Informal Notes: Zeno Contours, Parametric Forms, & Integrals

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Abstract: Elementary classroom notes on Zeno contours, streamlines, pathlines, and integrals

Definition: *Zeno contour*[1]. Let $g_{k,n}(z)=z+\eta_{k,n}\varphi(z)$ where $z\in S$ and $g_{k,n}(z)\in S$ for a convex set S in the complex plane. Require $\lim_{n\to\infty}\eta_{k,n}=0$, where (usually) $k=1,2,...,n$. Set $G_{1,n}(z)=g_{1,n}(z)$, $G_{k,n}(z)=g_{k,n}\left(G_{k-1,n}(z)\right)$ and $G_n(z)=G_{n,n}(z)$ with $G(z)=\lim_{n\to\infty}G_n(z)$, when that limit exists. The *Zeno contour* is a graph of this iteration. The word *Zeno* denotes the infinite number of actions required in a finite time period if $\eta_{k,n}$ describes a partition of the time interval [0,1]. Normally, $\varphi(z)=f(z)-z$ for a vector field, $\mathbb{F}=f$. The alternative notation $G_n(z)=\mathop{\mathcal{L}}_{k=1}^{n}g_{k,n}(z)$ is also available. Zeno contours are a natural and minor extension of *Euler's Method* for differential equations.

I. Begin with $\eta_{k,n}=\frac{1}{n}$ and $g_{k,n}(z)\equiv z+\frac{1}{n}\varphi(z)$ with $\varphi(z)$ continuous on a domain S , and $z\in S\Rightarrow g_{k,n}(z)\in S$. Then we have

$$G_{n,n}(z)=z+\frac{1}{n}\varphi(z)+\frac{1}{n}\varphi(G_{1,n}(z))+\frac{1}{n}\varphi(G_{2,n}(z))+\cdots+\frac{1}{n}\varphi(G_{n-1,n}(z))\,.$$

Now, imagine a function

$$\psi(z,t)\ ,\ t\in[0,1]\quad \underline{\text{and}}\quad \psi\left(z,\frac{k}{n}\right)\equiv\lim_{m\to\infty}\varphi\left(G_{mk-1,mn}(z)\right),\text{ with }\int_0^1\psi(z,t)\,dt\text{ defined}:$$

$$G_n(z)-z=\frac{1}{n}\psi\left(z,\frac{1}{n}\right)+\frac{1}{n}\psi\left(z,\frac{2}{n}\right)+\frac{1}{n}\psi\left(z,\frac{3}{n}\right)+\cdots+\frac{1}{n}\psi\left(z,\frac{n}{n}\right)\ \approx\int_0^1\psi(z,t)dt$$

And for t irrational, $\psi(z,t)=\lim_{t_r\to t}\psi(z,t_r)$ for rational t_r . The existence of this function (and the integral) is equivalent to the convergence of the Zeno contour. $\int_0^1\psi(z,t)dt$ is more a *virtual* integral since its analytical form can be murky at times, but very simple under the right circumstances.

(i) **Zeno Contour to Parametric Form**: Write the recurrence sequence as

$$z\left(\tfrac{k}{n}\right) = z\left(\tfrac{k-1}{n}\right) + \frac{1}{n}\varphi\left(z\left(\tfrac{k-1}{n}\right)\right), \ z(0) = z_0$$

Assuming $z = z(t)$, one concludes

$$\frac{\Delta_{k,n}z}{\Delta_n t} = \varphi\left(z\left(\tfrac{k-1}{n}\right)\right) \ \Rightarrow \ \frac{dz}{dt} = \varphi\left(z(t)\right), \ t \in [0,1].$$

When conditions allow: $\psi(z,t) = \varphi(z(t)) = \dfrac{dz}{dt}$ and $\int_0^1 \psi(z,t)dt = z(1) - z(0)$.

Example: $\varphi(z) = z^2$ produces $\dfrac{dz}{dt} = z^2$ which gives $z(t) = \dfrac{z_0}{1 - z_0 t}$, and that in turn gives $\int_0^1 \psi(z_0,t)\,dt = \lambda(z_0) = z_0^{\,2}/(1-z_0)$. Here $\psi(z_0,t) = z^2 = z_0^{\,2}/(1-z_0 t)^2$

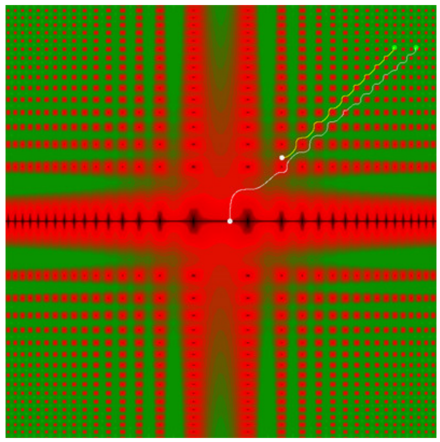
Example: $\varphi(z) = e^z$. Here $z(t) = z_0 - Ln(1 - e^{z_0}t)$ and $\psi(z_0,t) = e^{z_0}/(1 - e^{z_0}t)$.

