ON CONVERGENCE AND DIVERGENCE OF FOURIER EXPANSIONS WITH RESPECT TO SOME GEGENBAUER-SOBOLEV TYPE INNER PRODUCT

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Abstract. Let introduce the discrete Sobolev-type inner product

$$\langle f,g\rangle=\int_{-1}^{1}f(x)g(x)d\mu(x)+M[f(1)g(1)+f(-1)g(-1)]$$
$$+N[f'(1)g'(1)+f'(-1)g'(-1)],$$

where

$$d\mu(x)=\frac{\Gamma(2\alpha+2)}{2^{(2\alpha+1)}\Gamma^2(\alpha+1)}(1-x^2)^{\alpha}\,dx,\qquad M,N\geq 0,\qquad \alpha>-1.$$

In this paper we prove the failure of a.e. convergence of the Fourier expansion in terms of the orthonormal polynomials with respect to the above inner product. Necessary conditions for mean convergence are also discussed.

1. Introduction

Let us first introduce some notation. We shall say that $f\in L^p(d\mu)$ if f is measurable on $[-1,1]$ and $\|f\|_{L^p(d\mu)}<\infty$, where

$$\|f\|_{L^p(d\mu)}=\begin{cases}\left(\int_{-1}^{1}|f(x)|^p d\mu(x)\right)^{\frac{1}{p}} & \text{if } 1\leq p<\infty,\\ \underset{-1<x<1}{ess\ sup}\,|f(x)| & \text{if } p=\infty.\end{cases}$$

Now let introduce the Sobolev-type spaces

$$S_p=\{f\ :\ \|f\|_{S_p}^p=\|f\|_{L^p(d\mu)}^p+M[|f(1)|^p+|f(-1)|^p]$$
$$+N[|f'(1)|^p+|f'(-1)|^p]<\infty\},\qquad 1\leq p<\infty,$$

$$S_\infty=\{f\ :\ \|f\|_{S_\infty}=\|f\|_{L^\infty(d\mu)}<\infty\},\qquad p=\infty.$$

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If $f,\ g \in S_2$ we can introduce the discrete Sobolev-type inner product

$$(1) \quad \langle f,g\rangle = \int_{-1}^{1} f(x)g(x)d\mu(x) + M[f(1)g(1)+f(-1)g(-1)] \\ + N[f'(1)g'(1)+f'(-1)g'(-1)]$$

where $M, N \geq 0$. We denote by $\{\hat{B}_n^{(\alpha)}\}_{n\geq 0}$ the orthonormal polynomials with respect to the inner product (1) (see [1], [2], [11]). They are called Gegenbauer-Sobolev type polynomials. For $M = N = 0$, we get the classical Gegenbauer orthonormal polynomials that we will denote $\{p_n^{(\alpha)}\}_{n\geq 0}$. It is well known that, up to the Gegenbauer orthonormal polynomials, the polynomials $\hat{B}_n^{(\alpha)}$ for $M > 0$, $N \geq 0$ decay at the rate of $n^{-\alpha-3/2}$ at the end points 1 and -1.

For $f \in S_1$, the Fourier expansion in terms of Gegenbauer-Sobolev type polynomials is

$$(2) \qquad \sum_{k=0}^{\infty} \hat{f}(k)\hat{B}_k^{(\alpha)}(x),$$

where

$$\hat{f}(k) = \langle f, \hat{B}_k^{(\alpha)}\rangle, \qquad k = 0,\ 1,\ \dots\ .$$

The Cesàro means of order δ of the expansion (2) is defined by (see [19, p. 76-77])

$$\sigma_n^{\delta} f(x) = \sum_{k=0}^{n} \frac{A_{n-k}^{\delta}}{A_n^{\delta}} \hat{f}(k)\hat{B}_k^{(\alpha)}(x),$$

where $A_k^{\delta} = \binom{k+\delta}{k}$.

For $M = N = 0$ and $\alpha = 0$, Pollard [15] shows that for each $p < 4/3$ there exists a function $f \in L^p(dx)$ such that its Fourier expansion (2) diverges a.e. on $[-1,1]$. Later on, Meaney [9] extends the result to $p = 4/3$. Furthermore, he proved that this is a special case of a divergence result for Jacobi polynomials series. The failure of a.e. convergence of the expansions (2), for $M > 0$ and $N = 0$, has been discussed in [4].

