Powers of the fixed points of the Gauss map

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Abstract

This paper shows that any odd power of any fixed point of the Gauss map is also a fixed point of th Gauss map, and gives its specific continued fraction form.
This paper also shows that any even power of any fixed point of the Gauss map is an inverse image of a specific 2-cycle.

Key words 1. The Gauss map 2. Continued fraction 3.fixed point
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1. Preliminaries

The main purpose of this section is to introduce basic definitions and major properties of continued fractions and the Gauss map.

1.1 Continued Fractions

Every real number γ has a unique continued fraction representation of the form $\gamma = [a_0; a_1, a_2, \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$, where $i, a_i \in \mathbb{N}$.

If γ is a rational number, then it is represented as a finite continued fraction of the form $\gamma = [a_0; a_1, a_2, \cdots, a_k]$, where $a_1, a_2, \cdots, a_k \in \mathbb{N}$.

If $\gamma = [0; a_1, a_2, \cdots, a_k, a_1, a_2, \cdots, a_k, \cdots]$, then it is called a $k-cycle$, or a periodic continued fraction of period k, and we denote it by $[0; \overline{a_1, a_2, \cdots, a_k}\ ]$.

Lagrange's theorem : γ is a periodic continued fraction if and only if γ is an irrational quadratic .

The convergents of γ are denoted by $\{C_n\}_{n=1}^{\infty}$, where $C_n = \frac{A_n}{B_n}$, $A_0 = a_0$, $A_1 = a_0 a_1 + 1$, $A_n = a_n A_{n-1} + A_{n-2}$, for every $n \geq 2$, and $B_0 = 1$, $B_1 = a_1$, $B_n = a_n B_{n-1} + B_{n-2}$, for every $n \geq 2$.

For more details about continued fractions and their main properties, the reader can refer to [3], [4] and [5]

1.2 Gauss map

The Gauss map $G : [0, 1] \to [0, 1]$ is defined as follows :

$$G(x) = \begin{cases} 0 & if \quad x = 0 \ . \\ \\ \frac{1}{x} \ mod \ 1 & if \quad x \in (0, 1] \ . \end{cases}$$

If $\gamma = [0; a_1, a_2, \cdots]$ is any continued fraction in the interval $(0, 1]$, where $i, a_i \in \mathbb{N}$, then $G([0; a_1, a_2, \cdots]) = [0; a_2, a_3, \cdots]$, which means that the effect of the Gauss map on every continued fraction is shifting its entries to the right. Hence the fixed points of the Gauss map are the 1-cycles, or continued fractions of the form $\gamma = [0; \overline{n}] = \dfrac{-n + \sqrt{n^2 + 4}}{2}$, for every $n \in \mathbb{N}$.

For more details about the Gauss map, the reader can refer to [1] and [2].

2. Introduction

We know that the fixed points of the Gauss map are the $1 - cycles$, or continued fractions of the form $[0; \overline{n}] = \dfrac{-n + \sqrt{n^2 + 4}}{2}$, for every $n \in \mathbb{N}$.

Are the powers of any fixed point of the Gauss map also fixed points of the Gauss map?.
Let us calculate some of the powers of the fixed point $\gamma = \left[0; \overline{1}\right]$. $\gamma^3 = \left[0; \overline{4}\ \right]$, $\gamma^4 = \left[0; 6,\ \overline{1, 5}\ \right]$, $\gamma^5 = \left[0; \overline{11}\ \right]$, $\gamma^6 = \left[0; 17,\ \overline{1, 16}\ \right]$, ...
We notice that γ^3 and γ^5 are fixed points of the Gauss map, but γ^4 and γ^6 are not fixed points of the Gauss map.

Are the the odd powers of any fixed point of the Gauss map also fixed points of the Gauss map?.
In section 3, we prove that the odd powers of any fixed point of the Gauss map are also fixed points. Also, we give the continued fraction form of any odd power of any fixed point of the Gauss map in terms of the B_n of the fixed point itself.

What is the continued fraction form of any even power of any fixed point of the Gauss map?.
In section 4, we prove that any even power of any fixed point of the Gauss map is an inverse image of a specific $2 - cycle$, and we give its specific continued fraction form in terms of the B_n of the fixed point itself.

