Generalized Tchebychev Polynomials and Fibonacci numbers

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Abstract

First, we prove that normalized polynomials $\{\hat{P}_n\}_{n\geq 0}$ orthogonal on $[-1,1]$ with respect to the weight function $w_a(x) = \dfrac{a+1-x^2}{\sqrt{1-x^2}}$ fulfill $\hat{P}_{n+1}(x) = x\hat{P}_n(x) - \gamma_n(a)\hat{P}_{n-1}(x),\ n \geq 0,\ \hat{P}_{-1} = 0,\ \hat{P}_0 = 1$ with

$$\gamma_1(a) = \frac{\hat{T}_3(\sqrt{a+1})}{2\hat{T}_1(\sqrt{a+1})\hat{T}_2(\sqrt{a+1})},\ \gamma_n(a) = \frac{\hat{T}_{n-1}(\sqrt{a+1})\hat{T}_{n+2}(\sqrt{a+1})}{4\hat{T}_n(\sqrt{a+1})\hat{T}_{n+1}(\sqrt{a+1})},\ n \geq 1,$$

where $\hat{T}_n(x)$ computes the nth normalized Tchebychev polynomial of the first kind. Second, we prove that the sequence $\dfrac{1}{4(\gamma_n(\frac{1}{4}) - \frac{1}{4})}$ (respectively $\dfrac{3^{n-1}}{2(\gamma_n(\frac{1}{3}) - \frac{1}{4})}$, $\dfrac{1}{4(\gamma_n(\frac{1}{2}) - \frac{1}{4})}$, $\dfrac{1}{2(\gamma_n(1) - \frac{1}{4})}$ and $\dfrac{1}{(\gamma_n(2) - \frac{1}{4})}$) coincides with the sequence $A093467$ (Fibonacci sequence $F(2n-1)+1$) (respectively $A051406$, $A101907$, $A011900$ and $A054318$) found in the On-Line Encyclopedia of Integer Sequences by SLOANE N. J. A.

Subject classification. 33C45, 05E35.

Keywords and Phrases: orthogonal polynomials, Christoffel's formula, Sloane's on-line encyclopedia of integer sequences.

1 Introduction and basic notations

1.1 Introduction

Semi-classical orthogonal polynomials of class $s = 1, 2$ are studied by some authors, but from the non linearity of the system fulfilled by the recurrence coefficients, called Laguerre-Freud equations, only few recurrence coefficients are explicitly known [1, 2, 3, 4, 5, 6, 7, 8, 9].

The idea considered in this work gives the recurrence coefficients of the monic polynomials $\hat{P}_n$ orthogonal on $[-1,1]$ with respect to $\dfrac{a+1-x^p}{\sqrt{1-x^2}},\quad p = 1, 2$ and where $a+1$ is not necessary outside $[-1,1]$.

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These weights are, of course, polynomial modification of the Tchebychev weight of the first kind and therefore an explicit representation of the corresponding polynomials is given by a determinant [10]. But the elements in the Christoffel determinant ask to compute the values of these previous weights at the points $-\sqrt{a+1}$ and $\sqrt{a+1}$ and does not give directly representation of the recurrence coefficients as function of n.

In general, the recurrence coefficients have many applications. In [11], the author showed that the recurrence coefficients satisfy a non-linear recurrence relation which corresponds to the discrete Painlevé I, pointed out, earlier, by Magnus in [15]. Here, we show a relationship between orthogonal polynomials and the solutions of some combinatoric problems. In fact, we show that, when we give to a some real values, the sequence $\dfrac{c_1}{c_2(\gamma_n(a)-\frac{1}{4})}$ coincides with some integer Sequence found in the on-line Encyclopedia of Integer Sequences .

We hope that bringing together these examples of orthogonal polynomials and the corresponding integer sequences will be illuminating and encourage researchers in the field of orthogonal polynomials and researchers in other fields (combinatorics, Painlevé equations etc...) to talk to each other and that the interaction between areas of mathematics will shed some extra light on either subjects.

