/8
\end{aligned}$$

8.4 – Shear and Bending Moment Diagrams

Beams are structural elements primarily designed to support vertical loads.

- Locating the points of maximum shear and maximum moment and their magnitudes during beam design is important because that's where the beam is most likely to fail.

8.4.1 – Shear and Bending Moment Diagrams

- Shear and moment diagrams are graphs which show the internal shear and bending moment plotted along the length of the beam.
- Beams can be supported in a variety of ways as shown. The common support methods are

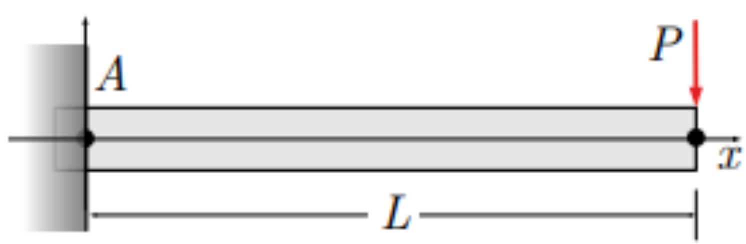
(a) Simply Supported (b) Cantilevered (c) Overhanging

8.5 – Section Cut Method

- To draw a shear and bending moment diagram, the procedure is similar except that the cut is taken at a variable position designated by *x* instead of at a specified point.
- The analysis produces equations for shear and bending moments as functions of *x*.

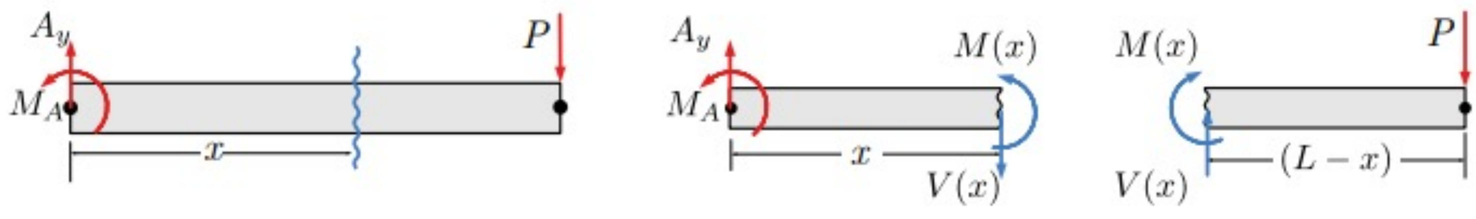
8.5 – Section Cut Method

- Consider a cantilevered beam fixed to a wall on its left end and subject to a vertical force P on its right end as an example.
- Global equilibrium requires that the reactions at the fixed support at A are a vertical force $A_y = P$, and a counterclockwise moment $M_A = P\,L$.



8.5 – Section Cut Method

- Cut at a distance *x* from the left and draw two free-body diagrams with lengths *x* and (*L* − *x*).

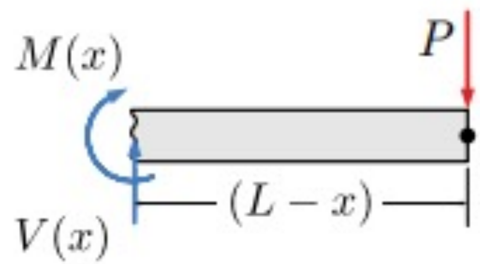


(a) Cut at position x. **(b)** Left FBD. **(c)** Right FBD.

8.5 – Section Cut Method

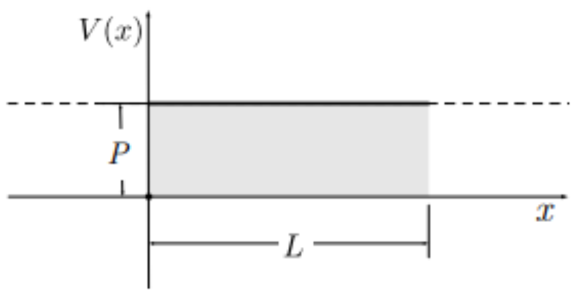
- To find the shear and bending moment functions, we apply the equilibrium to one of the free-body diagrams.

$$\Sigma F_y = 0 \quad \Sigma M_{\text{cut}} = 0$$
$$V(x) = P \quad M(x) = -P(L - x)$$