Generally, if we know the B_n of any fixed point of the Gauss map, then we can know the continued fraction form of all the powers of the fixed point.

3. Odd powers of the fixed points of the Gauss map

In this section, we show that any odd power of any fixed point of the Gauss map is also a fixed point.

More precisely, we show in propsition (3.5), that if $\gamma = [0; \overline{n}]$, where $n \in \mathbb{N}$, is a fixed point of the Gauss map, then $\gamma^{2k+1} = \left[0; \overline{B_{2k-1} + B_{2k+1}}\ \right]$, for every $k \in \mathbb{N}$.

But first, we present four easy lemmas which are helpful for the proof of propsition (3.5).

Lemma 3.1 : If $\gamma = [0; \overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $B_{2k+1} = n\,[B_{2k} + B_{2k-2} + \cdots + B_2 + B_0]$, for every $k \in \mathbb{N}$.
Proof : By induction

- The case $k = 1$, we want to show that $B_3 = n\,[B_2 + B_0]$.

$$\begin{aligned} B_3 &= n\ B_2 + B_1 \\ &= n\ B_2 + n\ B_0 + B_{-1} \\ &= n\ B_2 + n\ B_0 + 0 \\ &= n\,[B_2 + B_0]\ . \end{aligned}$$

- Suppose that the statement is true for $k = m$.
  That is, $B_{2m+1} = n\,[B_{2m} + B_{2m-2} + \cdots + B_2 + B_0]$.
- Now, we prove that the statement is true for the case $k = m + 1$.
  We want to show that $B_{2m+3} = n\,[B_{2m+2} + B_{2m} + \cdots + B_2 + B_0]$.

$$\begin{aligned} B_{2m+3} &= n\ B_{2m+2} + B_{2m+1} \\ &= n\ B_{2m+2} + n\,[B_{2m} + B_{2m-2} + \cdots + B_2 + B_0] \\ &\quad (\ \textit{by induction step}\ ) \\ &= n\,[B_{2m+2} + B_{2m} + B_{2m-2} + \cdots + B_2 + B_0]\ . \end{aligned}$$

Lemma 3.2 : If $\gamma = [0; \overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $(B_{2k+1}\ B_{2k-1}) + 1 = B_{2k}^2$, for every $k \in \mathbb{N}$.
Proof : By induction

- The case $k = 1$, note that $B_1 = n$, $B_2 = n^2 + 1$ and $B_3 = n^3 + 2n$.
  We want to show that $(B_3\ B_1) + 1 = B_2^2$.

$$\begin{aligned} (B_3\ B_1) + 1 &= \big(n\left(n^3 + 2n\right)\big) + 1 \\ &= n^4 + 2n^2 + 1 \\ &= \left(n^2 + 1\right)^2 \\ &= B_2^2\ . \end{aligned}$$

- Suppose that the statement is true for $k = m$.
  That is, $(B_{2m+1}\ B_{2m-1}) + 1 = B_{2m}^2$.

- Now, we prove that the statement is true for the case $k = m + 1$.
  We want to show that $(B_{2m+3}\ B_{2m+1}) + 1 = B_{2m+2}^2$.

$$\begin{aligned}
(B_{2m+3}\ B_{2m+1}) + 1 &= (n\,[B_{2m+2} + B_{2m} + B_{2m-2} + \cdots + B_2 + B_0]\,B_{2m+1}) + 1 \\
&\quad (\ \textit{by lemma}\ 3.1\ ) \\
&= n\ B_{2m+2}\ B_{2m+1} + n\ B_{2m+1}\ B_{2m} \\
&+ B_{2m+1}\ n\ [B_{2m-2} + \cdots + B_2 + B_0] + 1 \\
&= n\ B_{2m+1}\,(n\ B_{2m+1} + B_{2m}) + n\ B_{2m+1}\ B_{2m} \\
&+ (B_{2m+1}\ B_{2m-1}) + 1 \\
&\quad (\ \textit{by definition of}\ B_{2m+2}\ ,\ \textit{and lemma}\ 3.1\ ) \\
&= n^2 B_{2m+1}^2 + 2n\ B_{2m+1}\ B_{2m} + B_{2m}^2 \\
&\quad (\ \textit{by induction step}\ ) \\
&= (n\ B_{2m+1} + B_{2m})^2\ =\ B_{2m+2}^2\ .
\end{aligned}$$

