Zeno's Arrow: A Mathematical Speculation

John Gill

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"If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless" – Aristotle on Zeno

"Time is not composed of indivisible nows any more than any other magnitude is composed of indivisibles" – Aristotle's objection

"*Instants are not parts of time, for time is not made up of instants any more than a magnitude is made of points, as we have already proved. Hence it does not follow that a thing is not in motion in a given time, just because it is not in motion in any instant of that time" - Saint Thomas Aquinas commenting on Aristotle's objection of Zeno's Paradox of the Arrow in Flight*

Introduction: Let a set of points A in the complex plane $\mathcal{C}$ be considered an event that will change over time. For each z in A an *event evolution function* will transform the original event into its evolved form after a set period of time. An evolution function (EF), z(t) differentiable and thus continuous in t, will describe an instantaneous rate of change at time t_0 through its derivative $z'(t_0)$. The only time the derivative equals 0 is when the EF is "flat" I.e., there is no instantaneous change. If this were the case throughout a time interval P = [0,1] there would be no change over P. However, a different philosophical perspective might suggest that whereas a change does take place over P, over infinitesimal intervals the change is in fact 0, and still the original event changes over P in a "continuous" fashion. It is the purpose of this note to describe an interesting functional sequence – an infinite composition arising from extensions of Tannery's Theorem – that can be (humorously) used to mathematically model Zeno's Arrow. These sequences generate *Tannery's Series* that do *not* conform to Tannery's Theorem [1].

Proposition: Consider functions of a complex variable $g_{k,n}(z) = z + \varphi_{k,n}(z)$ where $z \in S \Rightarrow g_{k,n}(z) \in S$ and $\lim_{n \to \infty} \varphi_{k,n}(z) = 0$ for all $1 \le k \le n$ and all $z \in S$. Thus $g_{k,n}(z) \to z,$ for each k as $n \to \infty$. Partition the time interval P=[0,1] into n equal subintervals of the form $\left[\frac{k-1}{n}, \frac{k}{n}\right]$. Apply $g_{1,n}(z)$ to change an *event*, z , over the interval $\left[0, \frac{1}{n}\right]$, then apply $g_{2,n}(g_{1,n}(z)) = g_{2,n} \circ g_{1,n}(z)$ over $\left[\frac{1}{n}, \frac{2}{n}\right]$, etc. The total event evolution over P may then be written $G_{n,n}(z)$, where $G_{k,n}(z) = g_{k,n} \circ g_{k-1,n} \circ \cdots \circ g_{1,n}(z)$. To continuize the process, simply allow $n \to \infty$.

If $\lim_{n\to\infty} G_{n,n}(z) = G(z)$ exists, then $G(z)$ is the "*continuous" evolution* of event z (or set A) over P, and at each "instant" the change of (each part of) the event is 0.

Example 1: Set $g_{k,n}(z) = z + \frac{k}{n^2}\mathrm{C}$. Then $G_{n,n}(\mathrm{z}) \to \mathrm{G}(\mathrm{z}) = z + C\int_0^1 w dw = z + \frac{C}{2}$, a translation. In a larger sense, the set A becomes the set A + C/2.

Observe the following:

$$G_{n,n}(z) = z + \varphi_{1,n}(z) + \varphi_{2,n}(G_{1,n}(z)) + \varphi_{3,n}(G_{2,n}(z)) + \cdots + \varphi_{n,n}(G_{n-1,n}(z)) \quad ,$$

Which is a *Tannery Series*, and which would be of little consequence if the classical *Tannery's Theorem* [1] applied, for then $G(z) \equiv z$ and there would have been no change, reflecting – under the notions of classical calculus – an instant rate of change of 0 at each point of the entire interval P.

Thus we go outside the realm of 19th century theory into more intricate formulations that, in a sense, extend the notion of Riemann Integral while accommodating a different philosophical argument concerning event evolution. General convergence theory of *Zeno contours* is discussed in [2].

Example 1 illustrates perhaps the simplest scenario, that of a Riemann Integral. Moving into slightly more complex territory, there is the following :

Theorem 1: Set $g_{k,n}(z) = z + \frac{k}{n^2} f_k(z)$ where $\lim_{k\to\infty} f_k(z) = c$ uniformly

for all z in a set S. Assume $g_{k,n}(S) \subseteq S$. Then $G_{n,n}(z) \to z + \frac{c}{2}$ uniformly in S.

Sketch of Proof : Write $G_n = z + \frac{1}{n^2} f_1(z) + \frac{2}{n^2} f_2(G_{1,n}) + \cdots + \frac{n}{n^2} f_n(G_{n-1,n})$ and

$T_n = z + \frac{1}{n^2}c + \frac{2}{n^2}c + \cdots + \frac{n}{n^2}c \to z + \frac{c}{2}$. Set $M_k = |f_k(z) - c|$, so that

$|G_n - T_n| \le I + II$ where $I = \frac{1}{n^2}\sum_{k=1}^{p} kM_k$ and $II = \frac{1}{n^2}\sum_{k=1}^{r}(p+k)M_{p+k}$ with $n = p + r$.