The main goal of this contribution is to prove that, for $M \geq 0,\ N > 0$, and $p_0 = \frac{4\alpha+4}{2\alpha+3}$, there are functions $f \in L^{p_0}(d\mu)$ whose expansions in terms of the polynomials associated with the Sobolev inner product (1) are divergent almost everywhere on $[-1,1]$.

The structure of the paper is as follows. In Section 2, the background about the asymptotic behaviour of Gegenbauer-Sobolev type orthogonal polynomials is presented. In Section 3 we show that, for $M, N \geq 0$, $1 < p < p_0$ and $0 < \delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}$, there exists a function $f \in L^p(d\mu)$ with almost everywhere divergent Cesàro means of order δ. Finally, in Section 4 necessary conditions for the norm convergence of the Fourier expansion (2) are given.

Notice that the study of the convergence of the Fourier expansion in terms of the polynomials with respect to the discrete Sobolev inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)d\mu(x) + \sum_{k=1}^{K}\sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k)g^{(i)}(a_k), \qquad M_{k,i} > 0,$$

has been discussed in ([5], [6], [7], [16]).

2. Gegenbauer-Sobolev type polynomials

We summarize some properties of Gegenbauer-Sobolev type polynomials that we will need in the sequel (see [1], [2], [11]). Throughout the manuscript positive constants are denoted by $c,\ c_1,\ \dots$; unless specified, their values may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and notation $u_n \sim v_n$ means $c_1 u_n \leq v_n \leq c_2 u_n$ for n large enough.

The representation of $\hat{B}_n^{(\alpha)}$ in terms of Gegenbauer orthonormal polynomials is

$$\hat{B}_n^{(\alpha)}(x) = A_n(1-x^2)^2 p_{n-4}^{(\alpha+4)}(x) + B_n(1-x^2)p_{n-2}^{(\alpha+2)}(x) + C_n p_n^{(\alpha)}(x) \tag{3}$$

where

i) If $M = 0$, $N > 0$, then

$$A_n \cong \frac{2^{\alpha+1}\Gamma(\alpha+1)}{\alpha+2}\sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}},\ \ B_n \cong -2^{\alpha+1}\Gamma(\alpha+1)\sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}},\ \ C_n \cong -A_n,$$

ii) If $M > 0$, $N > 0$, then

$$A_n \cong 2^{\alpha+1}\Gamma(\alpha+1)\sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}},\ \ B_n \sim -n^{-2\alpha-2},\ \ C_n \sim -n^{-2\alpha-2}$$

iii) If $M > 0$, $N = 0$, then

$$A_n = 0,\ \ B_n \cong -2^{\alpha+1}\Gamma(\alpha+1)\sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}},\ \ C_n \sim n^{-2\alpha-2}.$$

The polynomials $\hat{B}_n^{(\alpha)}$ satisfy the estimate

$$|\hat{B}_n^{(\alpha)}(cos\theta)| = \begin{cases} O(\theta^{-\alpha-1/2}) & \text{if } c/n < \theta \leq \pi/2 \\ O(n^{\alpha+1/2}) & \text{if } 0 \leq \theta \leq c/n. \end{cases} \tag{4}$$

The inner strong asymptotic behaviour of $\hat{B}_n^{(\alpha)}$, for $\theta \in [\epsilon, \pi - \epsilon]$ and $\epsilon > 0$, is given by

$$\hat{B}_n^{(\alpha)}(cos\theta) = c\left(sin\frac{\theta}{2}cos\frac{\theta}{2}\right)^{-\alpha-1/2} cos(k\theta+\gamma) + O(n^{-1}), \tag{5}$$

where $k = n + \alpha + 1/2$ and $\gamma = -(\alpha + 1/2)\pi/2$.

The formula of Mehler-Heine type for Gegenbauer orthonormal polynomials is (see [17, Theorem 8.1.1] and [17, (4.3.4)])

$$\lim_{n\to\infty} n^{-\alpha-1/2} p_n^{(\alpha)}\left(cos(\frac{z}{n})\right) = z^{-\alpha} J_\alpha(z), \tag{6}$$

where α is a real number and $J_\alpha(z)$ is the Bessel function. This formula holds uniformly for $|z| \le R$, for R a given positive real number.

From (8) it can be shown that

$$\lim_{n\to\infty} n^{-\alpha-1/2} p_n^{(\alpha)}\left(cos(\frac{z}{n+j})\right) = z^{-\alpha} J_\alpha(z) \tag{7}$$

holds uniformly for $|z| \le R$, $R > 0$ fixed, and $j \in N \cup \{0\}$.