Lemma 3.3 : If $\gamma = [0; \overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $\gamma^{2k+1} = \gamma\ B_{2k} - B_{2k-1}$, for every $k \in \mathbb{N}$.
Proof : By induction

- The case $k = 1$, note that $B_1 = n$ and $B_2 = n^2 + 1$.
  We want to show that $\gamma^3 = \gamma\ B_2 - B_1$.

$$\begin{aligned}
\gamma &= \frac{1}{\gamma} - n \\
\gamma^2 &= 1 - n\gamma \\
\gamma^3 &= \gamma - n\ \gamma^2 \\
&= \gamma - n\ \ (1 - n\ \gamma) \\
&= \gamma - n + n^2\gamma \\
&= \left(n^2 + 1\right)\gamma - n \\
&= \gamma\ B_2 - B_1\ .
\end{aligned}$$

- Suppose that the statement is true for $k = m$.
  That is, $\gamma^{2m+1} = \gamma\ B_{2m} - B_{2m-1}$.

- Now, we prove that the statement is true for the case $k = m + 1$.

We want to show that $\gamma^{2m+3} = \gamma\ B_{2m+2} - B_{2m+1}$.

$$\begin{aligned}
\gamma^{2m+3} &= \gamma^2\ \gamma^{2m+1} \\
&= (1 - n\ \gamma)\ \ (\gamma\ B_{2m} - B_{2m-1}) \\
&\quad (\ by\ induction\ step\ ) \\
&= \gamma\ B_{2m} - B_{2m-1} - n\gamma^2\ B_{2m} + n\gamma\ B_{2m-1} \\
&= \gamma\ \ (B_{2m} + n\ B_{2m-1}) - n\ B_{2m}\ \ (1 - n\ \gamma) - B_{2m-1} \\
&\quad (\ since\ \gamma^2\ =\ 1 - n\gamma\ ) \\
&= \left(n^2 B_{2m} + n\ B_{2m-1} + B_{2m}\right)\ \gamma - (n\ B_{2m} + B_{2m-1}) \\
&= (n\ \ (n\ B_{2m} + B_{2m-1}) + B_{2m})\ \ \gamma - B_{2m+1} \\
&\quad (\ by\ definition\ of\ B_{2m+1}\ ) \\
&= (n\ B_{2m+1} + B_{2m})\ \ \gamma - B_{2m+1} \\
&= \gamma\ B_{2m+2} - B_{2m+1} \\
&\quad (\ by\ definition\ of\ B_{2m+2}\ )\ .
\end{aligned}$$

Lemma 3.4 : If $\gamma = [0; \overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $B_{2k}^2\ \left(n^2 + 4\right) = \left(B_{2k+1} + B_{2k-1}\right)^2 + 4$, for every $k \in \mathbb{N}$.
Proof : For every $k \in \mathbb{N}$,

$$\begin{aligned}
\left(B_{2k+1} + B_{2k-1}\right)^2 + 4 &= [n\ \ (B_{2k} + 2\,(B_{2k-2} + \cdots + B_2 + B_0))]^2 + 4 \\
&\quad (\ by\ lemma\ 3.1\ ) \\
&= n^2\ [B_{2k}^2 + 4B_{2k}\,(B_{2k-2} + \cdots + B_2 + B_0) \\
&+ 4\,(B_{2k-2} + \cdots + B_2 + B_0)^2] + 4 \\
&= n^2\ B_{2k}^2 + 4n^2 B_{2k}\ \ (B_{2k-2} + \cdots + B_2 + B_0) \\
&+ 4n^2\ \ (B_{2k-2} + \cdots + B_2 + B_0)^2 + 4 \\
&= n^2\ B_{2k}^2 + 4n^2\ \ (B_{2k-2} + \cdots + B_2 + B_0) \\
&\quad [B_{2k} + (B_{2k-2} + \cdots + B_2 + B_0)] + 4 \\
&= n^2\ B_{2k}^2 + 4\ B_{2k+1}\ B_{2k-1} + 4 \\
&\quad (\ by\ lemma\ 3.1\ ) \\
&= n^2 B_{2k}^2 + 4\,[(B_{2k+1}\ B_{2k-1}) + 1] \\
&= n^2 B_{2k}^2 + 4B_{2k}^2 \\
&\quad (\ by\ lemma\ 3.2\ ) \\
&= B_{2k}^2\ \left(n^2 + 4\right)\ .
\end{aligned}$$

