Abstract
First, we prove that normalized polynomials \( \{ \hat{P}_n \}_{n \geq 0} \) orthogonal on \([-1,1]\) with respect to the weight function \( w_a(x) = \frac{a + 1 - x^2}{\sqrt{1 - x^2}} \) fulfill
\[
\hat{P}_{n+1}(x) = x \hat{P}_n(x) - \gamma_n(a) \hat{P}_{n-1}(x), \quad n \geq 0, \quad \hat{P}_0 = 1 \text{ with }
\gamma_1(a) = \frac{\hat{T}_3(\sqrt{a+1})}{2\hat{T}_1(\sqrt{a+1})\hat{T}_2(\sqrt{a+1})}, \quad \gamma_n(a) = \frac{\hat{T}_{n-1}(\sqrt{a+1})\hat{T}_{n+2}(\sqrt{a+1})}{4\hat{T}_n(\sqrt{a+1})\hat{T}_{n+1}(\sqrt{a+1})}, \quad n \geq 1,
\]
where \( \hat{T}_n(x) \) computes the \( n \)th normalized Tchebychev polynomial of the first kind. Second, we prove that the sequence \( \frac{1}{4(\gamma_n(\frac{1}{2}) - \frac{1}{4})} \) (respectively \( \frac{1}{2(\gamma_n(\frac{1}{2}) - \frac{1}{4}), \frac{1}{4(\gamma_n(\frac{1}{2}) - \frac{1}{4}), \frac{1}{2(\gamma_n(1) - \frac{1}{4})}, \frac{1}{2(\gamma_n(2) - \frac{1}{4})} } \) coincides with the sequence A093467 (Fibonacci sequence \( F(2n - 1) + 1 \)) (respectively A051406, A101907, A011900 and A054318) found in the On-Line Encyclopedia of Integer Sequences by SLOANE N. J. A.

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1 Introduction and basic notations

1.1 Introduction
Semi-classical orthogonal polynomials of class \( s = 1, 2 \) are studied by some authors, but from the non-linearity of the system fulfilled by the recurrence coefficients, called Laguerre-Freud equations, only few recurrence coefficients are explicitly known [1, 2, 3, 4, 5, 6, 7, 8, 9]. The idea considered in this work gives the recurrence coefficients of the monic polynomials \( \hat{P}_n \) orthogonal on \([-1,1]\) with respect to \( \frac{a + 1 - x^p}{\sqrt{1 - x^2}}, \quad p = 1, 2 \) and where \( a + 1 \) is not necessary outside \([-1,1]\).
These weights are, of course, polynomial modification of the Tchebychev weight of the first kind and therefore an explicit representation of the corresponding polynomials is given by a determinant [10]. But the elements in the Christoffel determinant ask to compute the values of these previous weights at the points $-\sqrt{a+1}$ and $\sqrt{a+1}$ and does not give directly representation of the recurrence coefficients as function of $n$.

In general, the recurrence coefficients have many applications. In [11], the author showed that the recurrence coefficients satisfy a non-linear recurrence relation which corresponds to the discrete Painlevé I, pointed out, earlier, by Magnus in [15]. Here, we show a relationship between orthogonal polynomials and the solutions of some combinatoric problems. In fact, we show that, when we give to $a$ some real values, the sequence $\frac{c_1}{c_2(\gamma_n(a) - \frac{1}{4})}$ coincides with some integer Sequence found in the on-line Encyclopedia of Integer Sequences.

We hope that bringing together these examples of orthogonal polynomials and the corresponding integer sequences will be illuminating and encourage researchers in the field of orthogonal polynomials and researchers in other fields (combinatorics, Painlevé equations etc...) to talk to each other and that the interaction between areas of mathematics will shed some extra light on either subjects.

