A Splinter on Twin Primes

Haakon Waadeland

0.Splinters

The concept of a *splinter* in mathematics was introduced by Wolf Thron in one of the many great workshops we enjoyed on both sides of the ocean, and where he was the essential participant. A definition does not exist, but any small idea, whose possible value in mathematics is an open question, may definitely be called a splinter. As may be expected, some splinters end up in the waste basket, others, actually quite a few, have developed to mathematics worth working on, and in the end to interesting results. For the present splinter, however, the most likely terminating place will be the waste basket. But hopefully there will be some fun along the road.

1.Twin primes

Twin primes are pairs of primes which differ by 2, such as (3,5), (5,7), (11,13), (17,19), (29,31), ... It is not known whether or not there are infinitely many twin primes, but Viggo Brun proved in a paper from 1919 that the series

$$\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\left(\frac{1}{29}+\frac{1}{31}\right)+\cdots \tag{1}$$

converges to a value B_2, called *Brun's constant*. Several people have computed very large sections of the series, all giving lower bounds for B_2, since the terms in (1) are all positive. One particular example is due to Sebah and Demichel. Using all twin primes up to 10^{16} they found in 2002 the (lower) estimate

$$B_2 \approx 1.902160583104.$$

On the other hand: By using the prime twins up to 10^{14} Thomas Nicely estimated heuristicly Brun's constant to be

$$B_2 \approx 1.902160578,$$

see e.g. [1]. This reference is also the reference for the whole Section 1 and contains several further references.

In the following we shall use the fact, known from the remarks above, that $B_2 > 1.9$ (or better e.g. $B_2 > 1.9021605$). Since the series converges there is also an upper bound. So far no such bound is known, but there are conjectures, based upon e.g. the above digits.

2.Continued fraction expansion

The series (1) can be converted to a continued fraction of the form

$$\frac{c_1}{1} \underset{+}{} \frac{c_2}{1} \underset{+}{} \frac{c_3}{1} \underset{+}{} \frac{c_4}{1} \underset{+}{} \cdots = \operatorname*{K}_{n=1}^{\infty} \frac{c_n}{1} \tag{2}$$

by using the procedure described in [2], Problems 23 and 24, page 51. The first c-values are

$$c_1 = \frac{8}{15}, c_2 = \frac{-9}{23}, c_3 = \frac{-980}{4899}, c_4 = \frac{-20449}{76325}, c_5 = \frac{-104329}{463540}, c_6 = \frac{-808201}{3102176},$$

$$c_7 = \frac{-77704225}{323484942}, c_8 = \frac{-77716806}{290639701}, c_9 = \frac{-2283396565}{10116899023}, c_{10} = \frac{-23376040344}{82096392175}.$$

For any n the nth approximant P_n of the series (1) is equal to the nth approximant S_n of the continued fraction. Examples are

$$P_3 = \left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{11} + \frac{1}{13}\right) = \frac{15676}{15015} = 1.044022644\ldots,$$

$$S_3 = \frac{c_1}{1} \underset{+}{} \frac{c_2}{1} \underset{+}{} \frac{c_3}{1} = \frac{15676}{15015} = 1.044022644\ldots,$$

$$P_7 = \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\cdots+\left(\frac{1}{59}+\frac{1}{61}\right) = \frac{360524837502712224}{276644289263692350} = 1.3032072285\ldots,$$

$$S_7 = \frac{c_1}{1} \underset{+}{} \frac{c_2}{1} \underset{+}{} \frac{c_3}{1} \underset{+}{} \frac{c_4}{1} \underset{+}{} \frac{c_5}{1} \underset{+}{} \frac{c_6}{1} \underset{+}{} \frac{c_7}{1} = \frac{360524837502712224}{276644289263692350} = 1.3032072285\ldots.$$

But what is the point in transforming the series to a continued fraction when we get exactly the same sequence of approximants? The answer is to be found in the concepts of *modification* and of *tails*. Rather than studying

$$\frac{c_1}{1} \underset{+}{} \frac{c_2}{1} \underset{+}{} \frac{c_3}{1} \underset{+}{} \frac{c_4}{1} \underset{+}{} \cdots \underset{+}{} \frac{c_n}{1} \tag{3}$$

we study

$$\frac{c_1}{1} + \frac{c_2}{1} + \frac{c_3}{1} + \frac{c_4}{1} + \cdots + \frac{c_n}{(1+x)}. \tag{3'}$$

Here (3) is the nth approximant of the continued fraction for Brun's constant, whereas (3') is a *modified approximant.* If x is chosen to be the value of the tail

$$\frac{c_{n+1}}{1} + \frac{c_{n+2}}{1} + \cdots,$$

then (3') is the continued fraction for Brun's constant.

3.Observation

We choose in (3') n=3, and find

$$y := \frac{c_1}{1} + \frac{c_2}{1} + \frac{c_3}{(1+x)} = \frac{4(3919+4899x)}{105(143+213x)}. \tag{4}$$

The graph of this is a hyperbola in the xy-plane, with horizontal asymptote

$$y = y_0 = \frac{4 \cdot 4899}{105 \cdot 213} = \frac{92}{105} = 0.87619047619\ldots$$

and vertical asymptote

$$x = x_0 = \frac{-143}{213} = -0.6713615023\ldots.$$

One branch (the upper branch) of the hyperbola is located in the region $x > x_0, y > y_0$.

Since $B_2 > 1.9$ our interest in the hyperbola is restricted to the part of the upper branch where $y > 1.9$. On this part of the upper branch there is a unique point, where the y-value is B_2, and the corresponding x-value is equal to the value of the tail whose first term is $c_4/1$.