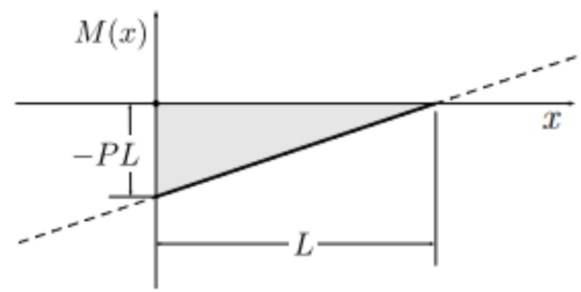


8.5 – Section Cut Method

- Plot the equations for *V(x)* and *M(x)*



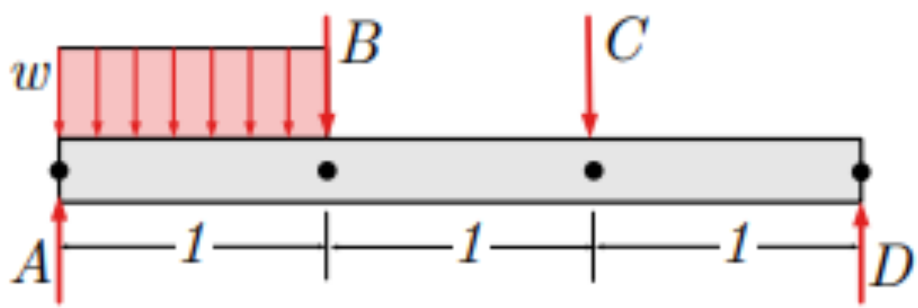
(a) $V(x)$ vs. x



(b) $M(x)$ vs. x

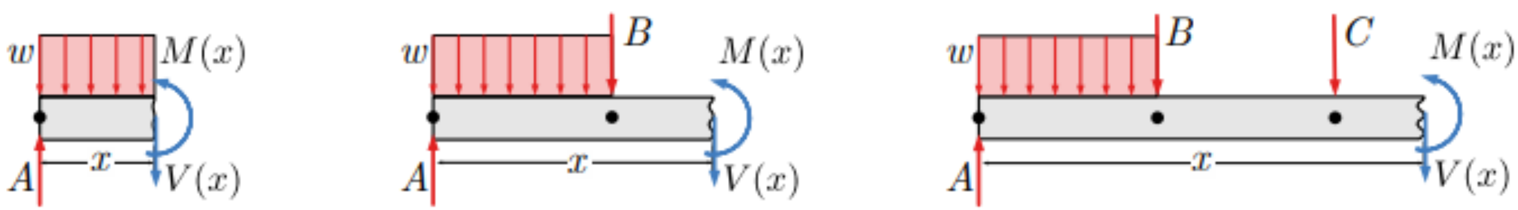
8.5 – Section Cut Method

- Beams with multiple loads must be divided into loading segments between the points where loads are applied or where distributed loads begin or end.
- Consider the simply supported beam *AD* with a uniformly distributed load w over the first segment from *A* to *B*, and two vertical loads *B* and *C*.



8.5 – Section Cut Method

- This beam has three loading segments so you must draw three free-body diagrams and analyze each segment independently.

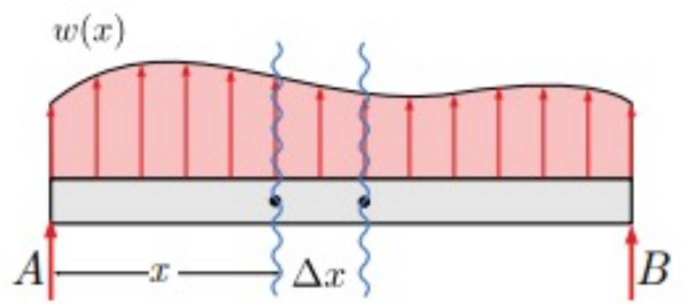


(a) $0 < x < 1$ (b) $1 < x < 2$ (c) $2 < x < 3$

- After the equilibrium equations are applied to each segment, the resulting equations *V(x)* and *M(x)* from each segment are joined to plot the shear and moment diagrams.