1.2 Basic notations

The sequence $\{P_n(x)\}_{n\geq 0}$ is orthogonal on $[\alpha,\beta]$ with respect to the weight $w(x)$ if

$$\int_\alpha^\beta w(x)P_n(x)P_m(x)dx = k_n\delta_{n,m}, \quad n,\ m\geq 0, \quad k_n\neq 0, \quad n\geq 0, \qquad (1)$$

and, in this case, the sequence $\{P_n(x)\}_{n\geq 0}$ satisfies the three term recurrence relation

$$\begin{gathered} P_0(x)=1, \quad P_1(x)=x-\beta_0, \\ P_{n+2}(x)=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x), \quad n\geq 0, \end{gathered} \qquad (2)$$

where for all $n \geq 0$, $\gamma_{n+1} \neq 0$, the sequence $\{P_n(x)\}_{n\geq 0}$ is symmetric if and only if $\beta_n = 0, \quad n \geq 0$, i.e.,

$$\begin{array}{c} P_0(x) = 1, \quad P_1(x) = x, \\ P_{n+2}(x) = xP_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \end{array} \tag{3}$$

in this case, we have

$$P_{2n+4}(x) = (x^2 - \gamma_{2n+2} - \gamma_{2n+3})P_{2n+2}(x) - \gamma_{2n+1}\gamma_{2n+2}P_{2n}(x), \quad n \geq 0. \tag{4}$$

Remark 1.1.
1) The expressions $\beta_n, \ \gamma_{n+1}, \ n \geq 0$ are called the recurrence coefficients.
2) Tchebychev polynomial of the first kind T_n are symmetric and fulfill

$$\begin{array}{c} T_0(x) = 1, \ \ T_1(x) = x, \\ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n \geq 2, \end{array} \tag{5}$$

3) Normalized Tchebychev polynomial of the first kind $\hat{T}_n$ are symmetric and fulfill

$$\begin{array}{c} \hat{T}_0(x) = 1, \ \hat{T}_1(x) = x, \hat{T}_2(x) = x^2 - \frac{1}{2}, \\ \hat{T}_n(x) = x\hat{T}_{n-1}(x) - \frac{1}{4}\hat{T}_{n-2}(x), \quad n \geq 3, \\ \hat{T}_0(x) = T_0(x) = 1, \ \hat{T}_n(x) = \dfrac{1}{2^{n-1}}T_n(x), \quad n \geq 1, \\ \hat{T}_0(x) = 1, \ \hat{T}_n(x) = \dfrac{(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n}{2^n}, \ n \geq 1. \end{array} \tag{6}$$

1.3 Christoffel's formula

Let us recall results given in [10]. If $\{P_n(x)\}_{n\geq 0}$ is orthogonal with respect to the weight $w(x)$ then $\{\tilde{P}_n(x) = (x-c)P_n\}_{n\geq 0}$ is orthogonal with respect to the weight $\tilde{w}(x) = (x-c)w(x)$ if and only if $P_{n+1}(c) \neq 0, \quad n \geq 0$. In this case we have

$$\begin{array}{l} (x-c)\tilde{P}_n(x) = P_{n+1}(x) - \dfrac{P_{n+1}(c)}{P_n(c)}P_n(x), \quad n \geq 0, \\ \tilde{\beta}_{n+1} = \beta_{n+1} + \dfrac{P_{n+2}(c)}{P_{n+1}(c)} - \dfrac{P_{n+1}(c)}{P_n(c)}, \quad n \geq 0, \\ \tilde{\gamma}_{n+1} = \dfrac{P_{n+2}(c)P_n(c)}{P^2_{n+1}(c)}\gamma_{n+1}, \quad n \geq 0, \end{array} \tag{7}$$

where $\beta_n, \gamma_{n+1}, \ n \geq 0$ are the recurrence coefficients of $\{P_n\}_{n\geq 0}$.

Remark 1.2. In general, it is difficult to get the explicit expression of $P_n(c)$, see [7], p 99.

1.4 Organization of this paper

Our paper will be organized as follows:
First, using Christoffell's formula, we give the recurrence coefficients of polynomials orthogonal on $[-1,1]$ with respect to $\frac{\sqrt{a+1}-x}{\sqrt{1-x^2}}$, and again, using Christoffell's formula, we give the recurrence coefficients of polynomials orthogonal with respect to $(\sqrt{a+1}+x)\frac{\sqrt{a+1}-x}{\sqrt{1-x^2}} = \frac{a+1-x^2}{\sqrt{1-x^2}}$.