Example: $\varphi(z) = \dfrac{z-\alpha}{z-\beta}$. However, $z(t) - z_0 = t + (\alpha - \beta)\cdot Ln\left(\dfrac{z(t)+\alpha}{z_0+\alpha}\right)$ allows no closed formulation of $\psi(z_0,t)$, although $\int_0^1 \psi(z_0,t)dt = 1 + (\alpha-\beta)\cdot Ln\left(\dfrac{z(1)+\alpha}{z_0+\alpha}\right)$.

Example: $\varphi(z) = (x+y) + i(x-y)$. We have $\dfrac{dx}{dt} = x + y$, $\dfrac{dy}{dt} = x - y$, which can be solved

$x(t) = C_1 e^{-t\sqrt{2}} + C_2 e^{t\sqrt{2}}$ and $y(t) = C_3 e^{-t\sqrt{2}} + C_4 e^{t\sqrt{2}}$, with $C_k = C_k(x_0, y_0)$

<u>*Example*</u>: $\varphi(z) = xCos(y^2) + i\, ySin(x^2)$ giving $\dfrac{dx}{dt} = xCos(y^2)$, $\dfrac{dy}{dt} = ySin(x^2)$, a system not solvable in closed form, if at all. The two contours use $\eta_n = \dfrac{1}{n}$, $\eta_n = \dfrac{1}{\sqrt{n}}$, the latter terminating at one of many attractors on the real axis.



(ii) **Parametric Form to Zeno Contour**: Given $z = z(t),\ 0 \le t \le 1$, describe $\varphi(z)$ or $\varphi(z,t)$. We must have $z(0) = z_0$.

<u>*Example*</u>: $z(t) = z_0 e^{\alpha t} \quad \Rightarrow \quad \dfrac{dz}{dt} = \alpha z = \varphi(z)$.

<u>*Example*</u>: $z(t) = z_0 e^{\alpha t} + \beta t$, $\dfrac{dz}{dt} = \alpha z_0 e^{\alpha t} + \beta = \alpha z + \beta(1 - \alpha t) = \varphi(z,t)$. Therefore the generating functions are $g_{k,n}(z) = z + \dfrac{1}{n}\left(\alpha z + \beta(1 - \alpha \tfrac{k}{n})\right)$, and the contour is a *pathline*, rather than a *streamline*, given by $G(z_0) = \lim\limits_{n\to\infty}\left(g_{n,n} \circ g_{n-1,n} \circ \cdots \circ g_{1,n}(z_0)\right)$. The underlying force field is

$$(1+\alpha)z + \beta(1 - \alpha t) = \varphi(z,t) + z = f(z,t) \rightarrow f(z) = (1+\alpha)z + \beta(1-\alpha)$$

Example: $z(t)=z_0(1+t)^2$, $\dfrac{dz}{dt}=\dfrac{2z}{1+t}=\varphi(z,t)$, giving a pathline, where the *vector fields*

$$f(z,t)=z\cdot\frac{3+t}{1+t}\quad\rightarrow\ f(z)=2z\,.\qquad g_{k,n}(z)=z\left(1+\frac{2}{k+n}\right)\ \Rightarrow\ G_n(z_0)\rightarrow 4z_0$$

II. A little visual background: Suppose $\left|f(z)-\alpha\right|<\rho\left|z-\alpha\right|$, $0\le\rho<1$ in a convex region S . By drawing a pair of concentric circles

$$c_1(z)=\{\zeta:\left|\zeta-\alpha\right|=\left|z-\alpha\right|\}\quad\text{and}\quad c_2(z)=\{\zeta:\left|\zeta-\alpha\right|=\rho\left|z-\alpha\right|\}$$

it is not difficult to establish the following: $\left|g_n(z)-\alpha\right|\le\left|z-\alpha\right|-\dfrac{C}{n}\left|f(z)-z\right|$, leading to