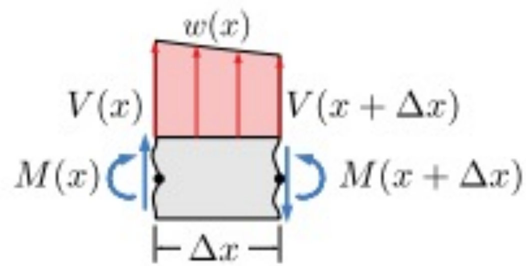
8.6 – Relation Between Loading, Shear and Moment

- Suppose that we have a simply supported beam upon which there is an applied load *w(x)* which is distributed on the beam by some function of position, *x*.



8.6 – Relation Between Loading, Shear and Moment

- FBD for a small section of this beam from *x* to *x* + *Δx*.



- Since *Δx* is infinitely narrow, we can assume that the distributed load over this small distance is constant and equal to the value at *x*, and call it *w*.

8.6 – Relation Between Loading, Shear and Moment

- Apply equilibrium equations

$$\sum F_y = 0$$
$$V + w(\Delta x) - (V + \Delta V) = 0$$
$$\frac{\Delta V}{\Delta x} = w$$

- Taking the limit of both sides as *Δx* approaches 0 and then integrating

$$\lim_{\Delta x \to 0} \left(\frac{\Delta V}{\Delta x} \right) = \lim_{\Delta x \to 0} (w)$$
$$\frac{dV}{dx} = w$$

$$\Delta V = \int w(x)\ dx$$

- Over a given distance, the change in the shear *V* between two points is the area under the loading curve between them.

8.6 – Relation Between Loading, Shear and Moment

- Similarly, for the internal bending moment

$$\sum M = 0$$

$$-\frac{\Delta x}{2}V - \frac{\Delta x}{2}(V + \Delta V) - M + (M + \Delta M) = 0$$

$$\frac{\Delta M}{\Delta x} = \frac{1}{2}(2V + \Delta V)$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta M}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(V + \frac{\Delta V}{2}\right)$$

$$\frac{dM}{dx} = V$$

$$\Delta M = \int V \, dx$$

- Over a given segment, the change in the moment value is the area under the shear curve.

8.6 – Relation Between Loading, Shear and Moment

Shear and bending moment diagram problems should include:

1. A neat, accurate, labeled free-body diagram of the entire structure, and the work to find the reactions.
2. A neat, properly scaled diagram of the beam showing its reactions and “true” loads. Distributed loads must be shown this diagram, because their distributed nature is significant.
3. A large graph of the shear and bending moment functions drawn directly below the scaled beam diagram.
4. The correct shape and curvature for each curve segment: zero, constant slope, polynomial. Changes in curve shapes should align with the load which causes them. Indicate the scale used for shear and moment, and use a straightedge.
5. Values of shear and moment at maximums, minimums and points of inflection.
6. Any other work need to justify your results.

8.7 – Graphical Method

Draw shear and bending moments efficiently and accurately using this procedure

1. First, determine the reaction forces and moments by drawing a free-body diagram of the entire beam and applying the equilibrium equations.
2. Establish the shear graph with a horizontal axis below the beam and a vertical axis to represent shear. Positive shears will be plotted above the *x* axis and negative below.
3. Make vertical lines at all the “interesting points”, i.e. points where concentrated forces or moments act on the beam and at the beginning and end of any distributed loads.

8.7 – Graphical Method

4. Draw the shear diagram by starting with a dot at $x = 0$, $V = 0$ then proceeding from left to right until you reach the end of the beam. Choose and label a scale which keeps the diagram a reasonable size.
   a. Whenever you encounter a concentrated force, jump up or down by that value
   b. Whenever you encounter a concentrated moment, do not jump.
   c. Whenever you encounter a distributed load, move up or down by the “area” under the loading curve over the length of the segment. The “area” is actually a force.
   d. Distributed loads cause the shear diagram to have a slope equal to value of the distributed load at that point. For unloaded segments of the beam, the slope is zero, i.e. the shear curve is horizontal. For segments with uniformly distributed load, the slope is constant. Downward loads cause downward slopes.
   e. The shear diagram should start and end at $V = 0$. If it doesn’t, recheck your work.