Then, for $a = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$ and 2, we show that the sequence $\frac{1}{4(\gamma_n(\frac{1}{4}) - \frac{1}{4})}$ (respectively $\frac{3^{n-1}}{2(\gamma_n(\frac{1}{3}) - \frac{1}{4})}$, $\frac{1}{4(\gamma_n(\frac{1}{2}) - \frac{1}{4})}$, $\frac{1}{2(\gamma_n(1) - \frac{1}{4})}$ and $\frac{1}{(\gamma_n(2) - \frac{1}{4})}$) coincides with the sequence $A093467$ (respectively $A051406$, $A101907$,$A011900$ and $A054318$).

Remark 1.2. For $a = 0$ we find Tchebychev polynomials of the second kind orthogonal on $[-1,1]$ with respect to weight $\sqrt{1-x^2}$.

2 Recurrence coefficients associated to $\frac{a+1-x^2}{\sqrt{1-x^2}}$

2.1 Recurrence coefficients associated to $\frac{\sqrt{a+1}-x}{\sqrt{1-x^2}}$

According to Christoffel's formula (7), to get the recurrence coefficients of polynomials orthogonal on $[-1,1]$ with respect to the weight $\frac{\sqrt{a+1}-x}{\sqrt{1-x^2}}$ we should evaluate $\hat{T}_n(\sqrt{a+1})$, $\hat{T}_n$ computes the nth normalized Tchebychev polynomial of the first kind.

Proposition 2.1. The recurrence coefficients of polynomials orthogonal on $[-1,1]$ with respect to $\frac{\sqrt{a+1}-x}{\sqrt{1-x^2}}$ are given by

$$\begin{aligned} \tilde{\gamma}_1 &= \frac{\hat{T}_2(\sqrt{a+1})}{2\hat{T}_1^2(\sqrt{a+1})}, \quad \tilde{\gamma}_n = \frac{\hat{T}_{n-1}(\sqrt{a+1})\hat{T}_{n+1}(\sqrt{a+1})}{4\hat{T}_n^2(\sqrt{a+1})}, \ n \geq 2. \\ \tilde{E}_0 &= \tilde{\beta}_0 = -\frac{1}{2\hat{T}_1(\sqrt{a+1})}, \ \tilde{E}_n = \sum_{k=0}^{n} \tilde{\beta}_k = -\frac{\hat{T}_n(\sqrt{a+1})}{4\hat{T}_{n+1}(\sqrt{a+1})}, \ n \geq 1. \end{aligned} \tag{8}$$

Proof. Using (7) we get $\tilde{\gamma}_n,\ n \geq 1$. For $n \geq 1$, with (7) – (6), one has

$$\tilde{E}_n = \sum_{k=0}^{n} \tilde{\beta}_n = \sum_{k=0}^{n} \beta_n + \frac{\hat{T}_{n+2}(\sqrt{a+1})}{\hat{T}_{n+1}(\sqrt{a+1})} - \frac{\hat{T}_1(\sqrt{a+1})}{\hat{T}_0(\sqrt{a+1})},\ n \geq 1.$$

$$\tilde{E}_n = \sum_{k=0}^{n} \tilde{\beta}_n = \frac{\hat{T}_{n+2}(\sqrt{a+1})}{\hat{T}_{n+1}(\sqrt{a+1})} - \hat{T}_1(\sqrt{a+1}) = -\frac{\hat{T}_n(\sqrt{a+1})}{4\hat{T}_{n+1}(\sqrt{a+1})},\ n \geq 1.$$

For $n = 0$ we have $\tilde{E}_0 = \tilde{\beta}_0 = -\dfrac{1}{2T_1(\sqrt{a+1})}$.

2.2 Recurrence coefficients associated to $\frac{a+1-x^2}{\sqrt{1-x^2}}$

According to Christoffel's formula (7), to get the recurrence coefficients of polynomials orthogonal on $[-1,1]$ with respect to the weight $(\sqrt{a+1} + x)\dfrac{\sqrt{a+1}-x}{\sqrt{1-x^2}} = \dfrac{a+1-x^2}{\sqrt{1-x^2}}$ we should evaluate $P_n(-\sqrt{a+1})$, where P_n are polynomials orthogonal on $[-1,1]$ with respect to $\dfrac{\sqrt{a+1}-x}{\sqrt{1-x^2}}$.