Choose and <u>fix</u> p so that $M_{p+k} < \frac{\varepsilon}{2}$ for $k \ge 1$. Then $II < \frac{\varepsilon}{2}$ if $n = p + r > R_1 = 3p + 1$.

Set $Sup_{1\le k\le p, z\in S} M_k = M$, so that $I < \frac{\varepsilon}{2}$ if $r > R_2 = \left[\!\left[\frac{2Mp^2}{\varepsilon} \right]\!\right]$.

Thus $\left|G_n - T_n\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ provided $n = p + r > N(\varepsilon) = p + \max\{R_1, R_2\}$. Etc. **|**

The simplest sequence-generating operators of the form $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$

for non-constant functions $f(z)$ is the subject of

Theorem 2: Set $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$ where $f(z) = \alpha z + \beta$, $\alpha \ge 0$.

Then $G_{n,n}(z) \to e^{\alpha/2} z + b\beta$ for all complex z.

Sketch of Proof: A little algebra gives

$$G_{n,n}(z) = z\prod_{k=1}^{n}\left(1 + \frac{k}{n^2}\alpha\right) \quad + \quad \frac{\beta}{n^2}\left[\sum_{k=1}^{n-1} k \prod_{t=k+1}^{n}\left(1 + \frac{t}{n^2}\alpha\right) \quad + \quad \frac{1}{n}\right]$$

$1 \le P_n(\alpha) \equiv \prod_{k=1}^{n}\left(1 + \frac{k}{n^2}\alpha\right) \to e^{\alpha/2}$. For $0 \le \alpha < 3$, P_n is monotonic decreasing, and for $3 < \alpha$,

P_n is monotonic increasing .

And $S_n(\alpha) = \frac{1}{n^2}\sum_{k=1}^{n-1} k \prod_{t=k+1}^{n}\left(1 + \frac{t}{n^2}\alpha\right) \le \left(e^{\alpha} + 1\right) \cdot \frac{1}{n^2}\sum_{k=1}^{n} k \le M$. Thus the monotonically increasing sequence $\{S_n(\alpha)\}$ converges. **|**

Theorem 2.1 : Set $g_{k,n}(z) = z + \frac{1}{n} f(z)$ with $f(z) = \alpha z + \beta$, $\alpha \geq 0$.

Then $$G_{n,n}(z) \to e^{\alpha}\left(z + \frac{\beta}{\alpha}\right) - \frac{\beta}{\alpha} \text{ as n} \to \infty$$

Proof: It is easily verified that $G_{n,n}(z) = z\left(1+\frac{\alpha}{n}\right)^n + \frac{\beta}{n}\left\{1+(1+\frac{\alpha}{n})+(1+\frac{\alpha}{n})^2+\cdots+(1+\frac{\alpha}{n})^{n-1}\right\}$,

from which the conclusion follows. **|**

An Interesting Observation: Set $F_{k,n}(z) = g_{k,n} \circ g_{k+1,n} \circ \cdots \circ g_{n,n}(z)$, an *Inner Composition*.

Set $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$. It is easy to see, assuming $f(z) \equiv C$, that

$$\lim_{n\to\infty} G_{n,n}(z) = \lim_{n\to\infty} F_{1,n}(z) = z + C\int_0^1 t\,dt = z + \frac{C}{2}$$

However, it is not so obvious that, in fact, $\lim_{n\to\infty} G_{n,n}(z) \approx \lim_{n\to\infty} F_{1,n}(z) \Rightarrow G(z) = F(z)$ for more general, non-constant functions $f(z)$ (see later developments in [5]).

The Associated Integral: In each example or theorem above the integral associated with the expansion is $\int_0^1 t\,dt$. However, virtually any proper integral on [0,1] may be used in this context.

