Allowable Rotation Numbers for the Siegel Cycles of a Class of Rational Maps

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Statement of Result

Theorem (Manlove)**.** *If f is a degree d rational map which has $2d-2$ critical point relations, attracting cycles, and/or irrationally indifferent cycles including a Siegel cycle with rotation number α and at least one attracting cycle, then α must be a Brjuno number.*

Definitions and Prior Results

Let f be a holomorphic function from $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with

$$f(z) = z_0 + \lambda(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + ...,$$

where $\lambda = e^{2\pi i\alpha}$, with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

Definition. For the fixed point z_0 and the map f as above we refer to α as the **rotation number**.

Definition. A non-Möbius function f as above is said to be **linearizable** at z_0 if there is a local conformal function $h(w) = w + O(w^2)$ with $h(0) = z_0$ so that on some open disc around 0 it holds that

$$f(h(w)) = h(\lambda w).$$

The largest possible image of h is called a **Siegel Disk**.

Definition. A (rotation) number α is said to be a **Brjuno number** if:

$$\sum_n \frac{\log(q_{n+1})}{q_n} < \infty$$

where $\frac{p_n}{q_n}$ is the nth convergent to α.

Conjecture (The Douady Conjecture)**.** [Dou87] A rational map of degree greater than or equal to two has a Siegel disk (or Siegel cycle), if and only if the rotation number is Brjuno.

If true, the implication of the Douady Conjecture would be that there is a reasonable division in the irrational real numbers between the dynamically quite irrational Brjuno numbers and the more nearly rational non-Brjuno numbers. The results of Brjuno, Yoccoz and Rüssman are the more concise statements of progress towards proving this result, for a summary see Milnor's text [Mil06]. There are also several results for special families of functions.

Theorem (Brjuno, Rüssman). *If α is Brjuno then any holomorphic germ with rotation number α is linearizable.*

Theorem (Yoccoz). *If a rotation number α is not Brjuno, then the quadratic map $f(z) = e^{2\pi i\alpha}z + z^2$ has a fixed point at 0 which is not linearizable. In other words, there is no neighborhood of 0 on which a linearization is possible.*

Sketch of Proof

First, the definition of what it means for two functions to "share dynamics" must be made precise. Imprecisely, it means that they have cycles of the same lengths and multipliers and/or critical point relations of the same type. In the simplest case, one might consider degree two rational functions with a Siegel disk and a critical point of period two. In this case, some difficulty can be avoided because a bit of calculus will demonstrate that there are four such maps up to Möbius transformation. For the more general case it must be demonstrated using tools from Algebraic Geometry [Sha74] and Epstein's recent paper [Eps] that the set of functions which share dynamics of the right kinds are discrete up to Möbius transformation.

Once discreteness has been established, one assumes that f is a function with some specified dynamics. The technique is then to perturb the function f in such a way that the resulting family f_ϵ has certain nice properties. These properties need to include shared dynamics with the original and convergence to the original as the parameter shrinks. There is also a requirement on the order of the perturbation (essentially quadratic) see [Gey01]. Then Shishikura's quasiconformal surgery technique [Shi87] is employed to discover that the perturbation family is quasiconformally conjugate to a rational map.

Finally, combining discreteness and this quasiconformal conjugacy, it can be see that the quadratic polynomial

$$f(z) = e^{2\pi i\alpha}z + z^2$$

is linearizable. Then from the result of Yoccoz it follows that the rotation number must be Brjuno.

References

[Sha74] I. R. Shafarevich. *Basic algebraic geometry.* Translated from the Russian by K. A. Hirsch, Die Grundlehren der mathematischen Wissenschaften, Band 213. Springer-Verlag, New York-Heidelberg, 1974, pp. xv+439.

[Dou87] Adrien Douady. “Disques de Siegel et anneaux de Herman”. In: *Astérisque* 152-153 (1987). Séminaire Bourbaki, Vol. 1986/87, 4, 151–172 (1988). ISSN: 0303-1179.

[Shi87] Mitsuhiro Shishikura. “On the quasiconformal surgery of rational functions”. In: *Ann. Sci. École Norm. Sup. (4)* 20.1 (1987), pp. 1–29. ISSN: 0012-9593. URL: `http://www.numdam.org/item?id=ASENS_1987_4_20_1_1_0`.

[Gey01] Lukas Geyer. “Siegel discs, Herman rings and the Arnold family”. In: *Trans. Amer. Math. Soc.* 353.9 (2001), pp. 3661–3683. ISSN: 0002-9947. DOI: `10.1090/S0002-9947-01-02662-9`. URL: `http://dx.doi.org/10.1090/S0002-9947-01-02662-9`.

[Mil06] John Milnor. *Dynamics in One Complex Variable.* Annals of Mathematics Studies. Princeton University Press, 2006. ISBN: 0-691-12488-4.

[Eps] A. Epstein. “Transversality in Holomorphic Dynamics”. Preprint, available at `http://homepages.warwick.ac.uk/ mases/`.