Now, we show that any odd power of $\gamma = [0; \overline{n}]$, where $n \in \mathbb{N}$ is a fixed point of the Gauss map, also we give the continued fraction form of any odd power of $\gamma = [0; \overline{n}]$ in terms of its B_n .

Proposition 3.5 : If $\gamma = [0; \overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $\gamma^{2k+1} = \left[0; \overline{B_{2k-1} + B_{2k+1}}\ \right]$, for every $k \in \mathbb{N}$.
Proof : For every $k \in \mathbb{N}$,

$$\begin{aligned}
\gamma^{2k+1} &= \gamma\ B_{2k} - B_{2k-1} \\
&\quad (\ by\ lemma\ 3.3\ ) \\
&= B_{2k}\ \left(\frac{-n + \sqrt{n^2+4}}{2}\right) - B_{2k-1} \\
&\quad \left(\ since\ \gamma = [0; \overline{n}] = \frac{-n + \sqrt{n^2+4}}{2}\ \right) \\
&= \frac{-\left(nB_{2k} + 2B_{2k-1}\right) + B_{2k}\sqrt{n^2+4}}{2} \\
&= \frac{-\left[\left(nB_{2k} + B_{2k-1}\right) + B_{2k-1}\right] + \sqrt{B_{2k}^2\left(n^2+4\right)}}{2} \\
&= \frac{-\left(B_{2k+1} + B_{2k-1}\right) + \sqrt{\left(B_{2k+1} + B_{2k-1}\right)^2 + 4}}{2} \\
&\quad (\ by\ definition\ of\ B_{2k+1}\ ,\ and\ lemma\ 3.4\ ) \\
&= \left[0; \overline{B_{2k+1} + B_{2k-1}}\ \right]\ .
\end{aligned}$$

Example

We used Mathematica to calculate few of the B_n of $\alpha = \left[0; \overline{3}\ \right]$, shown in the third and the fourth columns. Also, we calculated few of the odd powers of $\alpha = \left[0; \overline{3}\ \right]$, shown in the second column in a continued fraction form.
We notice that the entries of the continued fractions in the second column are the same as the numbers in the fifth column, as proved in proposition (3.5) .

k	α^{2k+1}	B_{2k+1}	B_{2k-1}	$B_{2k+1} + B_{2k-1}$
2	$\alpha^5 = \left[0; \overline{393}\ \right]$	360	33	393
3	$\alpha^7 = \left[0; \overline{4287}\ \right]$	3927	360	4287
10	$\alpha^{21} = \left[0; \overline{78788080764}\ \right]$	72171863277	6616217487	78788080764

4. Even powers of the fixed points of the Gauss map

In this section, we show that any even power of any fixed point of the Gauss map is an inverse image of a specific 2-cycle.

More precisely, we show in proposition (4.5), that if $\gamma = [0; \overline{n}\ ]$, where $n \in \mathbb{N}$, is any fixed point of the Gauss map, then for every $k \in \mathbb{N}$, we have $\gamma^{2k} = \Big[0; (B_{2k} + B_{2k-2} - 1)\,, \overline{1, (B_{2k} + B_{2k-2} - 2)}\Big]$.

But first, we present three easy lemmas which are helpful for the proof of proposition (4.5) .

Lemma 4.1 : If $\gamma = [0; \overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $B_{2k} = n\,[B_{2k-1} + B_{2k-3} + \cdots + B_1] + 1$, for every $k \in \mathbb{N}$.
Proof : Similar to the proof of Lemma (3.1) .