8.7 – Graphical Method

5. Add another interesting point wherever the shear diagram crosses the *x* axis, and determine the *x* position of the zero crossing.
6. After you have completed the shear diagram, calculate the area under the shear curve for each segment. Areas above the axis are positive, areas below the axis are negative. The areas represent moments and the sum of the areas plus the values of any concentrated moments should add to zero. If they don’t, then recheck work.
7. Establish the moment graph with a horizontal axis below the shear diagram and a vertical axis to represent moment. Positive moments will be plotted above the *x* axis and negative below.

8.7 – Graphical Method

8. Draw and label dots on the moment diagram by starting with a dot at $x = 0$, $M = 0$ then proceed from left to right placing dots until you reach the end of the beam. As you move over each segment move up or down from the current value by the "area" under the shear curve for that segment and place a dot on the graph.
   a. Positive areas cause the moment to increase, negative areas cause it to decrease.
   b. If you encounter a concentrated moment, jump straight up or down by the amount of the moment and place a dot. Clockwise moments cause upward jumps and counter-clockwise moments cause downward jumps.
   c. When you reach the end of the beam you should return to $M = 0$. If you don't, then recheck your work.
9. Connect the dots with correctly shaped lines. Segments under constant shear are straight lines, segments under changing shear are curves.

8.8 – Integration Method

There are times that the graphical technique falls short when the areas are more complicated than rectangles or triangles.

$$\Delta V = \int_a^b w(x)\, dx \qquad\qquad \Delta M = \int_a^b V(x)\, dx$$

8.8.1 – Determining Loading Functions

When determining equations for loading segments, you may choose either global equations, where all segments use the same origin, usually at the left end of the beam, or local equations, where each segment uses its own origin, usually at the left end of the segment.

8.8.1 – Determining Loading Functions

When determining equations for the loading segments from the load diagram, consider the following.

- *No load*.
- *Point Load*.
- *Uniformly Distributed Load.*
- *Uniformly Varying Load.*
- *Arbitrary Load.*

8.8.1 – Determining Loading Functions

No load.

Whenever there is no load at all on a segment there will be no change in the shear on the segment. On such sections the loading function is $w(x) = 0$. Note that this can only occur when the weight of the beam itself is neglected.

Point Load.

A point load is a concentrated force acting at a single point which causes a jump in the shear diagram.

8.8.1 – Determining Loading Functions

Uniformly Distributed Load.

A uniformly distributed load is constant over the segment and results in a linear slope, either a triangle or a trapezoid, on the shear diagram. The loading function on such sections is $w(x) = C$; $V(x) = Cx + b$. The constant value is negative if the load points down, and positive if it points upward.

8.8.1 – Determining Loading Functions

Uniformly Varying Load.

In this case the loading function is a straight, sloping lie forming a triangle or trapezoidal shape. The resulting shear function is parabolic. The general form of these functions are $w(x) = mx + 0;$ $V(x) = mx^2 + bx + c$. The slope m, intercept b, and constant c must be determined from the situation, and will depend on whether you are writing a global or local equation.

8.8.1 – Determining Loading Functions

Arbitrary Load.

The loading function will be a given function of x. $w(x) = f(x)$, and the shear and moment functions are found by integration.

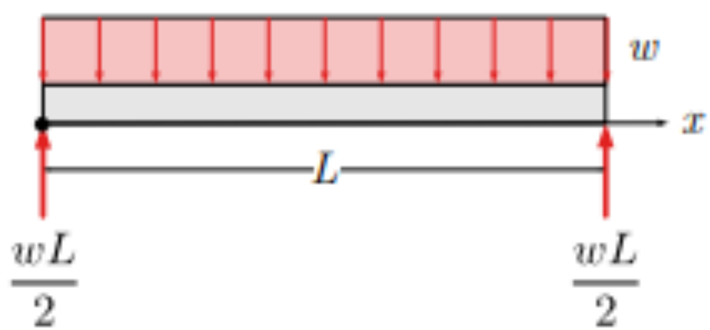
$V(x) = Z f(x)dx$

$M(x) = Z V(x)dx$

Most gravitational distributed loads are drawn with the arrows pointing down and resting on the beam. If you slide these along their line of action so that their tails are on the beam, the tips define the loading equation.

Example 8.3 (1/2)

Use the integration method to find the equations for shear and moment as a function of *x*, for a simply supported beam carrying a uniformly distributed load *w* over its entire length *L*.