1.2 Basic notations

The sequence $\{P_n(x)\}_{n \geq 0}$ is orthogonal on $[\alpha, \beta]$ with respect to the weight $w(x)$ if

$$\int_{\alpha}^{\beta} w(x)P_n(x)P_m(x)dx = k_n\delta_{n,m}, \quad n, \quad m \geq 0, \quad k_n \neq 0, \quad n \geq 0, \quad (1)$$

and, in this case, the sequence $\{P_n(x)\}_{n \geq 0}$ satisfies the three term recurrence relation

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0,$$
$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \quad (2)$$
where for all \( n \geq 0, \gamma_{n+1} \neq 0 \), the sequence \( \{P_n(x)\}_{n \geq 0} \) is symmetric if and only if \( \beta_n = 0, \ n \geq 0 \), i.e.,

\[
P_0(x) = 1, \quad P_1(x) = x, \quad P_{n+2}(x) = xP_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0,
\]

in this case, we have
\[
P_{2n+4}(x) = (x^2 - \gamma_{2n+2} - \gamma_{2n+3})P_{2n+2}(x) - \gamma_{2n+1}\gamma_{2n+2}P_{2n}(x), \quad n \geq 0.
\]  

**Remark 1.1.**
1) The expressions \( \beta_n, \gamma_{n+1}, n \geq 0 \) are called the recurrence coefficients.
2) Tchebychev polynomial of the first kind \( T_n \) are symmetric and fulfill
\[
T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n \geq 2,
\]
3) Normalized Tchebychev polynomial of the first kind \( \hat{T}_n \) are symmetric and fulfill
\[
\hat{T}_0(x) = 1, \quad \hat{T}_1(x) = x, \quad \hat{T}_2(x) = x^2 - \frac{1}{2}, \quad \hat{T}_n(x) = x\hat{T}_{n-1}(x) - \frac{1}{4}\hat{T}_{n-2}(x), \quad n \geq 3,
\]
\[
\hat{T}_0(x) = T_0(x) = 1, \quad \hat{T}_n(x) = \frac{1}{2^{n-1}}T_n(x), \quad n \geq 1,
\]
\[
\hat{T}_0(x) = 1, \quad \hat{T}_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2^n}, \quad n \geq 1.
\]

### 1.3 Christoffel’s formula

Let us recall results given in [10]. If \( \{P_n(x)\}_{n \geq 0} \) is orthogonal with respect to the weight \( w(x) \) then \( \{\tilde{P}_n(x) = (x - c)P_n(x)\}_{n \geq 0} \) is orthogonal with respect to the weight \( \tilde{w}(x) = (x - c)w(x) \) if and only if \( P_{n+1}(c) \neq 0, \ n \geq 0 \). In this case we have
\[
(x - c)\tilde{P}_n(x) = P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)}P_n(x), \quad n \geq 0,
\]
\[
\beta_{n+1} = \beta_{n+1} + \frac{P_{n+2}(c)}{P_{n+1}(c)} - \frac{P_{n+1}(c)}{P_n(c)}, \quad n \geq 0,
\]
\[
\gamma_{n+1} = \frac{P_{n+2}(c)P_n(c)}{P_{n+1}(c)^2} \gamma_{n+1}, \quad n \geq 0,
\]
where \( \beta_n, \gamma_{n+1}, n \geq 0 \) are the recurrence coefficients of \( \{P_n\}_{n \geq 0} \).

**Remark 1.2.** In general, it is difficult to get the explicit expression of \( P_n(c) \), see [7], p 99.
1.4 Organization of this paper

Our paper will be organized as follows:
First, using Christoffel’s formula, we give the recurrence coefficients of poly-
omials orthogonal on \([-1, 1]\) with respect to \(\sqrt{a + 1 - x} \sqrt{1 - x^2}\), and again, using
Christoffel’s formula, we give the recurrence coefficients of polynomials or-
thogonal with respect to \((\sqrt{a + 1} + x) \sqrt{1 - x^2}\) = \(a + 1 - x^2\).