Lemma 1. *For $|z| \le R$, $R > 0$ fixed, we get*

$$\lim_{n\to\infty} n^{-\alpha-1/2} \hat{B}_n^{(\alpha)}(cos(\frac{z}{n})) = z^{-\alpha}\left(c_1 J_{\alpha+4}(z) - c_2 J_{\alpha+2}(z) - c_3 J_\alpha(z)\right) \tag{8}$$

where

i) *If $M = 0$, $N > 0$, then $c_i > 0$, $i = 1, 2, 3$, $c_1 = c_3$*
ii) *If $M > 0$, $N > 0$, then $c_1 > 0$ and $c_2 = c_3 = 0$*
iii) *If $M > 0$, $N = 0$, then $c_2 > 0$ and $c_1 = c_3 = 0$*

Proof. From (3) we have

$$\begin{aligned} n^{-\alpha-1/2}\hat{B}_n^{(\alpha)}(cos\frac{z}{n}) = A_n\ sin^4(\frac{z}{n}) n^{-\alpha-1/2} p_{n-4}^{(\alpha+4)}(cos\frac{z}{n}) \\ + B_n\ sin^2(\frac{z}{n}) n^{-\alpha-1/2} p_{n-2}^{(\alpha+2)}(cos\frac{z}{n}) \\ + C_n n^{-\alpha-1/2} p_n^{(\alpha)}(cos\frac{z}{n}) \end{aligned}$$

where $j \in N \cup \{0\}$.

Finally, we take the limit $n \to \infty$ and use the fact that $sin\frac{z}{n} \cong \frac{z}{n}$ and (7) to obtain (8). □

Now we will estimate the S_p norms for Gegenbauer-Sobolev polynomials

$$\|\hat{B}_n^{(\alpha)}\|_{S_p}^p = \int_{-1}^{1} |\hat{B}_n^{(\alpha)}(x)|^p d\mu(x) + M\left[|(\hat{B}_n^{(\alpha)})(1)|^p + |(\hat{B}_n^{(\alpha)})(-1)|^p\right] + N\left[|(\hat{B}_n^{(\alpha)})'(1)|^p + |(\hat{B}_n^{(\alpha)})'(-1)|^p\right], \tag{9}$$

where $1 \leq p < \infty$. Hence, it is sufficient to estimate the $L^p(d\mu)$ norms for $\hat{B}_n^{(\alpha)}$. For $M = N = 0$ the calculation of this norm appears in [17, p.391. Exercise 91] (see also [8, (2.2)]).

Lemma 2. *Let $M \geq 0$ and $N \geq 0$. For $\alpha \geq -1/2$ and $1 \leq q < \infty$*

$$\int_0^1 (1-x)^\alpha |\hat{B}_n^{(\alpha)}(x)|^q dx \sim \begin{cases} c & \text{if } 2\alpha > q\alpha - 2 + q/2, \\ log\ n & \text{if } 2\alpha = q\alpha - 2 + q/2, \\ n^{q\alpha+q/2-2\alpha-2} & \text{if } 2\alpha < q\alpha - 2 + q/2. \end{cases}$$

Proof. In order to prove the lemma, we follow the same lines as in [17, Theorem 7.34]. From (4), for $q\alpha + q/2 - 2\alpha - 2 \neq 0$, we have

$$\begin{aligned} \int_0^1 (1-x)^\alpha |\hat{B}_n^{(\alpha)}(x)|^q dx &\sim \int_0^{\pi/2} \theta^{2\alpha+1}\ |\hat{B}_n^{(\alpha)}(cos\theta)|^q d\theta \\ &= O(1) \int_0^{n^{-1}} \theta^{2\alpha+1}\ n^{q\alpha+q/2} d\theta + O(1) \int_{n^{-1}}^{\pi/2} \theta^{2\alpha+1}\ \theta^{-q\alpha-q/2} d\theta \\ &= O(n^{q\alpha+q/2-2\alpha-2}) + O(1), \end{aligned}$$

and for $q\alpha + q/2 - 2\alpha - 2 = 0$ we have

$$\int_0^1 \theta^{2\alpha+1} |\hat{B}_n^{(\alpha)}(x)|^q dx = O(log\ n).$$

Now we will prove the lower estimates for integrals involving Gegenbauer-Sobolev type polynomials when $M = 0$ and $N > 0$. The proof of the other cases can be done in a similar way. According to Lemma 1 we have