Lemma 2.1. The nth polynomial P_n, evaluated at $-\sqrt{a+1}$ is given by

$$P_n(-\sqrt{a+1}) = \frac{(-1)^n \hat{T}_{n+1}(\sqrt{a+1})}{\sqrt{a+1}},\ n \geq 0.$$

Proof. By recurrence on n. For $n = 0$, (6) gives $\hat{T}_1(\sqrt{a+1}) = \sqrt{a+1}$. For $n = 1$, one has

$P_1(x) = x - \tilde{\beta}_0 = x + \dfrac{1}{2(\sqrt{a+1})}$, $P_1(-\sqrt{a+1}) = -\sqrt{a+1} + \dfrac{1}{2(\sqrt{a+1})}$

$= -\dfrac{2a+1}{2\sqrt{a+1}} = -2\dfrac{\hat{T}_2(\sqrt{a+1})}{\sqrt{a+1}}$.

We suppose the property true until $n+1$ and we prove that it still true for $n+2$. We use $P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), n \geq 0$ where $\beta_{n+1},\ \gamma_{n+1}$ are given by (8). Then, using (2), the nth polynomial P_{n+2}, evaluated at $-\sqrt{a+1}$ will be

$$P_{n+2}(-\sqrt{a+1}) = \Big(-\sqrt{a+1} - \beta_{n+1}\Big)P_{n+1}(-\sqrt{a+1}) - \gamma_{n+1}P_n(-\sqrt{a+1}),\ n \geq 0,$$

now, we replace $P_n(-\sqrt{a+1})$, we get

$$P_{n+2}(-\sqrt{a+1}) = (-\sqrt{a+1} - \beta_{n+1})\frac{(-1)^{n+1}\hat{T}_{n+2}(\sqrt{a+1})}{\sqrt{a+1}}$$

$$- \gamma_{n+1}\frac{(-1)^n\hat{T}_{n+1}(\sqrt{a+1})}{\sqrt{a+1}},\ n\geq 0,$$

which we write as

$$P_{n+2}(-\sqrt{a+1}) = \frac{(-1)^{n+2}}{\sqrt{a+1}}\Big(\sqrt{a+1}\hat{T}_{n+2}(\sqrt{a+1}) + \beta_{n+1}\hat{T}_{n+2}(\sqrt{a+1})$$

$$-\gamma_{n+1}\hat{T}_{n+1}(\sqrt{a+1})\Big),\ n\geq 0,$$

we write it also as

$$P_{n+2}(-\sqrt{a+1}) = \frac{(-1)^{n+2}}{\sqrt{a+1}}\Big(\sqrt{a+1}\hat{T}_{n+2}(\sqrt{a+1})-\frac{1}{4}\hat{T}_{n+1}(\sqrt{a+1})+\beta_{n+1}\hat{T}_{n+2}(\sqrt{a+1})$$

$$+(\frac{1}{4}-\gamma_{n+1})\hat{T}_{n+1}(\sqrt{a+1})\Big),\ n\geq 0,$$

using (6) we get

$$P_{n+2}(-\sqrt{a+1}) = \frac{(-1)^{n+2}}{\sqrt{a+1}}\Big(\hat{T}_{n+3}(\sqrt{a+1}) + \beta_{n+1}\hat{T}_{n+2}(\sqrt{a+1})$$

$$+(\frac{1}{4}-\gamma_{n+1})\hat{T}_{n+1}(\sqrt{a+1})\Big),\ n\geq 0,$$

now, using (8) to replace β_{n+1} and γ_{n+1} one has

$$P_{n+2}(-\sqrt{a+1}) = \frac{(-1)^{n+2}}{\sqrt{a+1}}\Big(\hat{T}_{n+3}(\sqrt{a+1})+(\frac{\hat{T}_n(\sqrt{a+1})}{4\hat{T}_{n+1}(\sqrt{a+1})}-\frac{\hat{T}_{n+1}(\sqrt{a+1})}{4\hat{T}_{n+2}(\sqrt{a+1})})$$

$$\times\hat{T}_{n+2}(\sqrt{a+1}) + (\frac{1}{4}-\frac{\hat{T}_n(\sqrt{a+1})\hat{T}_{n+2}(\sqrt{a+1})}{4\hat{T}^2_{n+1}(\sqrt{a+1})})\hat{T}_{n+1}(\sqrt{a+1})\Big),\ n\geq 0,$$

then

$$P_{n+2}(-\sqrt{a+1}) = \frac{(-1)^{n+2}}{\sqrt{a+1}}\Big(\hat{T}_{n+3}(\sqrt{a+1})+0\Big),\ n\geq 0.$$