Lemma 4.2 : If $\gamma = [0; \overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $(B_{2k}\ B_{2k-2}) - 1 = B_{2k-1}^2$, for every $k \in \mathbb{N}$.
Proof : By induction

- The case $k = 1$, note that $B_0 = 1$, $B_1 = n$ and $B_2 = n^2 + 1$.
  We want to show that $(B_2\ B_0) - 1 = B_1^2$.

$$\begin{aligned} (B_2\ B_0) - 1 &= \big(1\ (n^2+1)\big) - 1 \\ &= n^2 \\ &= B_1^2\ . \end{aligned}$$

- Suppose that the statement is true for $k = m$.
  That is, $(B_{2m}\ B_{2m-2}) - 1 = B_{2m-1}^2$.
- Now, we prove that the statement is true for the case $k = m + 1$.
  We want to show that $(B_{2m+2}\ B_{2m}) - 1 = B_{2m+1}^2$.

$$\begin{aligned} (B_{2m+2}\ B_{2m}) - 1 &= B_{2m}\,(n\,[B_{2m+1} + B_{2m-1} + B_{2m-3} + \cdots + B_1] + 1) - 1 \\ &\quad (\ \textit{by lemma } 4.1\ ) \\ &= n\ B_{2m+1}\ B_{2m} + n\ B_{2m-1}\ B_{2m} \\ &+ B_{2m}\ n\ [(B_{2m-3} + \cdots + B_1) + 1] - 1 \\ &= n\ B_{2m}\,(n\ B_{2m} + B_{2m-1}) + n\ B_{2m-1}\ B_{2m} \\ &+ (B_{2m}\ B_{2m-2}) - 1 \\ &\quad (\ \textit{by definition of } B_{2m+1}\ ,\ \textit{and lemma } 4.1\ ) \\ &= n^2 B_{2m}^2 + 2n\ B_{2m-1}\ B_{2m} + B_{2m-1}^2 \\ &\quad (\ \textit{by induction step}\ ) \\ &= (n\ B_{2m} + B_{2m-1})^2\ =\ B_{2m+1}^2 \\ &\quad (\ \textit{by definition of } B_{2m+1}\ )\ . \end{aligned}$$

Lemma 4.3 : If $\gamma = [0;\overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $B_{2k-1}^2 \left(n^2+4\right) = \left(B_{2k}+B_{2k-2}\right)^2 - 4$, for every $k \in \mathbb{N}$.
Proof : For every $k \in \mathbb{N}$,

$$\begin{aligned}
\left(B_{2k}+B_{2k-2}\right)^2 - 4 &= \left(n\ B_{2k-1}+2B_{2k-2}\right)^2 - 4 \\
&\quad (\ by\ definition\ of\ B_{2k}\ ) \\
&= n^2 B_{2k-1}^2 + 4n\ B_{2k-1}B_{2k-2} + 4B_{2k-2}^2 - 4 \\
&= n^2 B_{2k-1}^2 + 4\left[n\ B_{2k-1}B_{2k-2} + B_{2k-2}^2 - 1\right] \\
&= n^2 B_{2k-1}^2 + 4\left[B_{2k-2}\left(n\ B_{2k-1}+B_{2k-2}\right) - 1\right] \\
&= n^2 B_{2k-1}^2 + 4\left[\left(B_{2k}B_{2k-2}\right) - 1\right] \\
&\quad (\ by\ definition\ of\ B_{2k}\ ) \\
&= n^2 B_{2k-1}^2 + 4B_{2k-1}^2 \\
&\quad (\ by\ lemma\ 4.2\ ) \\
&= B_{2k-1}^2\ \left(n^2+4\right)\ .
\end{aligned}$$

Corollary 4.4 : If $\gamma = [0;\overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $B_{2k-1}^2 \left(n^2+4\right) = \left(B_{2k}+B_{2k-2}-2\right)^2 + 4\ \left(B_{2k}+B_{2k-2}-2\right)$, for every $k \in \mathbb{N}$.
Proof : For every $k \in \mathbb{N}$,

$$\begin{aligned}
\left(B_{2k}+B_{2k-2}-2\right)^2 + 4\left(B_{2k}+B_{2k-2}-2\right) &= B_{2k}^2 + 2B_{2k}B_{2k-2} + B_{2k-2}^2 - 4B_{2k} \\
&\quad -4B_{2k-2} + 4 + 4B_{2k} + 4B_{2k-2} - 8 \\
&= B_{2k}^2 + 2B_{2k}B_{2k-2} + B_{2k-2}^2 - 4 \\
&= \left(B_{2k}+B_{2k-2}\right)^2 - 4 \\
&= B_{2k-1}^2\ \left(n^2+4\right) \\
&\quad (\ by\ lemma\ 4.3\ )\ .
\end{aligned}$$