$$\begin{aligned} \int_0^{\pi/2} \theta^{2\alpha+1} |\hat{B}_n^{(\alpha)}(cos\theta)|^q d\theta &> \int_0^{n^{-1}} \theta^{2\alpha+1} |\hat{B}_n^{(\alpha)}(cos\theta)|^q d\theta \\ &\cong c \int_0^1 (z/n)^{2\alpha+1} n^{q\alpha+q/2}\ n^{-1} |z^{-\alpha} \left(c_1 J_{\alpha+4}(z) - c_2 J_{\alpha+2}(z) - c_1 J_\alpha(z)\right)|^q dz \\ &\sim n^{q\alpha+q/2-2\alpha-2}. \end{aligned}$$

Using a similar argument as above, for $2\alpha = q\alpha + q/2 - 2$, we have

$$\int_0^{\pi/2} \theta^{2\alpha+1} |\hat{B}_n^{(\alpha)}(cos\theta)|^q dx > c \int_0^{n^{-1/2}} \theta^{2\alpha+1} |\hat{B}_n^{(\alpha)}(cos\theta)|^q dx$$

$$\cong c \int_0^{n^{1/2}} z^{2\alpha+1} |z^{-\alpha} \left(c_1 J_{\alpha+4}(z) - c_2 J_{\alpha+2}(z) - c_1 J_\alpha(z)\right)|^q dz$$

$$\geq c\ n^{\alpha+1} \geq c\ log\ n.$$

Finally, from (5) we obtain

$$\int_0^{\pi/2} \theta^{2\alpha+1} |\hat{B}_n^{(\alpha)}(cos\theta)|^q d\theta > \int_{\pi/4}^{\pi/2} \theta^{2\alpha+1} |\hat{B}_n^{(\alpha)}(cos\theta)|^q d\theta \sim c.$$

The proof of Lemma 2 is complete. □

By using this lemma and [11, Proposition 4], we have:

Corollary 1. *Let* $M \geq 0$ *and* $N \geq 0$*. For* $\alpha \geq -1/2$ *and* $1 \leq q < \infty$

$$\|\hat{B}_n^{(\alpha)}(x)\|_{S_q} \sim \begin{cases} c & \text{if } 2\alpha > q\alpha - 2 + q/2, \\ (log\ n)^{1/q} & \text{if } 2\alpha = q\alpha - 2 + q/2, \\ n^{\alpha+1/2-(2\alpha+2)/q} & \text{if } 2\alpha < q\alpha - 2 + q/2. \end{cases}$$

3. Divergence almost everywhere

If the expansions (2) converges on a set, say E, of positive measure in [-1,1] then

$$|c_n(f)\hat{B}_n^{(\alpha)}(x)| \to 0 \qquad when\ n \to \infty$$

for $x \in E$. From Egorov's Theorem it follows that there is a subset $E_1 \subset E$ of positive measure such that

$$|c_n(f)\hat{B}_n^{(\alpha)}(x)| \to 0, \qquad when\ n \to \infty,$$

uniformly for $x \in E_1$. Hence, from (5), we have

$$|c_n(f)\left(cos(k\theta + \gamma) + O(n^{-1})\right)| \to 0, \qquad for\ n \to \infty,$$

uniformly for $cos\theta \in E_1$. Using the Cantor-Lebesgue Theorem, as described in [10, Subsection 1.5](see also [19, p.316]), we obtain

$$|c_n(f)| \to 0, \qquad when\ n \to \infty. \tag{10}$$

From Lemma 2, for $\alpha > -1/2$ and $1 \le q < \infty$, we have

$$(11) \quad ||\hat{B}_n^{(\alpha)}||_{L^q(d\mu)} = \left(\frac{\Gamma(2\alpha+2)}{2^{2\alpha}\Gamma^2(\alpha+1)} \int_0^1 |\hat{B}_n^{(\alpha)}(x)|^q (1-x)^\alpha dx \right)^{1/q}$$

$$\sim \begin{cases} (log\ n)^{1/q_0} & \text{if } p = p_0, \\ n^{\alpha+1/2-2\alpha/q-2/q} & \text{if } p < p_0. \end{cases}$$

where p is the conjugate of q i.e. $1/p + 1/q = 1$.

Now we can prove our first result

Theorem 1. *There is an $f \in S_{p_0}$ whose Fourier expansion* (2) *diverges almost everywhere on* $[-1,1]$.