Then, for \(a = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\) and 2, we show that the sequence \(\frac{3^{n-1}}{2(\gamma_n(\frac{1}{4}) - \frac{1}{4})}\)
(respectively \(\frac{1}{4(\gamma_n(\frac{1}{2}) - \frac{1}{4})}\), \(\frac{1}{2(\gamma_n(1) - \frac{1}{4})}\) and \(\frac{1}{(\gamma_n(2) - \frac{1}{4})}\))
coincides with the sequence A093467 (respectively A051406, A101907, A011900
and A054318).

Remark 1.2. For \(a = 0\) we find Tchebychev polynomials of the second
kind orthogonal on \([-1, 1]\) with respect to weight \(\sqrt{1 - x^2}\).

2 Recurrence coefficients associated to \(\frac{a + 1 - x^2}{\sqrt{1 - x^2}}\)

2.1 Recurrence coefficients associated to \(\frac{\sqrt{a + 1 - x}}{\sqrt{1 - x^2}}\)

According to Christoffel’s formula (7), to get the recurrence coefficients of
polynomials orthogonal on \([-1, 1]\) with respect to the weight \(\sqrt{a + 1 - x} \sqrt{1 - x^2}\)
we should evaluate \(\hat{T}_n(\sqrt{a + 1})\), \(\hat{T}_n\) computes the nth normalized Tchebychev
polynomial of the first kind.

Proposition 2.1. The recurrence coefficients of polynomials orthogonal on
\([-1, 1]\) with respect to \(\frac{\sqrt{a + 1 - x}}{\sqrt{1 - x^2}}\) are given by
\[
\tilde{\gamma}_n = \frac{\hat{T}_n(\sqrt{a + 1})}{2T_1(\sqrt{a + 1})}, \quad \tilde{\gamma}_n = \frac{\hat{T}_{n-1}(\sqrt{a + 1}\hat{T}_{n+1}(\sqrt{a + 1})}{2T_1(\sqrt{a + 1})}, \quad \frac{4T_n^2(\sqrt{a + 1})}{4T_n^2(\sqrt{a + 1})}, \quad n \geq 1.
\]

(8)
Proof. Using (7) we get \( \tilde{\gamma}_n, \ n \geq 1 \). For \( n \geq 1 \), with (7) – (6), one has
\[
\tilde{E}_n = \sum_{k=0}^{n} \tilde{\beta}_n = \sum_{k=0}^{n} \beta_n + \frac{T_{n+2}(\sqrt{a+1})}{T_{n+1}(\sqrt{a+1})} - \frac{T_1(\sqrt{a+1})}{T_0(\sqrt{a+1})}, \ n \geq 1.
\]
\[
\tilde{E}_n = \sum_{k=0}^{n} \tilde{\beta}_n = \frac{T_{n+2}(\sqrt{a+1})}{T_{n+1}(\sqrt{a+1})} - \frac{T_1(\sqrt{a+1})}{4T_{n+1}(\sqrt{a+1})}, \ n \geq 1.
\]
For \( n = 0 \) we have \( \tilde{E}_0 = \tilde{\beta}_0 = -\frac{1}{2T_1(\sqrt{a+1})} \).

2.2 Recurrence coefficients associated to \( \frac{a + 1 - x^2}{\sqrt{1-x^2}} \)

According to Christoffel’s formula (7), to get the recurrence coefficients of polynomials orthogonal on \([-1, 1]\) with respect to the weight \( (\sqrt{a+1} + x) \sqrt{a+1} - x \sqrt{1-x^2} = \frac{a + 1 - x^2}{\sqrt{1-x^2}} \), we should evaluate \( P_n(-\sqrt{a+1}) \), where \( P_n \) are polynomials orthogonal on \([-1, 1]\) with respect to \( \frac{\sqrt{a+1} - x}{\sqrt{1-x^2}} \).