Example 8.3 (2/2)

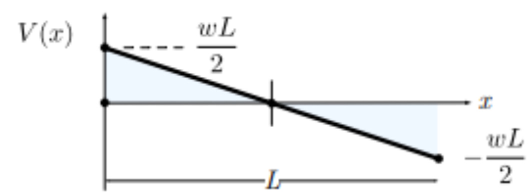
This beam has only one load section, and on that section the load is constant so, *w(x)* = −*w* . The initial conditions there are $V(0) = wL/2$, and $M(0) = 0$.

$$\Delta V = -\int_0^x w(x)\, dx$$

$$V(x) - \cancelto{wL/2}{V(0)} = -wx$$

$$V(x) = \frac{wL}{2} - wx$$

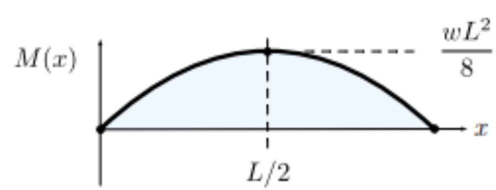
$$= w\left(\frac{L}{2} - x\right)$$



$$\Delta M = \int_0^x V(x)\, dx$$

$$M(x) - \cancelto{0}{M(0)} = \int_0^x w\left(\frac{L}{2} - x\right)\, dx$$

$$M(x) = \frac{w}{2}(Lx - x^2)$$



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Baker, Daniel W., and Haynes, William. “Chapter 8 – Internal Forces.” *Engineering Statics: Open and Interactive*. May 21, 2023. https://engineeringstatics.org/

Chapter 9 – Friction

Engineering Mechanics: Statics and Structures

Chapter Outline

9.1 Dry Friction

9.2 Slipping vs. Tipping

9.3 Wedges

9.5 Flexible Belts

Chapter 9 – Friction

Friction is the force which resists relative motion between surfaces in contact with each other.

Friction is categorized by the nature of the surfaces in contact and the conditions under which they are interacting.

1. **Dry friction**, which is the force that opposes one solid surface sliding across another solid surface.
2. **Rolling friction** is the force that opposes motion of a rolling wheel or ball.
3. **Fluid friction** is the friction between layers of a viscous fluid in motion.
4. **Skin friction**, also called drag, is the friction that occurs between a fluid and a moving surface.
5. **Internal friction** is the force resisting the internal deformation of a solid material.

9.1 – Dry Friction

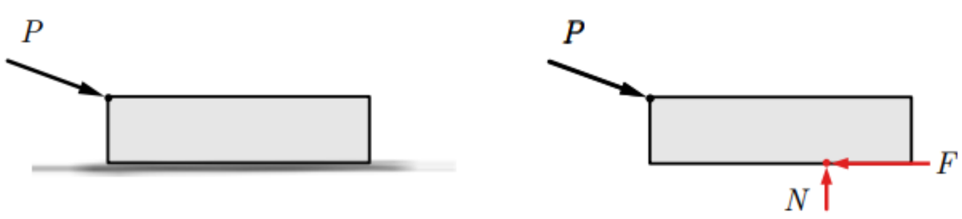
- Dry friction, also called Coulomb friction, is a force which appears between two solid surfaces in contact.
- This force is distributed over the contact area and always acts in whichever direction opposes relative motion between the surfaces.
  - Static Friction - hold the object in equilibrium
  - Kinetic Friction - retard but not prevent motion

9.1.1 – Coulomb Friction

- The force of friction is proportional to the normal force, where the normal force is the force acting perpendicular to the contacting surface.

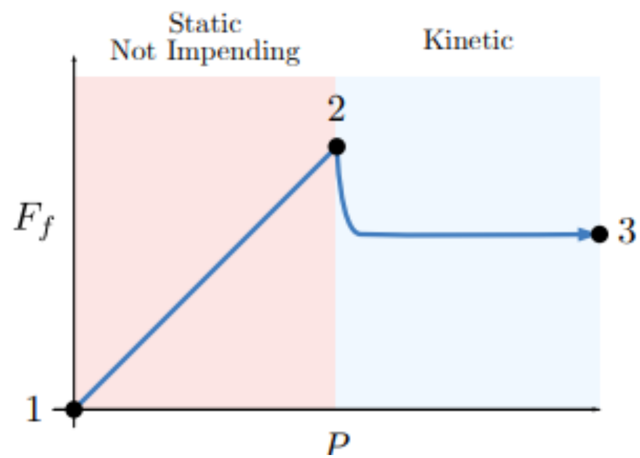
$$F_f = \mu N$$

- μ, is called the friction coefficient. μ is always greater than zero and commonly less than one.