Proposition 2.2. The recurrence coefficients of polynomials orthogonal on $[-1,1]$ with respect to $\frac{a+1-x^2}{\sqrt{1-x^2}}$ are given by

$$\begin{aligned}&\beta_n = 0,\ n\geq 0.\ \gamma_1 = \frac{\hat{T}_3(\sqrt{a+1})}{2\hat{T}_1(\sqrt{a+1})\hat{T}_2(\sqrt{a+1})},\\ &\gamma_n = \frac{\hat{T}_{n-1}(\sqrt{a+1})\hat{T}_{n+2}(\sqrt{a+1})}{4\hat{T}_n(\sqrt{a+1})\hat{T}_{n+1}(\sqrt{a+1})},\ n\geq 2.\end{aligned} \tag{9}$$

Proof. First, $\beta_n = 0$ because the weight $w(x) = \dfrac{a+1-x^2}{\sqrt{1-x^2}}$ is even on $[-1,1]$. Second, using (7), we get

$$\tilde{\gamma}_{n+1} = \frac{P_{n+2}(-\sqrt{a+1})P_n(-\sqrt{a+1})}{P_{n+1}^2(-\sqrt{a+1})}\gamma_{n+1}, \quad n \geq 0,$$

where γ_{n+1} is given by (8), then, with Lemma 2.1., one has

$$\tilde{\gamma}_{n+1} = \frac{(-1)^{n+2}\hat{T}_{n+3}(\sqrt{a+1})}{\sqrt{a+1}} \times \frac{(-1)^n\hat{T}_{n+1}(\sqrt{a+1})}{\sqrt{a+1}} \times \frac{(\sqrt{a+1})^2}{(-1)^{2n+2}\hat{T}_{n+2}^2(\sqrt{a+1})}$$

$$\times \frac{\hat{T}_n(\sqrt{a+1})\hat{T}_{n+2}(\sqrt{a+1})}{4\hat{T}_{n+1}^2(\sqrt{a+1})}, \quad n \geq 1,$$

this gives

$$\tilde{\gamma}_{n+1} = \frac{\hat{T}_n(\sqrt{a+1})\hat{T}_{n+3}(\sqrt{a+1})}{4\hat{T}_{n+1}(\sqrt{a+1})\hat{T}_{n+2}(\sqrt{a+1})}, \quad n \geq 1,$$

with

$$\tilde{\gamma}_1 = \frac{\hat{T}_3(\sqrt{a+1})}{2\hat{T}_1(\sqrt{a+1})\hat{T}_2(\sqrt{a+1})}.$$

3 Applications.

3.1 Application 1.

When $a = -\dfrac{3}{4}$, the recurrence coefficients of polynomials orthogonal on $[-1,1]$ with respect to $\dfrac{\frac{1}{4}-x^2}{\sqrt{1-x^2}}$ are given by

$$\gamma_{3n+1}(\tfrac{-3}{4}) = 1,\ n \geq 0,\ \gamma_{3n+2}(\tfrac{-3}{4}) = \gamma_{3n+3}(\tfrac{-3}{4}) = -\frac{1}{8},\ n \geq 0. \qquad (10)$$

Proof. Using (6) one gets

$$\hat{T}_n(-\frac{3}{4}) = \frac{\frac{(1+i\sqrt{(3)})^n}{2} + \frac{(1-i\sqrt{(3)})^n}{2}}{2^n},\ i^2 = -1.$$

With Moivre's formula we obtain

$$\hat{T}_n(-\frac{3}{4}) = \frac{[1, \frac{\pi}{3}]^n + [1, -\frac{\pi}{3}]^n}{2^n} = \frac{[1, \frac{n\pi}{3}] + [1, -\frac{n\pi}{3}]}{2^n}.$$

Consequently, for $n = 3m$, $n = 3m + 1$ and $n = 3m + 2$ we obtain

$$\hat{T}_{3m}(-\frac{3}{4}) = \frac{2(-1)^m}{2^{3m}},\ T_{3m+1}(-\frac{3}{4}) = \frac{(-1)^m}{2^{3m+1}},\ T_{3m+2}(-\frac{3}{4}) = \frac{(-1)^{m+1}}{2^{3m+2}},$$

therefore, with (9) we get

$$\gamma_{3m} = \frac{\hat{T}_{3n-1}(-\frac{3}{4})\hat{T}_{3n+2}(-\frac{3}{4})}{4\hat{T}_{3n}(-\frac{3}{4})\hat{T}_{3n+1}(-\frac{3}{4})} = \frac{1}{4}\frac{\frac{(-1)^m}{2^{3m-2}}\frac{(-1)^{m+1}}{2^{3m+2}}}{\frac{2(-1)^m}{2^{3m}}\frac{(-1)^m}{2^{3m+1}}} = -\frac{1}{8}.$$