Lemma 2.1. The \( n \)th polynomial \( P_n \), evaluated at \( -\sqrt{a+1} \) is given by
\[
P_n(-\sqrt{a+1}) = \frac{(-1)^n \tilde{T}_{n+1}(\sqrt{a+1})}{\sqrt{a+1}}, \ n \geq 0.
\]

Proof. By recurrence on \( n \). For \( n = 0 \), (6) gives \( \tilde{T}_1(\sqrt{a+1}) = \sqrt{a+1} \). For \( n = 1 \), one has
\[
P_1(x) = x - \tilde{\beta}_0 = x + \frac{1}{2(\sqrt{a+1})}, \ P_1(-\sqrt{a+1}) = -\sqrt{a+1} + \frac{1}{2(\sqrt{a+1})} = -\frac{2 \sqrt{a+1} + 1}{2 \sqrt{a+1}}.
\]
We suppose the property true until \( n + 1 \) and we prove that it still true for \( n + 2 \). We use \( P_{n+2}(x) = (x - \tilde{\beta}_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), n \geq 0 \) where \( \beta_{n+1}, \gamma_{n+1} \) are given by (8). Then, using (2), the \( n \)th polynomial \( P_{n+2} \), evaluated at \(-\sqrt{a+1} \) will be
\[
P_{n+2}(-\sqrt{a+1}) = \left(-\sqrt{a+1} - \beta_{n+1}\right)P_{n+1}(-\sqrt{a+1}) - \gamma_{n+1}P_n(-\sqrt{a+1}), \ n \geq 0,
\]
now, we replace \( P_n(-\sqrt{a+1}) \), we get
\[
P_{n+2}(-\sqrt{a+1}) = \left(-\sqrt{a+1} - \beta_{n+1}\right) \frac{(-1)^{n+1} \tilde{T}_{n+2}(\sqrt{a+1})}{\sqrt{a+1}}.
\]
which we write as
\[
P_{n+2}(-\sqrt{a} + 1) = \frac{(-1)^{n+2}}{\sqrt{a} + 1} \left( \beta_{n+1} T_{n+3}(\sqrt{a} + 1) + \beta_n T_{n+1}(\sqrt{a} + 1) \right)
\]
we write it also as
\[
P_{n+2}(-\sqrt{a} + 1) = \frac{(-1)^{n+2}}{\sqrt{a} + 1} \left( \beta_{n+1} T_{n+1}(\sqrt{a} + 1) + \frac{1}{4} \gamma_{n+1} T_{n+3}(\sqrt{a} + 1) \right), n \geq 0,
\]
using (6) we get
\[
P_{n+2}(-\sqrt{a} + 1) = \frac{(-1)^{n+2}}{\sqrt{a} + 1} \left( \beta_{n+1} T_{n+3}(\sqrt{a} + 1) + \frac{1}{4} \gamma_{n+1} T_{n+1}(\sqrt{a} + 1) \right), n \geq 0,
\]
now, using (8) to replace \( \beta_{n+1} \) and \( \gamma_{n+1} \) one has
\[
P_{n+2}(-\sqrt{a} + 1) = \frac{(-1)^{n+2}}{\sqrt{a} + 1} \left( \beta_{n+1} T_{n+1}(\sqrt{a} + 1) + \frac{1}{4} \gamma_{n+1} T_{n+3}(\sqrt{a} + 1) + 0 \right), n \geq 0,
\]
\textbf{Proposition 2.2.} The recurrence coefficients of polynomials orthogonal on }([-1, 1]\text{ with respect to } \frac{a + 1 - x^2}{\sqrt{1 - x^2}} \text{ are given by}
\[
\begin{align*}
\beta_n &= 0, \quad n \geq 0, \quad \gamma_1 = \frac{T_3(\sqrt{a} + 1)}{2T_1(\sqrt{a} + 1)T_2(\sqrt{a} + 1)}, \\
\gamma_n &= \frac{T_{n-1}(\sqrt{a} + 1)T_{n+2}(\sqrt{a} + 1)}{4T_n(\sqrt{a} + 1)T_{n+1}(\sqrt{a} + 1)}, \quad n \geq 2.
\end{align*}
\]
Proof. First, $\beta_n = 0$ because the weight $w(x) = \frac{a + 1 - x^2}{\sqrt{1 - x^2}}$ is even on $[-1, 1]$. Second, using (7), we get