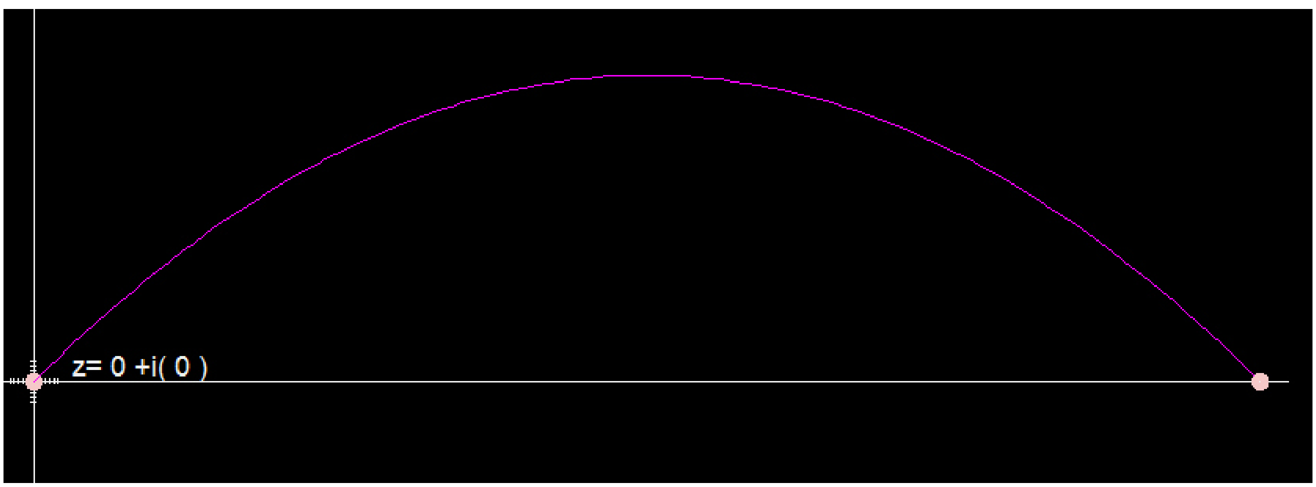
Example 3: Set $g_{k,n}(z) = z + \varphi_{k,n}(z) = z + \frac{1}{n+k} f(z)$, $f(z) = z^2$. Here the associated integral is $\int_0^1 \frac{1}{t+1} dt = \ln 2$. Thus $G_{n,n}(z) = z + \frac{1}{n+1} f(z) + \frac{1}{n+2} f(G_{1,n}) + \cdots + \frac{1}{n+n} f(G_{n-1,n})$, and

for instance, $G(.5+.5i) = .28045... + .91394...\, i$. Also, $F(.5+.5i) = .28045... + .91394...\, i$.

Zeno's Arrow: Standard calculus provides the position of a projectile launched at ground level at an optimum angle of 45 degrees with an initial velocity of V_0, ignoring air resistance. It can be shown that the corresponding evolution generating functions are

$$g_{k,n}(z) = z + \frac{2\upsilon^2}{gn}\left[1 + i\left(1 + \frac{1-2k}{n}\right)\right] \quad ,$$

where $\upsilon = \frac{V_0}{\sqrt{2}} = \frac{g}{2}\eta$ and g = acceleration due to gravity.



$V_0 = 100, n = 50$ $\qquad$ $z = 312.5 + i(0)$

The arrow travels over the time interval $[0,\eta]$, which is divided into subintervals $\left\{\left[\frac{\eta(k-1)}{n}, \frac{\eta k}{n}\right]\right\}$.

Flight begins at z=0 and ends at $x + iy = \frac{V_0^{\,2}}{g}$. That is to say, $G_{n,n}(0) \to \frac{V_0^{\,2}}{g}$.

This is a somewhat trivial example of the theory described above, since the second term does not involve the variable z.

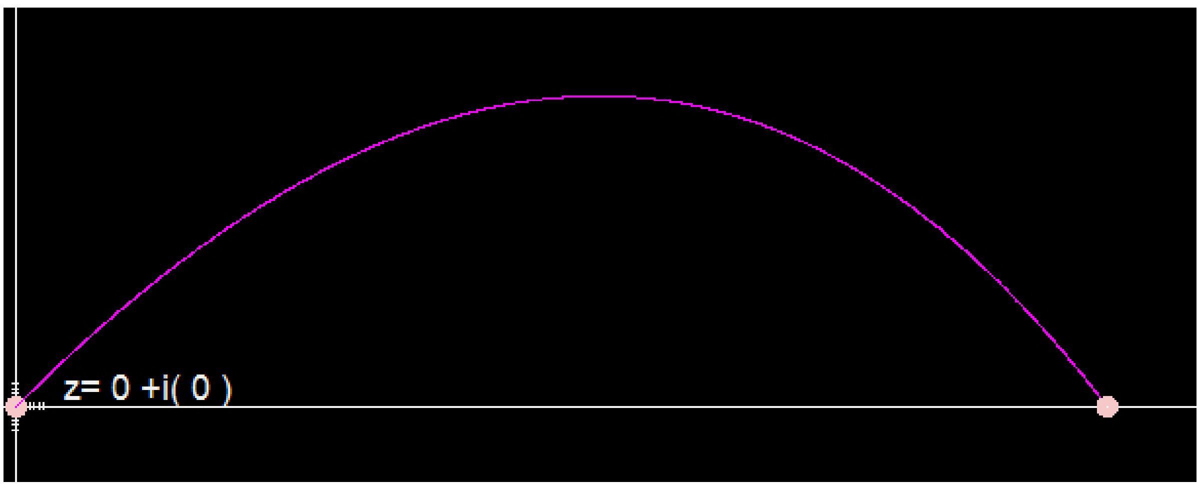
If the force exerted by air resistance is proportional to the speed of the projectile the resulting function looks a bit different. The angle of launch is kept more general here:

Suppose $Force_{air} = \rho V(t)$. Assume m = the mass of the projectile. Now, assume the time interval from launch to impact at ground level is $[0,\eta]$. Divide this interval into subintervals

$\left\{\left[\frac{\eta k}{n}, \frac{\eta(k+1)}{n}\right]\right\}$. Set $\upsilon_{k,n} = e^{-\frac{\rho}{m}\frac{\eta k}{n}}\left(e^{\frac{\rho}{m}\frac{\eta}{n}} - 1\right)$. Then the generating evolution functions are

$$g_{k,n}(z) = z + \frac{m}{\rho}\left[\upsilon_{k,n} V_0 \cos\theta + i\left(\upsilon_{k,n}\left(V_0 \sin\theta + \frac{mg}{\rho}\right) - \frac{\eta g}{n}\right)\right],$$

where $\upsilon_{k,n} \to 0$ as $n \to \infty$ for $1 \le k \le n$.



$V_0 = 100, n = 2000, \theta = \frac{\pi}{4}$ $z = 239.47 + i(0)$

Is the motion at an "Instant" actually 0? Consider the simple case where the time interval is

[0,1] and $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$, with f bounded. Suppose $t_0 \in (0,1)$.

Then there exists a sequence of intervals $\left\{\left[\frac{k}{n},\frac{k+1}{n}\right]\right\}_{n\to\infty}$ such that

$\frac{k}{n} < t_0 < \frac{k+1}{n}$ and the intervals collapse to the point $t_0 \in (0,1)$. On these intervals , given $\varepsilon > 0$, $|g_{k,n}(z) - z| < \varepsilon$ for n sufficiently large.

What about Continuity? In the context of Zeno's Arrow or similar motion of a point through space continuity essentially means that, for a small increment of time on the time axis, there is observed a similar small increment of motion of the point or projectile. That is to say

$|G_{k,n}(z) - G_{k-1,n}(z)| = |g_{k,n}(G_{k-1,n}(z)) - G_{k-1,n}(z)| < \varepsilon$ for sufficiently large values of n.

However, assuming f is uniformly bounded by M over a set S, and $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$,

$|g_{k,n}(G_{k-1,n}(z)) - G_{k-1,n}(z)| \le \frac{k}{n^2}|f(z)| \le \frac{1}{n}\cdot\frac{k}{n}M \le \frac{M}{n} < \varepsilon$ if $n > \frac{M}{\varepsilon}$.

Clearly, the condition is satisfied uniformly over the set S. A similar argument suffices for the evolution function describing Zeno's arrow.

Theorem 7 Let $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$, $z \in S \Rightarrow g_{k,n}(z) \in S$ Suppose $f(z)$ is analytic on S and that it satisfies a Lipshitz Condition: $|f(z_1) - f(z_2)| \le \rho |z_1 - z_2|$ and also $f(\alpha) = 0$.

Then α is a fixed point of $g_{k,n}(z)$ and $|G_{n,n}(z) - \alpha| \le |z - \alpha| \cdot \prod_{k=1}^{n}\left(1 + \frac{k}{n^2}\rho\right) \le \eta(\rho)\cdot|z - \alpha|$

and, if $\lim_{n\to\infty} G_{n,n}(z) = G(z)$ exists, then $|G'(\alpha)| \le \eta(\rho)$

.

Sketch of Proof: The results follow easily from $\left|g_{k,n}(z)-\alpha\right| \le \left|z-\alpha\right| + \frac{k}{n^2}\rho\cdot\left|f(z)-f(\alpha)\right|$.|

Some values for $\eta(\rho) \approx e^{\rho/2}$ are: $\eta(\frac{1}{4}) \approx 1.133$, $\eta(\frac{1}{2}) \approx 1.284$, $\eta(1) \approx 1.648$, $\eta(2) = e$, $\eta(3) \approx 4.481$, $\eta(4) = \infty$

An Integral function arising from these ideas?

Start with $g_{k,n}(z) \equiv z + \frac{1}{n}f(z)$ with $f(z)$ analytic on a domain S, and $z \in S \Rightarrow g_{k,n}(z) \in S$.

Then we have $G_{n,n}(z) = z + \frac{1}{n}f(z) + \frac{1}{n}f(G_{1,n}(z)) + \frac{1}{n}f(G_{2,n}(z)) + \cdots + \frac{1}{n}f(G_{n-1,n}(z))$.

Now, imagine a function

(1) $\boxed{\varphi(z,t)\ ,\ t \in [0,1] \ \ \underline{\text{and}} \ \ \varphi\left(z,\frac{k}{n}\right) \equiv f\left(G_{k-1,n}(z)\right), \text{with } \int_0^1 \varphi(z,t)\,dt \text{ defined}}$

Set $\Phi_n(z) = G_{n,n}(z) - z = \frac{1}{n}\varphi\left(z,\frac{1}{n}\right) + \frac{1}{n}\varphi\left(z,\frac{2}{n}\right) + \frac{1}{n}\varphi\left(z,\frac{3}{n}\right) + \cdots + \frac{1}{n}\varphi\left(z,\frac{n}{n}\right)$.