$\left|g_n(z)-\alpha\right|\le\left(1-\dfrac{C}{n}(1-\rho)\right)\cdot\left|z-\alpha\right|$. Thus $\left|G_n(z)-\alpha\right|\le\left(1-\dfrac{C}{n}(1-\rho)\right)^n\cdot\left|z-\alpha\right|$. Writing , for large n ,

$$\left(1-\frac{C}{n}(1-\rho)\right)^n=\left[\left(1-\frac{1}{n\left(\frac{1}{C(1-\rho)}\right)}\right)^{n\left(\frac{1}{C(1-\rho)}\right)}\right]^{C(1-\rho)}\sim\left(\frac{1}{e}\right)^{C(1-\rho)}$$

If $C=C_n=\sqrt{n}$ then the last expression approximates $\left(\dfrac{1}{e}\right)^{(1-\rho)\sqrt{n}}\rightarrow 0$ and convergence occurs.

Thus $\eta_{k,n}=\dfrac{1}{\sqrt{n}}$ forces the contour to terminate at an attractor. For $\eta_{k,n}=\dfrac{1}{n}$, when the differential equations described previously can be solved early termination follows easily. Zeno contours follow streamlines of course.

Writing $g_n(z)=z+\dfrac{1}{n}(f(z)-z)$ and then $g_n(z)-\alpha=\left(1-\dfrac{1}{n}\right)(z-\alpha)+\dfrac{1}{n}(f(z)-\alpha)$,

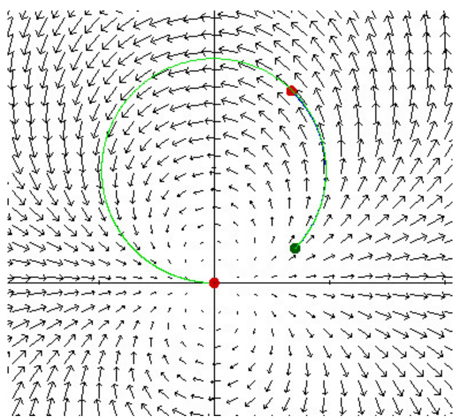
for $\left|f(z)-\alpha\right|<\rho\left|z-\alpha\right|$, a simple vector analysis shows that, in an approximate sense,

$$\left(\frac{1}{e}\right)^{1+\rho} \cdot |z-\alpha| \;\le\; |G_n(z)-\alpha| \;\le\; \left(\frac{1}{e}\right)^{1-\rho} \cdot |z-\alpha|$$

For large values of n. Thus the leading tip of the contour is trapped in an annulus that is quite small if z is close to α.

Example: $f(z)=z^2+z$, $z_0=.7+.3i$, $n=100{,}000$, $\eta_n=\dfrac{1}{\sqrt{n}}$ vs $\eta_n=\dfrac{1}{n}$.

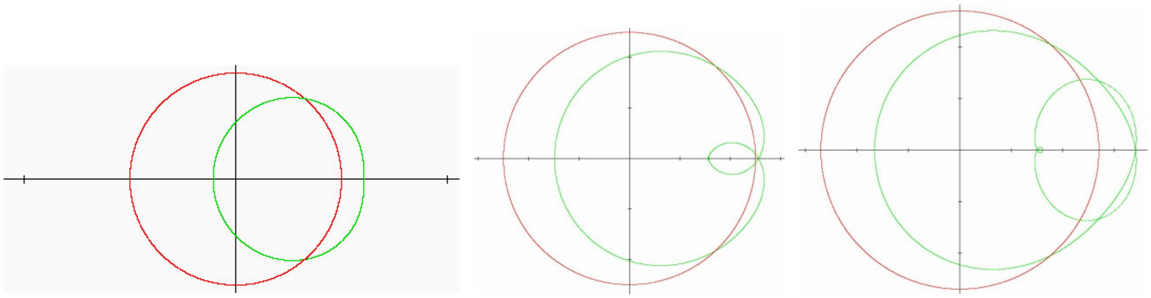
$$Z=Z(z_0,t)=\frac{1}{1/z_0 \; - \; t} \quad \Rightarrow \; Z(z_0,1)=\frac{z_0}{1-z_0}.$$



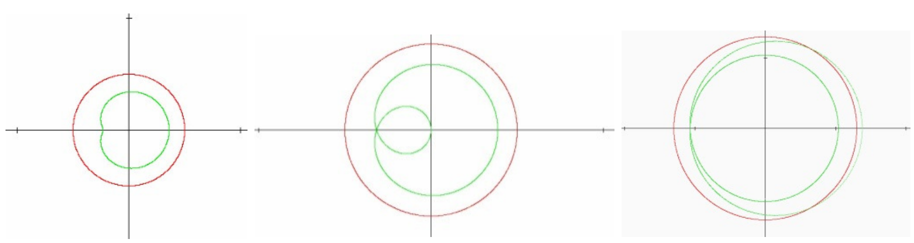
III. Consider the function defined by $\lambda(z)=\int_0^1 \psi(z,t)\,dt$.