Proposition 4.5 : If $\gamma = [0;\overline{n}]$ is any fixed point of the Gauss map, where $n \in \mathbb{N}$, then $\gamma^{2k} = \left[0;\left(B_{2k}+B_{2k-2}-1\right),\overline{1,\left(B_{2k}+B_{2k-2}-2\right)}\right]$, for every $k \in \mathbb{N}$.
Proof :
First we use induction to prove that $\gamma^{2k} = \dfrac{\left(B_{2k}+B_{2k-2}\right) - B_{2k-1}\sqrt{n^2+4}}{2}$, for every $k \in \mathbb{N}$.

- The case $k = 1$, note that $B_0 = 1$, $B_1 = n$ and $B_2 = n^2 + 1$.

$$\begin{aligned}
\gamma^2 &= \left(\frac{-n+\sqrt{n^2+4}}{2}\right)^2 = \frac{2n^2+4-2n\sqrt{n^2+4}}{4} \\
&= \frac{n^2+2-n\sqrt{n^2+4}}{2} = \frac{(B_2+B_0)-B_1\sqrt{n^2+4}}{2} .
\end{aligned}$$

- Suppose that the statement is true for $k = m$.
  That is, $\gamma^{2m} = \dfrac{(B_{2m}+B_{2m-2})-B_{2m-1}\sqrt{n^2+4}}{2}$.
- Now, we prove that the statement is true for the case $k = m + 1$.
  We want to show that $\gamma^{2m+2} = \dfrac{(B_{2m+2}+B_{2m})-B_{2m+1}\sqrt{n^2+4}}{2}$.

$$\begin{aligned}
\gamma^2\ \gamma^{2m} &= \left(\frac{n^2+2-n\sqrt{n^2+4}}{2}\right)\left(\frac{B_{2m}+B_{2m-2}-B_{2m-1}\sqrt{n^2+4}}{2}\right) \\
&= \frac{\left(n^2+2\right)\left(B_{2m}+B_{2m-2}\right)+nB_{2m-1}\left(n^2+4\right)}{4} \\
&- \frac{\left[n\left(B_{2m}+B_{2m-2}\right)+\left(n^2+4\right)B_{2m-1}\right]\sqrt{n^2+4}}{4} .
\end{aligned}$$

Now, note that $\left(n^2+2\right)\left(B_{2m}+B_{2m-2}\right)+nB_{2m-1}\left(n^2+4\right)$.

$$\begin{aligned}
&= n^2B_{2m}+n^2B_{2m-2}+2B_{2m}+2B_{2m-2}+n^3B_{2m-1}+4nB_{2m-1} \\
&= n^2B_{2m}+n^2\left(nB_{2m-1}+B_{2m-2}\right)+2B_m+4nB_{2m-1}+2B_{2m-2} \\
&= 2\left(n^2B_{2m}+2nB_{2m-1}+B_{2m}+B_{2m-2}\right) \\
&\quad (\ by\ definition\ of\ B_{2m}\ ) \\
&= 2\left[n\left(nB_{2m}+B_{2m-1}\right)+nB_{2m-1}+B_{2m}+B_{2m-2}\right] \\
&= 2\left(nB_{2m+1}+2B_{2m}\right) \\
&\quad (\ by\ definitions\ of\ B_{2m}\ and\ B_{2m+2}\ ) \\
&= 2\left(nB_{2m+1}+B_{2m}+B_{2m}\right) \\
&= 2\left(B_{2m+2}+B_{2m}\right) \\
&\quad (\ by\ definition\ of\ B_{2m+2}\ ) .
\end{aligned}$$