Then $\Phi_n(z) \to \int_0^1 \varphi(z,t)dt \equiv F(z)$, by the definition of the Riemann Integral.

Becoming more specific , let $S = (|z| < R)$ and $S_1 = (|z| < R_1)$ with $R_1 < R$.

Now define $R_2 = \frac{R_1 + R}{2}$ and choose $z \in S_2 = (|z| < R_2)$. Assume $f(S) \subset \overline{S_1}$.

Then $\left|G_{k,n}(z)\right| < R$, and each $\left|\varphi\left(z,\frac{k}{n}\right)\right| = \left|f\left(G_{k-1,n}(z)\right)\right| < R_1$.

Since $\{\Phi_n(z)\}$ converges (and contracts) the fixed points $\{\alpha_n\}$ of $\{\Phi_n(z)\}$ converge: $\alpha_n \to \alpha$.

Define $F_n(z) = \Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_1(z)$. Theorem 6 implies $F_n(z) \to \alpha = \int_0^1 \varphi(\alpha,t)dt$ if and only if the sequence of fixed points of $\{\Phi_n(z)\}$ converges to that limit.

Example 4: Somewhat trivial, but is a rare case when the closed form of $\varphi(z,t)$ can be approximated.

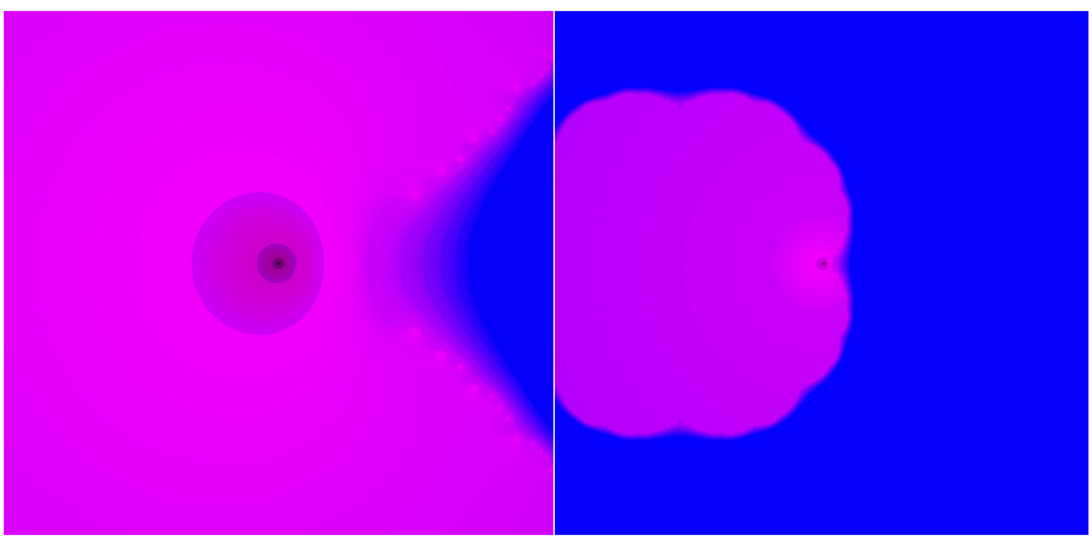
Set $g_n(z) = z - \frac{1}{n} f(z)$, $f(z) = \frac{1}{4} z.$ $R_1 = \frac{1}{4},\ R_2 = \frac{1}{2},\ R = 1$.

Then $$G_{k-1,n}(z) = z\left(1 - \frac{1}{4n}\right)^k = z\left(\left(1 - \frac{1}{4n}\right)^{4n}\right)^{\frac{1}{4}\cdot\frac{k}{n}} \approx ze^{-\frac{1}{4}\frac{k}{n}}$$ for large values of n ,

with $\left|G_{k-1,n}(z)\right| < R_2$.

Thus $$\varphi\left(z, \frac{k}{n}\right) = f(G_{k-1,n}(z)) \approx \frac{1}{4} z\, e^{-\frac{1}{4}\cdot\frac{k}{n}} \Rightarrow \varphi(z,t) \approx \frac{1}{4} z\, e^{-\frac{1}{4}t} ,$$

and $$\int_0^1 \varphi(z,t)dt \approx z\left(1 - \sqrt{e}\right) = z \Leftrightarrow z = 0 = \alpha .$$



Convergence behavior of $G_{n,n}(z)$ for $g_{k,n}(z) = z + \frac{k}{n^2} \cdot z^2$ on [-6,6] and [-60,60]. N<10

Very dark means $\left|G_{n,n}(z) - z\right|$ is very small, tapering out to blue, representing either extremely high values or, more likely, divergence.