$$\tilde{\gamma}_{n+1} = \frac{P_{n+2}(-\sqrt{a + 1})P_n(-\sqrt{a + 1})}{P_{n+1}^2(-\sqrt{a + 1})}\gamma_{n+1}, \quad n \geq 0,$$

where $\gamma_{n+1}$ is given by (8), then, with Lemma 2.1., one has

$$\tilde{\gamma}_{n+1} = n \frac{(n+1)(n+2) \hat{T}_n(\sqrt{a + 1}) \hat{T}_{n+1}(\sqrt{a + 1})}{4 \hat{T}_{n+1}(\sqrt{a + 1}) \hat{T}_{n+2}(\sqrt{a + 1})}, \quad n \geq 1,$$

this gives

$$\tilde{\gamma}_{n+1} = \frac{\hat{T}_n(\sqrt{a + 1}) \hat{T}_{n+3}(\sqrt{a + 1})}{4 \hat{T}_{n+1}(\sqrt{a + 1}) \hat{T}_{n+2}(\sqrt{a + 1})}, \quad n \geq 1,$$

with

$$\tilde{\gamma}_1 = \frac{\hat{T}_3(\sqrt{a + 1})}{2 \hat{T}_1(\sqrt{a + 1}) \hat{T}_2(\sqrt{a + 1})}.$$  

3 Applications.

3.1 Application 1.

When $a = -\frac{3}{4}$, the recurrence coefficients of polynomials orthogonal on $[-1, 1]$ with respect to $\frac{\frac{1}{4} - x^2}{\sqrt{1 - x^2}}$ are given by

$$\gamma_{3n+1}(\frac{-3}{4}) = 1, \quad n \geq 0, \quad \gamma_{3n+2}(\frac{-3}{4}) = \gamma_{3n+3}(\frac{-3}{4}) = \frac{1}{8}, \quad n \geq 0. \quad (10)$$

Proof. Using (6) one gets

$$\hat{T}_n(\frac{-3}{4}) = \frac{(1+i\sqrt{3})^n + (1-i\sqrt{3})^n}{2^n}, \quad i^2 = -1.$$
With Moivre’s formula we obtain
\[ T_n(-\frac{3}{4}) = \frac{[1, \frac{\pi}{3}]^n + [1, -\frac{\pi}{3}]^n}{2^n} = \frac{[1, \frac{n\pi}{3}] + [1, -\frac{n\pi}{3}]}{2^n}. \]
Consequently, for \( n = 3m, n = 3m + 1 \) and \( n = 3m + 2 \) we obtain
\[ T_{3m}(-\frac{3}{4}) = \frac{2(-1)^m}{2^{3m}}, \quad T_{3m+1}(-\frac{3}{4}) = \frac{(-1)^m}{2^{3m+1}}, \quad T_{3m+2}(-\frac{3}{4}) = \frac{(-1)^{m+1}}{2^{3m+2}}, \]
therefore, with (9) we get
\[ \gamma_{3m} = \frac{T_{3n-1}(-\frac{3}{4}) T_{3n+2}(-\frac{3}{4})}{4T_{3n}(-\frac{3}{4})T_{3n+1}(-\frac{3}{4})} = \frac{1}{4} \frac{(-1)^m (-1)^{m+1}}{2^{3m-2} 2^{3m+2}} = -\frac{1}{8}. \]
Similary, we get
\[ \gamma_{3m+1}(-\frac{3}{4}) = 1, \quad \gamma_{3m+2}(-\frac{3}{4}) = -\frac{1}{8}. \]

### 3.2 Application 2.

When \( a = \frac{1}{4} \), the recurrence coefficients of polynomials orthogonal on \([-1, 1]\]
with respect to \( \frac{\sqrt{5} - x^2}{\sqrt{1 - x^2}} \) are \( \beta_n = 0, n \geq 0 \) and \( \gamma_1(\frac{1}{4}) = \frac{T_3(\sqrt{\frac{5}{4}})}{2T_1(\sqrt{\frac{5}{4}})T_2(\sqrt{\frac{5}{4}})} \).