Similary, we get

$$\gamma_{3m+1}(-\frac{3}{4}) = 1,\ \gamma_{3m+2}(-\frac{3}{4}) = -\frac{1}{8}.$$

3.2 Application 2.

When $a = \frac{1}{4}$, the recurrence coefficients of polynomials orthogonal on $[-1, 1]$ with respect to $\frac{\frac{5}{4} - x^2}{\sqrt{1-x^2}}$ are $\beta_n = 0,\ n \geq 0$ and $\gamma_1(\frac{1}{4}) = \frac{\hat{T}_3(\sqrt{\frac{5}{4}})}{2\hat{T}_1(\sqrt{\frac{5}{4}})\hat{T}_2(\sqrt{\frac{5}{4}})}$, $\gamma_n(\frac{1}{4}) = \frac{\hat{T}_{n-1}(\sqrt{\frac{5}{4}})\hat{T}_{n+2}(\sqrt{\frac{5}{4}})}{4\hat{T}_n(\sqrt{\frac{5}{4}})\hat{T}_{n+1}(\sqrt{\frac{5}{4}})}$ such that, for $n \geq 1$, $v_n = \frac{1}{4(\gamma_n(\frac{1}{4}) - \frac{1}{4})}$ coincides with the sequences $A032908$ and $A093467$[12, 13] [18]. Vladeta Jovovic proved, Mar 19, 2003, that $v_{n-1} = Fibonacci(2n-1) + 1,\ n \geq 2$. Robert G. Wilson, proved, Apr 08 2004, that v_n fulfil
$v_{n+2} = 3v_{n+1} - v_n - 1, n > 0,\ v_{-1} = 1,\ v_0 = 2,\ v_1 = 3.$

Proof. $\gamma_1(\frac{1}{4}) = \frac{\hat{T}_3(\sqrt{\frac{5}{4}})}{2\hat{T}_1(\sqrt{\frac{5}{4}})\hat{T}_2(\sqrt{\frac{5}{4}})} = \frac{1}{3}$, then $v_1 = \frac{1}{4(\gamma_1(\frac{1}{4}) - \frac{1}{4})} = 3$. $\gamma_2(\frac{1}{4}) = \frac{\hat{T}_1(\sqrt{\frac{5}{4}})\hat{T}_4(\sqrt{\frac{5}{4}})}{4\hat{T}_2(\sqrt{\frac{5}{4}})\hat{T}_3(\sqrt{\frac{5}{4}})} = \frac{7}{24}$, then $v_2 = \frac{1}{4(\gamma_2(\frac{1}{4}) - \frac{1}{4})} = 6$. Using (6) we get

$$\hat{T}_n(\sqrt{\frac{5}{4}}) = \frac{(\sqrt{5}-1)^n + (\sqrt{5}+1)^n}{2^{2n}},$$

and then

$$v_n = \frac{1}{4(\gamma_n - \frac{1}{4})} = \frac{2^{2n+1}}{\sqrt{5}}\hat{T}_n(\sqrt{\frac{5}{4}})\hat{T}_{n+1}(\sqrt{\frac{5}{4}}),$$

we, easily, prove that $v_{n+2} - 3v_{n+1} + v_n + 1 = 0$.

3.3 Application 3.

When $a = \dfrac{1}{2}$, the recurrence coefficients of polynomials orthogonal on $[-1, 1]$ with respect to $\dfrac{\frac{3}{2} - x^2}{\sqrt{1-x^2}}$ are $\beta_n = 0,\ n \geq 0$ and $\gamma_1(\frac{1}{2}) = \dfrac{\hat{T}_3(\sqrt{\frac{3}{2}})}{2\hat{T}_1(\sqrt{\frac{3}{2}})\hat{T}_2(\sqrt{\frac{3}{2}})}$, $\gamma_n = \dfrac{\hat{T}_{n-1}(\sqrt{\frac{3}{2}})\hat{T}_{n+2}(\sqrt{\frac{3}{2}})}{4\hat{T}_n(\sqrt{\frac{3}{2}})\hat{T}_{n+1}(\sqrt{\frac{3}{2}})}$ such that, for $n \geq 2$, $v_n = \dfrac{1}{4(\gamma_n(\frac{1}{2}) - \frac{1}{4})}$ coincides with the sequences $A101879$ and by $A101265$ [18]. Lambert Klasen and Gary W. Adamson, on Jan 28 2005, gave the expression of v_n: $v_0 = 1,\ v_1 = 1,\ v_2 = 2,\ for\ n > 2,\ v_n = 5v_{n-1} - 5v_{n-2} + v_{n-3}$.