The image in the z-plane (red) is $\gamma(s)$ and the image in the w-plane (green) is $\lambda(\gamma(s))$.

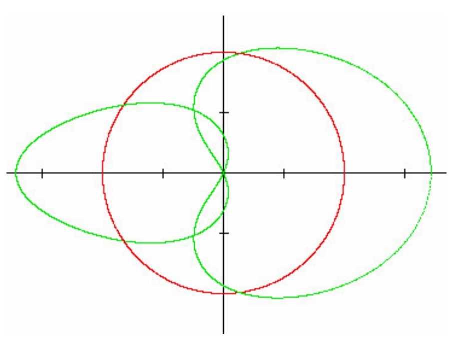
<u>Example</u>: $\varphi(z) = Cos(z) - z$ provides the following mappings of circles (scales are one unit):



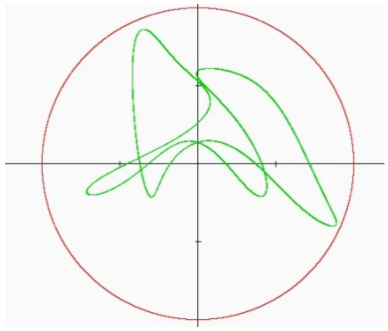
Another example is $\varphi(z) = z^2 - z$, with $\lambda(z) = \int_0^1 \Psi(z,t)dt = \frac{z(1-z)(1-e)}{z(1-e)+e}$.



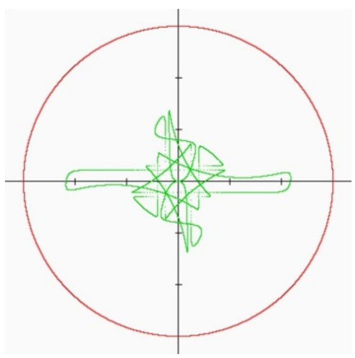
A non-analytic example: $\varphi(z) = \varphi(x+iy) = xCos(y) + ixSin(y)$



And another: $\varphi(z) = \varphi(x+iy) = xCos(x+y) + iySin(y-x)$



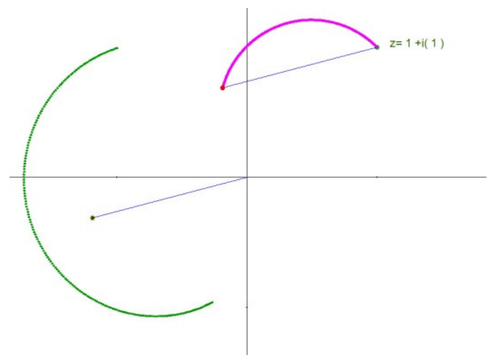
Also, $\varphi(z) = \varphi(x+iy) = yCos(x^2) + ixSin(y^2)$ for $(|z| = 3)$



V. The *Zeno integral* is loosely associated with *Ergodic theory* in the following way:

$\int_0^1 \psi(z,t)dt \approx G_{n,n}(z) - z = \frac{1}{n}\sum_{k=0}^{n-1} \varphi\left(G_{k,n}(z)\right) \rightarrow$ an average value of $\left\{\varphi\left(G_{k,n}(z)\right)\right\}_{k=1}^{n-1}$ where the sequence $\left\{G_{k,n}(z)\right\}_{k=1}^{n}$ are points distributed along the emerging Zeno contour.

Example: Set $\varphi(z)=z^2-z$ for $z=1+i$ and $n=200$:



The (approximate) Zeno contour is in purple, while the distribution $\left\{\varphi\left(G_{k,n}(z)\right)\right\}_{k=1}^{n}$ is in green. The point $\lambda(z)$ lies in quadrant III within the arc of the distribution: $\lambda(1+i)\approx -1.187+i(-.312)$

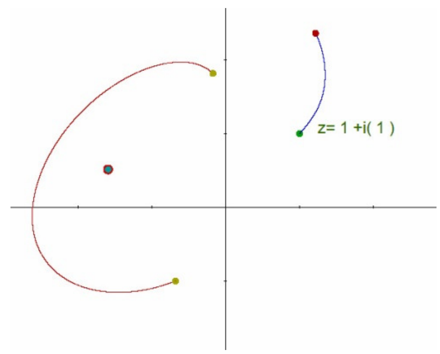
Another integral worthy of mention and easily obtained [1] is

$$\int_{\gamma(\varphi(z))}\varphi(z)dz=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varphi^2(G_{k,n}(z))$$

so that the integral, where the integrand is basic for generating the Zeno contour upon which it is evaluated, provides the average value of the sequence $\left\{\varphi^2\left(G_{k,n}(z)\right)\right\}_{k=0}^{n-1}$. Note that the sequence $\left\{G_{k,n}(z)\right\}_{k=0}^{n-1}$ is generated using the function φ, not φ^2.