Theorem 4: Consider $g_{k,n}(z) = z + \frac{k}{n^2} \cdot f(z)$, $f(0) = 0$, $|f(z)| \le R$ for $|z| \le R$.

Then $|G_{n,n}(z)| < R$, $\forall n$ if $z \in S = \left(|z| \le \frac{R}{2}\right)$.

Sketch of proof:

Schwarz's Lemma implies $|f(z)| \le |z|$ for $|z| \le R$.

Hence $|G_{1,n}(z)| \le |z| + \frac{1}{n^2}|f(z)| \le \frac{R}{2}\left(1 + \frac{1}{n^2}\right)$ if $z \in S = \left(|z| \le \frac{R}{2}\right)$

And . . . $|G_{n,n}(z)| \le \frac{R}{2}\prod_{k=1}^{n}\left(1 + \frac{k}{n^2}\right) < R$ for $z \in S$. ❙

And, a little wider scope . . .

Theorem 5: Suppose $|z| < R \Rightarrow |f(z)| < M$ where $M < 2R$. Choose $\varepsilon > 0$ such that

$R_0 = R - M\left(\frac{1}{2} + \varepsilon\right) > 0$. Then $N = N(\varepsilon) = \left[\!\left[\frac{1}{2\varepsilon}\right]\!\right] + 1$ and

$|z| < R_0 \Rightarrow |G_{k,n}(z)| < R$ for $n > N$.

Sketch of proof: $n > \frac{1}{2\varepsilon} \Rightarrow R_0 + \left(\frac{1}{2} + \frac{1}{2n}\right)M < R_0 + \left(\frac{1}{2} + \varepsilon\right)M < R$. Thus

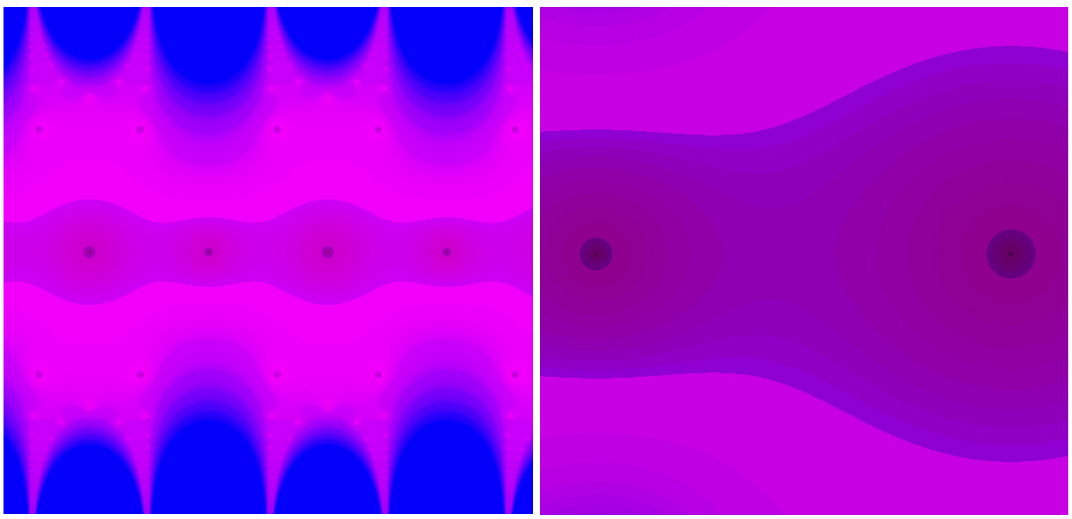
$$|G_{1,n}(z)| \le |z| + \frac{1}{n^2}|f(z)| < R_0 + \frac{1}{n^2}M \le R_0 + M\sum_{k=1}^{n}\frac{k}{n^2} = R_0 + \left(\frac{1}{2} + \frac{1}{2n}\right)M < R,$$

$$\left|G_{2,n}(z)\right| \le |z| + \frac{1}{n^2}|f(z)| + \frac{2}{n^2}\left|f\left(G_{1,n}(z)\right)\right| < R_0 + \left(\frac{1}{n^2}+\frac{2}{n^2}\right)M \le R_0 + M\sum_{k=1}^{n}\frac{k}{n^2} = R_0 + \left(\frac{1}{2}+\frac{1}{2n}\right)M < R,$$