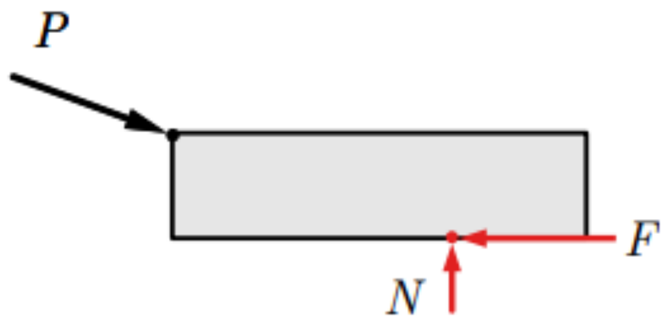


9.1.1 – Coulomb Friction

- Friction has two distinct regions



- The region from point 1 to point 2, where the force of friction increases linearly with load is called the ***static friction*** region. Here you must use the coefficient of static friction $\boldsymbol{\mu_s}$.
- The region from point 2 to point 3, where the friction remains roughly constant is called the ***kinetic friction*** region. In this region you must use the coefficient of kinetic friction $\boldsymbol{\mu_k}$.
- Point 2 is called the point of ***impending motion***



When force P is gradually increased from zero, the normal force N and the frictional force F_f both change in response.

- Initially both P and F_f are zero and the object is in equilibrium. The interaction between the two surfaces in contact means that friction is available but it is not engaged $F_f = P = 0$.
- As P increases, the opposing friction force F_f increases as well to match and hold the object in equilibrium. In this static-but-not-impending phase $F_f = P$.
- When P reaches point 2, motion is impending because friction has reached its maximum value. $F_{f\,max} = \mu_s N = P$. If force P increases slightly beyond $F_{f\,max}$, the friction force suddenly drops to the kinetic value $F_f = \mu_k N$.

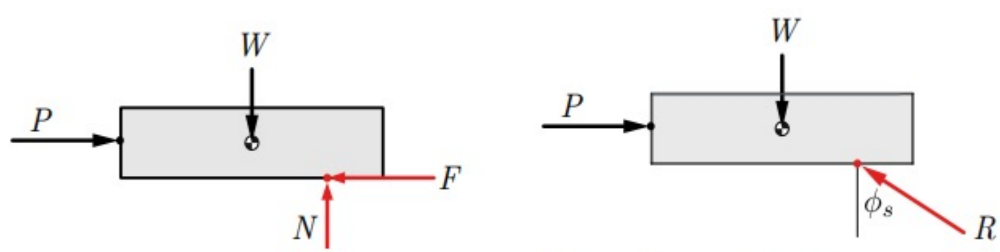
9.1.2 – Friction Angle and Friction Resultant

- The friction resultant is the vector sum of the friction and normal forces.

$$R = \sqrt{F_f^2 + N^2}$$

- The friction angle ϕ_s is defined as the angle between the friction resultant and the normal force.

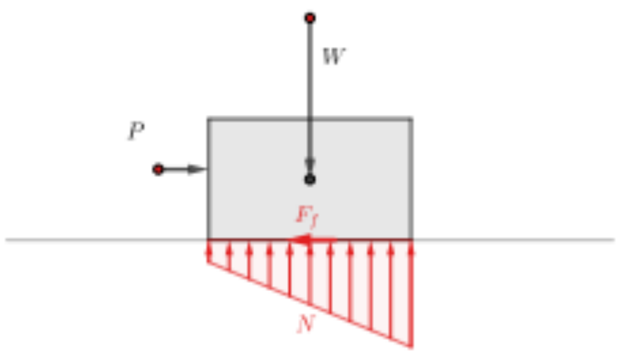
$$\tan \phi_s = \frac{F}{N}$$



(a) Force components. (b) Resultant and Friction angle.

9.1.3 – Normal Forces

- The normal force supporting the object is distributed over the entire contact surface, however it is common on two dimensional problems to replace the distributed force with an equivalent concentrated force acting at a particular spot on the contacting surface.