We have $y > 1.9$ iff

$$\frac{4(3919+4899x)}{105(143+213x)} > 1.9,$$

from which follows $x < -0.5613058194\ldots$. Since moreover we have to be to the right of the asymptote we have $x > x_0 = -0.6713615023\ldots$ and hence

$$-0.6713615023\ldots < x < -0.5613058194\ldots.$$

We state this as a triviem, a *twin prime tail triviem* :

TPT Triviem

For the series (1) and it's continued fraction (2) the following holds:

$$-0.6713615023\ldots < \mathop{\mathrm{K}}_{k=4}^{\infty} \frac{c_k}{1} < -0.5613058194\ldots . \qquad (5)$$

Observe that the length l of the interval in the triviem is

$$l = 0.11\ldots$$

4.Some general remarks

If A_n and B_n are the numerator and denominator of the nth approximant,

$$S_n(0) = \frac{A_n}{B_n},$$

determined by using the standard recurrence relations from e.g. [2], Lemma 1.1, page 6, we have

$$S_n(x) = \frac{c_1}{1} + \frac{c_2}{1} + \frac{c_3}{1} + \frac{c_4}{1} + \cdots + \frac{c_n}{(1+x)} = \frac{A_n + xA_{n-1}}{B_n + xB_{n-1}} \quad . \qquad (6)$$

(B_2 in $S_2(x)$ or $S_3(x)$ not to be confused with Brun's constant B_2.) The graph of (6) is a hyperbola. Horizontal axis is $y = y_0 = \frac{A_{n-1}}{B_{n-1}} = S_{n-1}(0)$, vertical axis is $x = x_0 = -\frac{B_n}{B_{n-1}}$.

Then the value of the continued fraction

$$\mathop{\mathrm{K}}_{k=n+1}^{\infty} \frac{c_k}{1}$$

is located in the interval between $S_n^{-1}(\infty)$ and $S_n^{-1}(1.9)$, i.e.

$$S_n^{-1}(\infty) < \mathop{\mathrm{K}}_{k=n+1}^{\infty} \frac{c_k}{1} < S_n^{-1}(1.9), \quad l = S_n^{-1}(1.9) - S_n^{-1}(\infty), \qquad (7)$$

where l is the length of the interval. More explicit formulas are

$$S_n^{-1}(\infty) = -\frac{B_n}{B_{n-1}}, \quad S_n^{-1}(1.9) = \frac{1.9B_n - A_n}{A_{n-1} - 1.9B_{n-1}}.$$

and, by simple computation

$$l_n = \frac{B_n}{B_{n-1}} \left(\frac{\frac{A_{n-1}}{B_{n-1}} - \frac{A_n}{B_n}}{\frac{A_{n-1}}{B_{n-1}} - 1.9} \right) \quad . \tag{8}$$

For the first n-values we find:

$n = 1$:

$$-1 < \mathop{\mathrm{K}}_{k=2}^{\infty} \frac{c_k}{1} < -0.719\ldots, \quad l = 0.28\ldots$$

$n = 2$:

$$-0.60869\ldots < \mathop{\mathrm{K}}_{k=3}^{\infty} \frac{c_k}{1} < -0.45599\ldots, \quad l = 0.15\ldots$$

$n = 3$

$$-0.6713615023\ldots < \mathop{\mathrm{K}}_{k=4}^{\infty} \frac{c_k}{1} < -0.5613058194\ldots, \quad l = l_3 = 0.11\ldots$$

$n = 4$

$$-0.60093\ldots < \mathop{\mathrm{K}}_{k=5}^{\infty} \frac{c_k}{1} < -0.52268\ldots, \quad l = l_4 = 0.078\ldots$$

$n = 5$

$$-0.62546\ldots < \mathop{\mathrm{K}}_{k=6}^{\infty} \frac{c_k}{1} < -0.56939\ldots, \quad l = l_5 = 0.056\ldots$$

$n = 6$

$$-0.58346\ldots < \mathop{\mathrm{K}}_{k=7}^{\infty} \frac{c_k}{1} < -0.54244(9)\ldots, \quad l = l_6 = 0.041\ldots$$

$n = 10$

$$-0.5142883082 < \mathop{\mathrm{K}}_{k=11}^{\infty} \frac{c_k}{1} < -0.4969518033, \quad l = l_{10} = 0.0173\ldots$$

5. Looking ahead

In the last case, $n = 10$ we find

$$\mathop{\mathrm{K}}_{k=11}^{\infty} \frac{c_k}{1} \approx -0.50562,$$

with an error bound 0.0087. Related results hold in the other cases. So, whereas the continued fraction (2), representing Brun's constant B_2, can not be approximated in a way related to the ones above (unless we *believe* in some heuristicly produced bounds), all tails (except for (2) itself) can be squeezed in between bounds in a proper mathematical way. The length l_n of the interval around the nth tail cannot be chosen in advance. We have to "take what we get".

Asymptotic properties of

$$\mathop{\mathrm{K}}_{k=n+1}^{\infty}(c_k/1)$$

and of l_n are of interest. For l_n we find from (8), since $A_n/B_n \to B_2$ when $n \to \infty$, that

$$l_n = \frac{B_n}{B_{n-1}} \cdot \epsilon_n \quad , \tag{9}$$

where $\epsilon_n \to 0$ when $n \to \infty$. If the sequence $\{B_n/B_{n-1}\}$ is bounded, then $l_n \to 0$ when $n \to \infty$. But so far this is an open question (temporarily?), and may perhaps be discussed later, together with asymptotic properties of the tails. (The latter may be difficult, but you may get some guesses for free.)

References

1. [*http : //en.wikipedia.org/wiki/Brun's − constant*].

2. Lisa Lorentzen and Haakon Waadeland, *Continued Fractions, Volume 1: Convergence theory,* Atlantis Press and World Scientific, Amsterdam, Paris 2008.