\[ \gamma_n(\frac{1}{4}) = \frac{T_{n-1}(\sqrt{\frac{5}{4}})T_{n+2}(\sqrt{\frac{5}{4}})}{4T_n(\sqrt{\frac{5}{4}})T_{n+1}(\sqrt{\frac{5}{4}})} \] such that, for \( n \geq 1 \), \( v_n = \frac{1}{4(\gamma_n(\frac{1}{4}) - \frac{1}{4})} \) coincides with the sequences A032908 and A093467[12, 13] [18]. Vladeta Jovovic proved, Mar 19, 2003, that \( v_{n-1} = \text{Fibonacci}(2n - 1) + 1, n \geq 2. \) Robert G. Wilson, proved, Apr 08 2004, that \( v_n \) fulfil
\[ v_{n+2} = 3v_{n+1} - v_n - 1, n > 0, v_1 = 1, v_0 = 2, v_1 = 3. \]

**Proof.** \( \gamma_1(\frac{1}{4}) = \frac{T_3(\sqrt{\frac{5}{4}})}{2T_1(\sqrt{\frac{5}{4}})T_2(\sqrt{\frac{5}{4}})} = \frac{1}{3} \) then \( v_1 = \frac{1}{4(\gamma_1(\frac{1}{4}) - \frac{1}{4})} = 3. \)
\[ \gamma_2(\frac{1}{4}) = \frac{T_4(\sqrt{\frac{5}{4}})T_1(\sqrt{\frac{5}{4}})}{4T_2(\sqrt{\frac{5}{4}})T_3(\sqrt{\frac{5}{4}})} = \frac{7}{24}, \text{ then } v_2 = \frac{1}{4(\gamma_2(\frac{1}{4}) - \frac{1}{4})} = 6. \] Using (6) we get
\[ T_n(\sqrt{\frac{5}{4}}) = \frac{(\sqrt{5} - 1)^n + (\sqrt{5} + 1)^n}{2^{2n}}, \]
and then
\[ v_n = \frac{1}{4(\gamma_n - \frac{1}{4})} = \frac{2^{2n+1}}{\sqrt{5}} \hat{T}_n(\sqrt{\frac{5}{4}})\hat{T}_{n+1}(\sqrt{\frac{5}{4}}), \]
we, easily, prove that \( v_{n+2} - 3v_{n+1} + v_n + 1 = 0 \).

3.3 Application 3.

When \( a = \frac{1}{2} \), the recurrence coefficients of polynomials orthogonal on \([-1, 1]\) with respect to \( \frac{3}{2} - x^2 \) are \( \beta_n = 0 \), \( n \geq 0 \) and \( \gamma_1(\frac{1}{2}) = \frac{\hat{T}_3(\sqrt{\frac{3}{2}})}{2\hat{T}_1(\sqrt{\frac{3}{2}})\hat{T}_2(\sqrt{\frac{3}{2}})} \),

\[ \gamma_n = \frac{\hat{T}_{n-1}(\sqrt{\frac{3}{2}})\hat{T}_{n+2}(\sqrt{\frac{3}{2}})}{4\hat{T}_n(\sqrt{\frac{3}{2}})\hat{T}_{n+1}(\sqrt{\frac{3}{2}})} \]
such that, for \( n \geq 2 \), \( v_n = \frac{1}{4(\gamma_n(\frac{1}{2}) - \frac{1}{4})} \) coincides with the sequences A101879 and by A101265 [18]. Lambert Klasen and Gary W. Adamson, on Jan 28 2005, gave the expression of \( v_n \): \( v_0 = 1 \), \( v_1 = 1 \), \( v_2 = 2 \), for \( n > 2 \), \( v_n = 5v_{n-1} - 5v_{n-2} + v_{n-3} \).