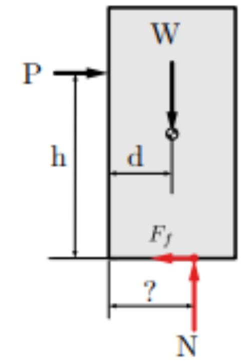
9.2 – Slipping vs. Tipping

- The point at which an object starts to move is called the point of impending motion.
- Two possible outcomes
  - the maximum static friction force will be reached and the box will begin to slide,
  - or the pushing force and the friction force will create a sufficient couple to cause the box tip on its corner.

9.2 – Slipping vs. Tipping

- The easiest way to determine whether the box will slip, tip, or stay put is to solve for the maximum load force P twice, once assuming slipping and a second time expecting tipping, then compare the actual load to these maximums.
- Three steps to determine which outcomes occurs
  1. *Check slipping.*
  2. *Check tipping.*
  3. *Compare the results.*

9.2 – Slipping vs. Tipping



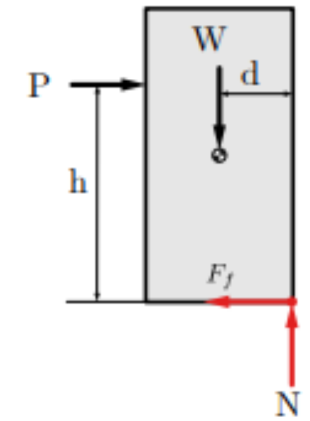
1. *Check slipping*.

The maximum friction force is equal to the static coefficient of friction times the normal force

$$F_{f\,max} = \mu_s N$$

Assume that the maximum normal force N is acting at an unknown location and solve for the applied force which will maintain equilibrium. If the load exceeds this value than this the body will slip or tip.

9.2 – Slipping vs. Tipping



2. Check tipping.

The object will tip when the resultant normal force *N* shifts off the end of the object, because it no longer acts on the object so it can't contribute to equilibrium.

Create a free-body diagram assuming that the normal force *N* acts at the far corner of the box and solve for the applied force which will maintain equilibrium. Any greater force will make the body tip, unless it is already slipping. At tipping, the friction force is static-but-not impending as it has not reached impending motion for slipping.

9.2 – Slipping vs. Tipping

3. Compare the results.

If *P* exceeds the smaller of the limiting values, it will initiate the corresponding impending motion.

9.3 – Wedges

- A wedge is a tapered object which converts a small input force into a large output force using the principle of an inclined plane.
- Wedges are used to separate, split or cut objects, lift weights, or fix objects in place.
- The mechanical advantage of a wedge is determined by the angle of its taper; narrow tapers have a larger mechanical advantage.

9.3 – Wedges

Wedges are used in two primary ways:

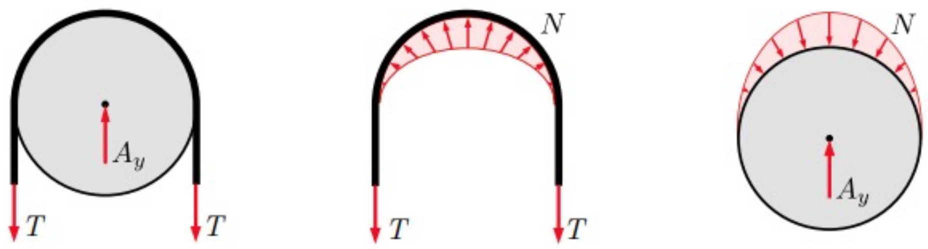
1. Low friction wedges are a simple machines which allows users to create large output forces to move objects using comparatively small input forces.
2. High-friction (self-locking) wedges control the location of objects or hold them in place.

9.5 – Flexible Belts

- When a belt, rope, or cable is wrapped around an object, there is potential for flexible belt friction

9.5.1 – Frictionless Belts

A non-uniform distributed normal force acts at points of contact with the cylinder to oppose the tension in the belt and maintain equilibrium.