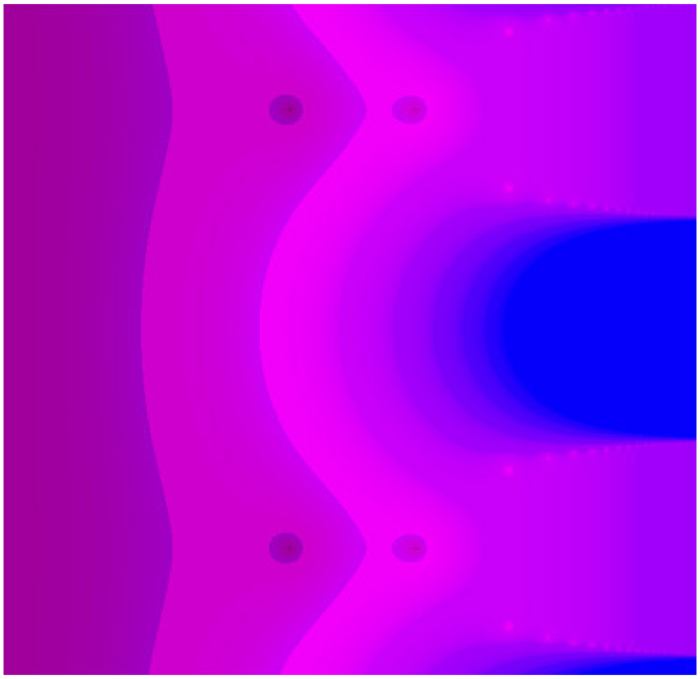
Etc. |

Example: $|z| < R = 1 \Rightarrow \left|\frac{e^z}{10}\right| < .28 = M$. Hence $\varepsilon < 3.08$. Choose $\varepsilon = .10 \Rightarrow N = 6$.

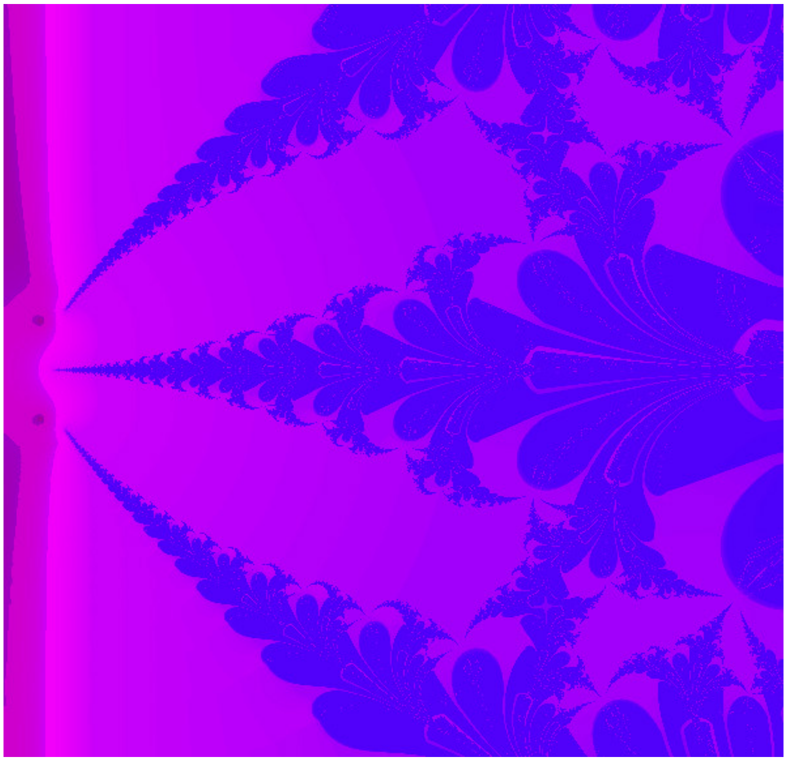
Then $R_0 \approx .83$. Thus $|z| < .83 \Rightarrow \left|G_{k,n}(z)\right| < 1$ for $n > 6$.



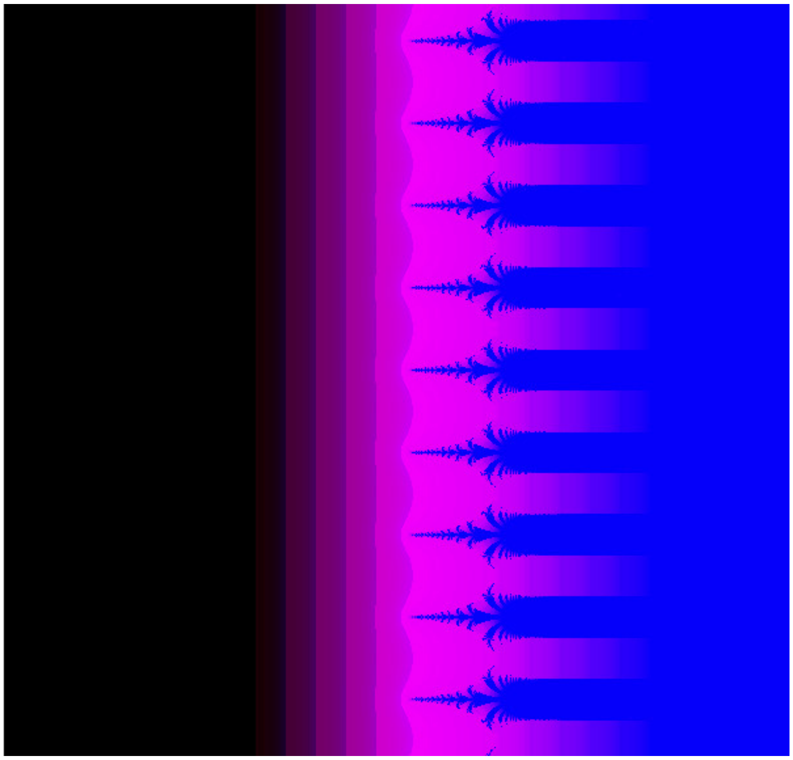
Behavior of $\left|G_{n,n}(z) - z\right|$ for $g_{k,n}(z) = z + \frac{k}{n^2}\cdot cos(z)$ on [-7,7] and [-2,2]. N<5