Also, note that $n\left(B_{2m}+B_{2m-2}\right)+\left(n^2+4\right)B_{2m-1}$

$$\begin{aligned}
&= nB_{2m}+nB_{2m-2}+n^2B_{2m-1}+2B_{2m-1}\\
&= nB_{2m}+n\left(nB_{2m-1}+B_{2m-2}\right)+2B_{2m-1}\\
&= 2nB_{2m}+2B_{2m-1}\\
&\quad (\ by\ definition\ of\ B_{2m}\ )\\
&= 2\left(nB_{2m}+B_{2m-1}\right)\ =\ 2B_{2m+1}\\
&\quad (\ by\ definition\ of\ B_{2m+1}\ )\ .
\end{aligned}$$

Therefore, $\gamma^{2m+2}=\dfrac{\left(B_{2m+2}+B_{2m}\right)-B_{2m+1}\sqrt{n^2+4}}{2}$.

Finally, for every $k\in\mathbb{N}$, we have γ^{2k}

$$= \frac{\left(B_{2k}+B_{2k-2}\right)-\sqrt{B_{2k-1}^2\left(n^2+4\right)}}{2}$$

$$= \frac{\left[\left(B_{2k}+B_{2k-2}-2\right)+2\right]-\sqrt{\left(B_{2k}+B_{2k-2}-2\right)^2+4\left(B_{2k}+B_{2k-2}-2\right)}}{2}$$

$(\ by\ Proposition\ (4.4)\ )$

$$= \left[0;\left(B_{2k}+B_{2k-2}-1\right),\overline{1,\left(B_{2k}+B_{2k-2}-2\right)}\right]$$

$$\left(\ since\ \ \frac{(s+2)-\sqrt{s^2+4s}}{2}\ =\ \left[0;(s+1)\ ,\ \overline{1,s}\ \right]\ \right)\ .$$

Example
We used Mathematica to calculate few of the B_n of $\alpha=\left[0;\overline{2}\ \right]$, shown in the third and the fourth columns. Also, we calculated few of the even powers of $\alpha=\left[0;\overline{2}\ \right]$, shown in the second column in a continued fraction form.
We notice the relation between the entries of the continued fractions in the second column and the numbers in the fifth column, as proved in proposition (4.5) .

k	α^{2k}	B_{2k}	B_{2k-2}	$B_{2k}+B_{2k-2}-2$
2	$\alpha^4=\left[0;33,\ \overline{1,32}\ \right]$	29	5	32
3	$\alpha^6=\left[0;197,\ \overline{1,196}\ \right]$	169	29	196
10	$\alpha^{20}=\left[0;45239073,\ \overline{1,45239072}\ \right]$	38613965	625109	45239072

Conclusions

Let $\gamma = [0; \overline{n}\ ]$ be any fixed point of the Gauss map, where $n \in \mathbb{N}$, then :

1. $\gamma^{2k+1} = \left[0; \overline{B_{2k-1} + B_{2k+1}}\right]$, for every $k \in \mathbb{N}$.

   Any odd power of any fixed point of the Gauss map is also a fixed point.

2. $\gamma^{2k} = \left[0; (B_{2k} + B_{2k-2} - 1)\,, \overline{1, (B_{2k} + B_{2k-2} - 2)}\right]$, for every $k \in \mathbb{N}$.

   Any even power of any fixed point of the Gauss map is an inverse image of a specific 2-cycle.

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References

[1] Barrow ,J.D. (2000) *"Chaos in number land: The secret life of continued farctions"* , Plus Magazine , Isuue 11 , Millennium Mathematics Project , United Kingdom. http://plus.maths.org/issue11/features/cfractions/index.html

[2] Corless, R.M. (1992) *"Continued Fractions and Chaos"* , American Mathematical Monthly , No. 3, pp. 203-215.

[3] Khinchin, A. Y. (1964) *"Continued Fractions"* , The University of Chicago Press.

[4] Olds, C.D. (1963) *"Continued Fractions"* , The Mathematical Association of America.

[5] Rockett, A. M. and Szusz, P. (1998) *"Continued Fractions"* , World Scientific Publishing Co. Pte. Ltd. (New Jersey).