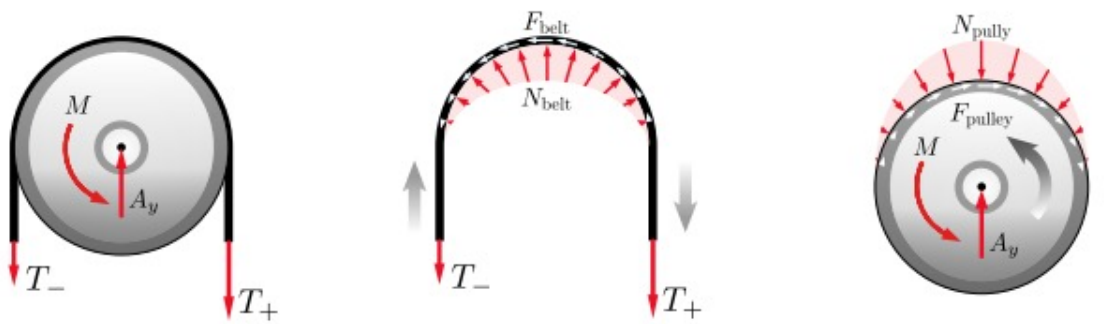


(a) FBD Belt and pulley **(b)** FBD of belt **(c)** FBD of cylinder

Without friction, the two tensions must be equal otherwise the belt would slip around the cylinder.

9.5.2 – Friction in Flat Belts

- Without friction, the two tensions must be equal otherwise the belt would slip around the cylinder.

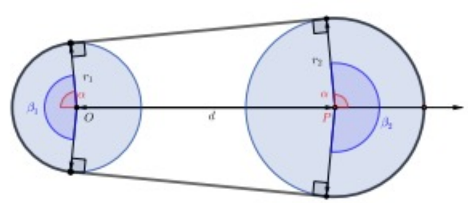


(a) FBD of Belt and Pulley **(b)** FBD of Belt **(c)** FBD of pulley

9.5.2 – Friction in Flat Belts

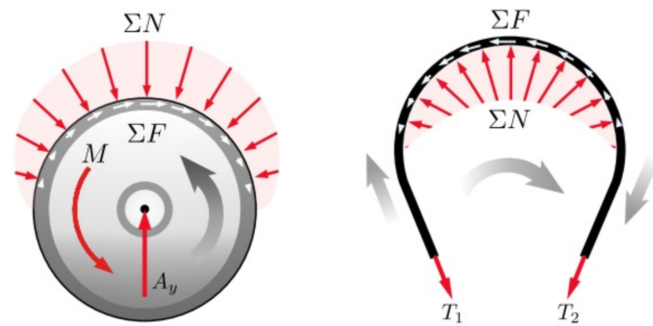
Contact Angle β

- The belt will depart the pulley at a point of tangency, which is always perpendicular to a radius.
- To find β create one or more right triangles using the incoming and outgoing belt paths and apply complementary angles to relate the belt geometry to the contact angle



9.5.2 – Friction in Flat Belts

Belt Tension



(a) Problem Diagram **(b)** FBD of Pulley **(c)** FBD of Belt

Summing forces along the belt, we find that the tension T_1 plus the distributed friction force ΣF must equal T_2 for equilibrium.

$\Sigma F_{belt} = 0$

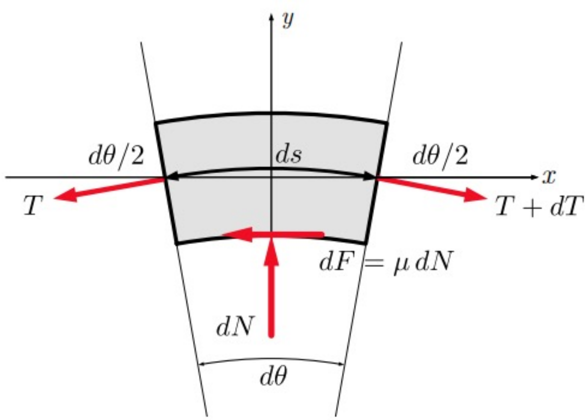
$T_1 + \Sigma F - T_2 = 0$

$T_2 = T_1 + \Sigma F$

Therefore, the larger tension is $T+ = T_2$ and the smaller tension is $T- = T_1$

Change in Belt Tension due to Friction

- Applying the equilibrium equations to a free-body diagram of a differential element of the belt enables us to derive the relation between the two belt tensions, the contact angle β, and the friction coefficient μ_s.



$$\frac{T_+}{T_-} = e^{\mu\beta}$$

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Baker, Daniel W., and Haynes, William. "Chapter 9 – Friction." *Engineering Statics: Open and Interactive*. May 21, 2023. https://engineeringstatics.org/