Proof. Taking into account (1), the Fourier coefficients of the series (2) can be written as

$$(12) \quad c_n(f) = c_n'(f) + M\left[f(1)\hat{B}_n^{(\alpha)}(1) + f(-1)\hat{B}_n^{(\alpha)}(-1)\right]$$
$$+ N\left[f'(1)\left(\hat{B}_n^{(\alpha)}\right)'(1) + f'(-1)\left(\hat{B}_n^{(\alpha)}\right)'(-1)\right],$$

where

$$c_n'(f) = \int_{-1}^1 f(x)\hat{B}_n^{(\alpha)}(x)d\mu(x).$$

The uniform boundedness principle and (11) yield the existence of functions $f \in S_{p_0}$, supported on $[0,1]$, such that the linear functional $c_n'(f)$ satifies

$$\frac{c_n'(f)}{(log\ n)^{1/(2q_0)}} \to \infty, \quad when\ n \to \infty.$$

Hence, from (12) and [11, Proposition 4], we obtain

$$\frac{c_n(f)}{(log\ n)^{1/(2q_0)}} \to \infty, \quad when\ n \to \infty.$$

Since this result contradicts (12) then for this f the Fourier-Sobolev series diverges almost everywhere on $[-1,1]$. □

Next we show that, for some values of δ, there are functions with a.e. divergent Cesàro means of order δ.

Theorem 2. *Let α, p, and δ be real numbers such that $\alpha > -1/2$,*

$$1 < p < \frac{4\alpha+4}{2\alpha+3} \qquad if\ M > 0,\ N \ge 0,$$

$$1 \le p < \frac{4\alpha+4}{2\alpha+3}, \qquad if\ M=0,\ N\ge 0,$$
$$0 \le \delta < \frac{2\alpha+2}{p} - \frac{2\beta+3}{2},$$

then there exists $f \in S_p$ whose Cesàro means $\sigma_n^\delta f(x)$ is divergent almost everywhere on $[-1,1]$.

Proof. Let $M, N \ge 0$. If the expansion (2) is Cesàro summable of order δ on a set, say E, of positive measure in $[-1,1]$, then from [19, Theorem 3.1.22] (see also [10, Lemma 1.1]) it follows that

$$|c_n(f)\hat{B}_n^{(\alpha)}(x)| = O(n^\delta)$$

for $x \in E$. Again, from Egorov's Theorem it follows that there is a subset $E_1 \subset E$ of positive measure such that

$$|c_n(f)\hat{B}_n^{(\alpha)}(x)| = O(n^\delta)$$

uniformly for $x \in E_1$. Hence, from (5), we have

$$|n^{-\delta}c_n(f)\left(cos(k\theta+\gamma)+O(n^{-1})\right)| \le c.$$

uniformly for $x = cos\theta \in E_1$. Using again the Cantor-Lebesgue Theorem we obtain

$$|\frac{c_n(f)}{n^\delta}| \le c, \qquad \forall n \ge 1. \tag{13}$$

Suppose that

$$0 \le \delta < \frac{2\alpha+2}{p} - \frac{2\beta+3}{2}.$$

If q is the conjugate of p, then from the last inequality, we get

$$\delta < \alpha + \frac{1}{2} - \frac{2\beta}{q} - \frac{2}{q}.$$

Using the argument given in [10, Subsection 1.4], (11), and [11, Proposition 4], for the linear functional $c_n'(f) = \int_{-1}^{1} f(x)\hat{B}_n^{(\alpha)}(x)d\mu(x)$, it follows that there is an $f \in S_p$, where

$$1 < p < p_0 \quad if\ M>0,\ N\ge 0,$$
$$1 \le p < p_0, \quad if\ M=0,\ N\ge 0,$$

supported on $[0,1]$, such that

$$\frac{c_n'(f)}{n^\delta} \to \infty, \quad when\ n\to\infty.$$

So, from (12) and [11, Proposition 4], we obtain

$$\frac{c_n(f)}{n^\delta} \to \infty, \quad \textit{when } n \to \infty.$$

Taking into account (13), for this f the Cesàro means $\sigma_n^\delta f(x)$ diverges almost everywhere. □

Remark 1. *Using formulae in* [3]*, relating the Riesz and Cesàro means of order* $\delta \geq 0$, *we conclude that Theorem 2 also holds for the Riesz means.*

4. Necessary conditions for the norm convergence

The problem of the norm convergence of partial sums of the Fourier expansions in terms of Gegenbauer polynomials has been discussed by many authors. See [12], [13], [14], and the references therein.