**Proof.** The same proof as done in application 2., by taking into account
\[ \hat{T}_n(\sqrt{\frac{3}{2}}) = (\sqrt{3} - 1)^n + (\sqrt{3} + 1)^n \]
\[ \hat{T}_n(\sqrt{\frac{3}{2}})\hat{T}_{n+1}(\sqrt{\frac{3}{2}}) \]
we, easily, prove that \( v_{n+2} - 5v_{n+1} + 5v_n - v_{n-1} = 0 \).

3.4 Application 4.

When \( a = \frac{1}{3} \), the recurrence coefficients of polynomials orthogonal on \([-1, 1]\) with respect to \( \frac{4}{3} - x^2 \) are given by (9) such that, for \( n \geq 1 \), \( v_n = \) \[ \frac{3^{n-1}}{2(\gamma_n(\frac{1}{3}) - \frac{1}{4})} \]
coincides with the sequences \( v_n = \frac{(3^n + 1)(3^{n+1} + 1)}{8} \), \( n \geq 1 \), are given in [18] (A 051406).

**Proof.** The same proof as done in application 2., by taking into account
\[ \hat{T}_n(\sqrt{\frac{3}{3}}) = \frac{\sqrt{3^n} + (\frac{1}{\sqrt{3}})^n}{2^n}, \]
\[ v_n = 2^{2n-2}3^{n+\frac{1}{2}}\hat{T}_n(\sqrt{\frac{4}{3}})\hat{T}_{n+1}(\sqrt{\frac{4}{3}}), \]
we, easily, prove that \[ v_n = \frac{(3^n + 1)(3^{n+1} + 1)}{8}, \quad n \geq 1. \]

### 3.5 Application 5.

When \( a = 1 \), the recurrence coefficients of polynomials orthogonal on \([-1, 1]\) with respect to \( \frac{2 - x^2}{\sqrt{1 - x^2}} \) are given by (9) such that, for \( n \geq 1 \), 
\[
\frac{1}{2(\gamma_n(1) - \frac{1}{4})} \text{ coincides with the sequence A011900 [18]},
\]
In [14], it is proved that 
\[
v_n = \frac{(\sqrt{2} + 1)^{2n+1} + (\sqrt{2} - 1)^{2n+1} + 2\sqrt{2}}{4\sqrt{2}}, \quad n \geq 1.
\]

**Proof.** The same proof as done in application 2., by taking into account 
\[
\hat{T}_n(\sqrt{2}) = \frac{(\sqrt{2} - 1)^n + (\sqrt{2} + 1)^n}{2^n},
\]
the expression of \( v_n = \frac{1}{2(\gamma_n(1) - \frac{1}{4})} \) becomes 
\[
v_n = \frac{(\sqrt{2} + 1)^{2n+1} + (\sqrt{2} - 1)^{2n+1} + 2\sqrt{2}}{4\sqrt{2}}, \quad n \geq 1.
\]

### 3.6 Application 6.

When \( a = 2 \), the recurrence coefficients of polynomials orthogonal on \([-1, 1]\) with respect to \( \frac{3 - x^2}{\sqrt{1 - x^2}} \) are given by (13) such that, for \( n \geq 1 \), 
\[
\frac{1}{(\gamma_n - \frac{1}{4})} \text{ coincides with the sequence A054318 [18]}.
\]
Ignacio Larrosa Canestro, on Feb 27, 2000, proved that 
\[
v_n = \frac{1}{2} + \frac{(3 - \sqrt{6})}{12}(5 + 2\sqrt{6})^{n+1} + \frac{(3 + \sqrt{6})}{12}(5 - 2\sqrt{6})^{n+1}.
\]
Remark 2.3. All the Sequences A032908, A101897, A051406, A011900 and A054318 come from the sequence

$$\frac{c_1}{c_2} \left( \frac{T_{n-1}(\sqrt{a+1})T_{n+2}(\sqrt{a+1})}{4T_n(\sqrt{a+1})T_{n+1}(\sqrt{a+1})} - \frac{1}{4} \right),$$

where $c_1$ and $c_2$ are constants.

References