Behavior of $\left|G_{n,n}(z)-z\right|$ for $g_{k,n}(z)=z+\frac{1}{n}e^{\left(\frac{z}{2}\cdot\frac{k}{n}\right)}$ on [-15,25], N=3



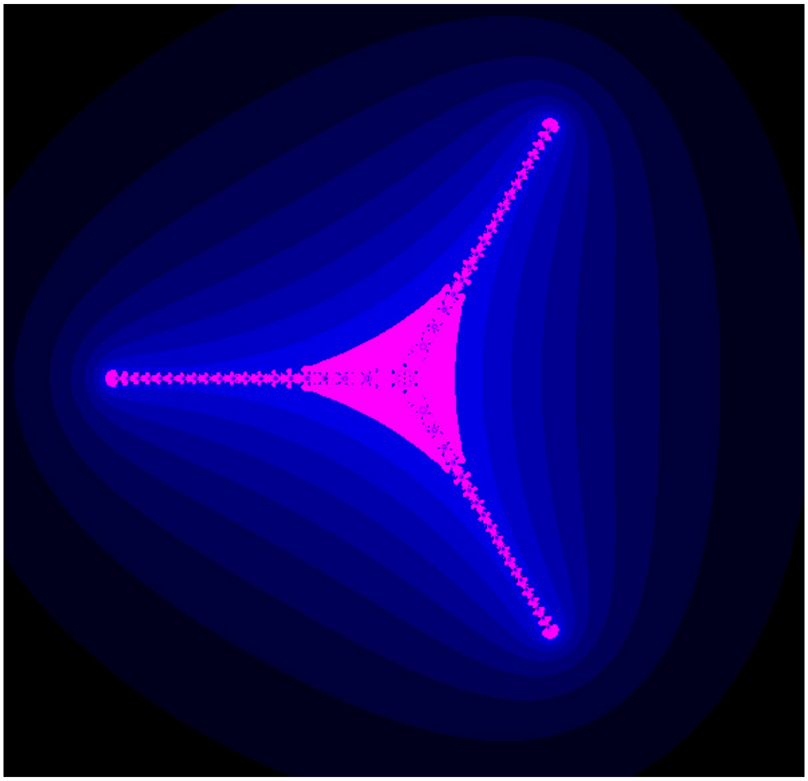
The complexity of convergence: $\left|G_{n,n}(z)-z\right|$ for $g_{k,n}(z)=z+\frac{1}{n}e^{\left(\frac{z}{2}\cdot\frac{k}{n}\right)}$ on [-10,190] , N=30. Here the process of iteration has been altered to replace the absolute value at each stage by a uniform constant when the absolute value is extremely high: the blue fractal “flowers” (Julia Set)



Example: $\left|G_{n,n}(z)-z\right|$ for $g_{k,n}(z)=z+\frac{k}{n^2}e^z$ and $\mathrm{Re}(z)<-R, R>0$. On [-30,30] for n=15.

A little complex algebra produces $\left|G_{k,n}(z)\right|\le|z|+e^{\mathrm{Re}(z)}\cdot\frac{1}{n^2}\sum_{j=1}^{k}j\le|z|+e^{\mathrm{Re}(z)}$,

seen graphically as the black area on the above picture.



Example: $g_{k,n}(z) = z + \frac{k}{n^2} \cdot \frac{1}{z^2}$. [-1.2,1.2], n=20. For $R > 1$, the following is not difficult to show:

$|z| > R + \frac{1}{R^2} \quad \Rightarrow \quad |G_{k,n}(z) - z| < \frac{1}{R^2} \cdot \frac{1}{n^2} \sum_{j=1}^{k} j < \frac{1}{R^2} \cdot \left(\frac{1}{2} + \frac{1}{2n} \right) \leq \frac{1}{R^2}$. Hence the surrounding black region where $|G_{k,n}(z) - z|$ is quite small. The odd, bright limbs at multiples of $\frac{2\pi}{3}$ show points that

move from one branch to another under the iteration.

References:

[1] J. Gill, Generalizations of the Classical Tannery's Theorem, www.johngill.net, 2011

[2] J. Gill, *Zeno Contours in the Complex Plane*, Comm. Anal. Th. Cont. Frac. XIX (2012)