Let $S_n f$ be the n-th partial sum of the expansion (2)

$$S_n(f, x) = \sum_{k=0}^{n} \hat{f}(k) \hat{B}_k^{(\alpha)}(x).$$

Theorem 3. *Let* $\alpha > -1/2$ *and* $1 < p < \infty$. *If there exists a constant* $c > 0$ *such that*

$$\|S_n f\|_{S_p} \leq c \|f\|_{S_p} \tag{14}$$

for every $f \in S_p$ *and* $n \geq 0$, *then* $p \in (p_0\ q_0)$.

Proof. For the proof, we apply the same argument as in [13] (see also [18]). Assume that (4.1) holds. Then

$$\|\langle f, \hat{B}_n^{(\alpha)} \rangle \hat{B}_n^{(\alpha)}(x)\|_{S_p} \leq 2c \|f\|_{S_p}.$$

Therefore

$$\|\hat{B}_n^{(\alpha)}(x)\|_{S_p} \|\hat{B}_n^{(\alpha)}(x)\|_{S_q} < \infty,$$

where p is the conjugate of q. From Corollary 1, it follows that the last inequality holds if and only if $p \in (p_0\ q_0)$.

The proof of Theorem 4.1 is completed.

□

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References

[1] H. Bavinck and H. G. Meijer, *Orthogonal polynomials with respect to a symmetric inner product involving derivatives*, Appl. Anal., **33** (1989), 103-117.

[2] H. Bavinck and H. G. Meijer, *On orthogonal polynomials with respect to an inner product involving derivatives: zeros and recurrence relations*, Indag. Math. N. S., **1** (1990), 7-14.

[3] J. J. Gergen, *Summability of double Fourier series*, Duke Math. J., **3** (1937), 133-148.

[4] J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, *Two notes on convergence and divergence a.e. of Fourier series with respect to some orthogonal system*, Proc. Amer. Math. Soc., **116** (1992), 457-464.

[5] J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, *Weighted norm inequalities for polynomial expansions associated to some measures with mass points*, Constr. Approx., **12** (1996), 341-360.

[6] F. Marcellán, B. P. Osilenker, and I. A. Rocha, *On Fourier series of Jacobi-Sobolev orthogonal polynomials*, J. Inequal. Appl., **7** (2002), 673-699.

[7] F. Marcellán, B. P. Osilenker, and I. A. Rocha, *On Fourier series of a discrete Jacobi-Sobolev inner product*, J. Approx. Theory, **117** (2002), 1-22.

[8] C. Markett, *Cohen type inequalities for Jacobi, Laguerre and Hermite expansions*, SIAM J. Math. Anal., **14** (1983), 819-833.

[9] Ch. Meaney, *Divergent Jacobi polynomial series*, Proc. Amer. Math. Soc., **87** (1983), 459-462.

[10] Ch. Meaney, *Divergent Cesàro and Riesz means of Jacobi and Laguerre expansions*, Proc. Amer. Math. Soc., **131** (2003), 3123-3128.

[11] A. F. Moreno, F. Marcellán, and B. P. Osilenker, *Estimates for polynomials orthogonal with respect to some Gegenbauer-Sobolev type inner product*, J. Inequal. Appl., **3** (1999), 401-419.

[12] B. Muckenhoupt, *Mean convergence of Jacobi series*, Proc. Amer. Math. Soc., **23** (1969), 306-310.

[13] J. Newman and W. Rudin, *Mean convergence of orthogonal series*, Proc. Amer. Math. Soc., **3** (1952), 219-222.

[14] H. Pollard, *The mean convergence of orthogonal series III*, Duke Math. J. **16** (1949), 189-191.

[15] H. Pollard, *The convergence almost everywhere of Legendre series*, Proc. Amer. Math. Soc, **35** (1972), 442-444.

[16] I. A. Rocha, F. Marcellán, and L. Salto, *Relative asymptotics and Fourier Series of orthogonal polynomials with a discrete Sobolev inner product*, J. Approx. Theory, **121** (2003), 336-356.

[17] G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Pub. 23, Amer. Math. Soc., Providence, RI (1975), 4th Edition.

[18] J. L. Varona, *Condiciones necesarias para la convergencia de series de Fourier*, Matemátiques, **3** (2005), 135-153.

[19] A. Zygmund, *Trigonometric series: Vols. I, II*, Cambridge University Press, London (1968).

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