Example: $\varphi(x+iy)=xSin(x+y)+iyCos(x-y)$, $z_0=1+i$. Then

$$\int_\gamma \varphi \approx -1.594+.515i \approx \frac{1}{n}\sum_{k=0}^{n-1}\varphi^2\left(G_{k,n}(z)\right)$$



Pathlines coincide with streamlines if the underlying force field remains constant. When it does not pathlines may or may not lie close to streamlines. In the language of Zeno contours, pathlines are computable in the following format: $g_{k,n}(z)=z+\frac{C}{n}\varphi_{k,n}(z)=z+\frac{C}{n}\left(f_{k,n}(z)-z\right)$, $G_{n,n}(z)=g_{n,n}\circ g_{n-1,n}\circ\cdots\circ g_{1,n}(z)$. Again, when φ is well-behaved, $\frac{dz}{dt}=\varphi(z(t),t)$ may be solvable and t eliminated from the equations.

And if $f_{k,n}\to f$ in the sense that $\left|f_{k,n}(z)-f(z)\right|<\varepsilon_{k,n}$ where $\sigma_n=\frac{C}{n}\sum_{k=1}^{n}\varepsilon_{k,n}\to 0$, then when $\left|f_{k,n}(z)-\alpha_{k,n}\right|<\rho_{k,n}\left|z-\alpha_{k,n}\right|$, $\alpha_{k,n}\to\alpha$ and $0\le\rho_{k,n}\le\rho<1$, $G_{n,n}(z)\to\alpha$ as before:

$g_{k,n}(z)-\alpha=z-\alpha+\frac{C}{n}\left(f_{k,n}(z)-f(z)+f(z)-z\right)$,

$\left|g_{k,n}(z)-\alpha\right|\le\left(1-\frac{C}{n}\right)\left|z-\alpha\right|+\rho\frac{C}{n}\left|z-\alpha\right|+\frac{C}{n}\varepsilon_{k,n}$, $\left|g_{k,n}(z)-\alpha\right|\le\mu_n\left|z-\alpha\right|+\frac{C}{n}\varepsilon_{k,n}$,

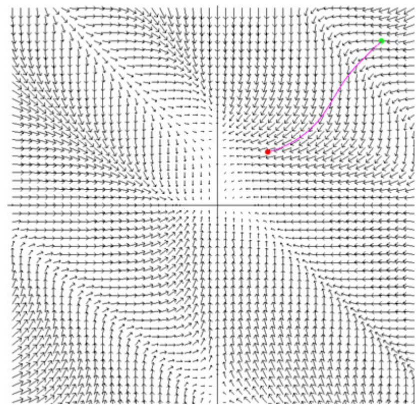
$\mu_n=\left(1-\frac{C}{n}(1-\rho)\right)$. Thus $\left|G_{n,n}(z)-\alpha\right|\le\left[\mu_n^{\,n/C(1-\rho)}\right]^{C(1-\rho)}\left|z-\alpha\right| + \frac{C}{n}\sum_{k=1}^{n}\varepsilon_{k,n} \sim \left(\frac{1}{e}\right)^{C(1-\rho)} +\sigma_n$

And $C=\sqrt{n}$ provides convergence as before.

Example: $\varphi=\frac{x}{1+t}+i\frac{y}{1+2t}=\frac{dZ}{dt}$ leading to $y=\frac{y_0}{\sqrt{x_0}}\sqrt{x-x_0}$.

Example: $\varphi(z,t) = xCos(x+y)t + iySin(x+y)t$. With vector fields

$$f(z,t) = \left(xCos(x+y)t + x\right) + i\left(ySin(x+y)t + y\right) \;\; \rightarrow \;\; f(z) = x(1+Cos(x+y)) + iy(1+Sin(x+y))$$



Reference:

[1] J. Gill, *Zeno Contours in the Complex Plane*, Comm. Anal. Th. Cont. Frac. XIX (2012), www.johngill.net