Proof. The same proof as done in application 2., by taking into account

$$\hat{T}_n(\sqrt{\frac{3}{2}}) = \frac{(\sqrt{3}-1)^n + (\sqrt{3}+1)^n}{2^{\frac{3n}{2}}},$$

$$v_n = \frac{2^{2n-1}}{\sqrt{3}}\hat{T}_n(\sqrt{\frac{3}{2}})\hat{T}_{n+1}(\sqrt{\frac{3}{2}}),$$

we, easily, prove that $v_{n+2} - 5v_{n+1} + 5v_n - v_{n-1} = 0$.

3.4 Application 4.

When $a = \dfrac{1}{3}$, the recurrence coefficients of polynomials orthogonal on $[-1, 1]$ with respect to $\dfrac{\frac{4}{3} - x^2}{\sqrt{1-x^2}}$ are given by (9) such that, for $n \geq 1$, $v_n = \dfrac{3^{n-1}}{2(\gamma_n(\frac{1}{3}) - \frac{1}{4})}$ coincides with the sequences $v_n = \dfrac{(3^n+1)(3^{n+1}+1)}{8}$, $n \geq 1$, are given in [18] (A 051406).

Proof. The same proof as done in application 2., by taking into account

$$\hat{T}_n(\sqrt{\frac{4}{3}}) = \frac{\sqrt{3}^n + (\frac{1}{\sqrt{3}})^n}{2^n},$$

$$v_n = 2^{2n-2}3^{n+\frac{1}{2}}\hat{T}_n(\sqrt{\frac{4}{3}})\hat{T}_{n+1}(\sqrt{\frac{4}{3}}),$$

we, easily, prove that $v_n = \dfrac{(3^n+1)(3^{n+1}+1)}{8}$, $n \geq 1$.

3.5 Application 5.

When $a = 1$, the recurrence coefficients of polynomials orthogonal on $[-1, 1]$ with respect to $\dfrac{2-x^2}{\sqrt{1-x^2}}$ are given by (9) such that, for $n \geq 1$, $v_n = \dfrac{1}{2(\gamma_n(1) - \frac{1}{4})}$ coincides with the sequence $A011900$ [18],
In [14], it is proved that

$$v_n = \frac{(\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1} + 2\sqrt{2}}{4\sqrt{2}}, \ n \geq 1.$$

Proof. The same proof as done in application 2., by taking into account

$$\hat{T}_n(\sqrt{2}) = \frac{(\sqrt{2}-1)^n + (\sqrt{2}+1)^n}{2^n},$$

the expression of $v_n = \dfrac{1}{2(\gamma_n(1) - \frac{1}{4})}$ becomes

$$v_n = \frac{(\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1} + 2\sqrt{2}}{4\sqrt{2}}, \ n \geq 1.$$

3.6 Application 6.

When $a = 2$, the recurrence coefficients of polynomials orthogonal on $[-1, 1]$ with respect to $\dfrac{3-x^2}{\sqrt{1-x^2}}$ are given by (13) such that, for $n \geq 1$, $v_n = \dfrac{1}{(\gamma_n - \frac{1}{4})}$ coincides with the sequence $A054318$ [18].
Ignacio Larrosa Canestro, on Feb 27, 2000, proved that

$$v_n = \frac{1}{2} + \frac{(3-\sqrt{6})}{12}(5+2\sqrt{6})^{n+1} + \frac{(3+\sqrt{6})}{12}(5-2\sqrt{6})^{n+1}.$$

Remark 2.3. All the Sequences $A032908$, $A101897$, $A051406$, $A011900$ and $A054318$ come from the sequence

$$\frac{c_1}{c_2\left(\frac{\hat{T}_{n-1}(\sqrt{a+1})\hat{T}_{n+2}(\sqrt{a+1})}{4\hat{T}_n(\sqrt{a+1})\hat{T}_{n+1}(\sqrt{a+1})}-\frac{1}{4}\right)},$$

where c_1 and c_2 are constants.

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