Progress Report: Zeno Contours in the Complex Plane

John Gill **August 26, 2012**

Abstract: Zeno contours – sequences of complex numbers generated from a process that defines a path (contour) of a point z in a continuous fashion, yet demonstrating virtually no movement at a hypothetical instant – arise from explorations involving Tannery's series (adding the next term of the series changes all the previous terms) [1] in the context of infinite compositions of complex functions. Specialized finite sequences (in the Euclidean plane) of this type occur in image processing in the construction of "active contours" or "snakes", but theory of infinite extensions – certainly in the complex plane - is not necessary there. "Zeno behavior" (infinite number of actions in a finite time period) occurs in these contour sequences and is a feature of hybrid dynamical systems, as well; see. E.g., [5]. "Zeno contours" is also a phenomenon occurring in certain physical processes, and the expression has a different meaning there. The presentation here is informal, even casual, and more an exploratory stream of minor discovery rather than a refined paper. Nothing important here – just musings of an oldtimer!

Define: $g_{k,n}(z)=z+\eta_{k,n}\varphi(z)$ where $z\in S$ and $g_{k,n}(z)\in S$ for a convex set S in the complex plane. Require $\lim\limits_{n\to\infty}\eta_{k,n}=0$, where (usually) $k=1,2,...,n$. Set $G_{1,n}(z)=g_{1,n}(z)$, $G_{k,n}(z)=g_{k,n}\left(G_{k-1,n}(z)\right)$ and $G_n(z)=G_{n,n}(z)$ with $G(z)=\lim\limits_{n\to\infty}G_n(z)$, when that limit exists.

The simplest example is $\eta_{k,n}\equiv\dfrac{C_n}{n}$. In particular, let $C_n=C\geq 1$ for now. It is not required the functions so described be analytic in this region, although usually they will be. If $\varphi(z)\equiv\lambda$, a constant, the resulting sequence is a simple Riemann sum.

Assume that all aspects of the system delineated above are well behaved. This system nevertheless exhibits "Zeno behavior".

Consider a time interval $I=[0,1]$ partitioned by $\left[\dfrac{(k-1)}{n},\dfrac{k}{n}\right]_{k=1}^{n}$. Then a point z is "continuously" transformed via a path $\gamma(z)$ into $G(z)$ in such a manner that at each "instant" of the time interval the motion of the point is virtually zero [2]. Such a pattern is analogous to *Zeno's Arrow* [3], where the arrow traverses its path but exhibits no motion at any "instant".

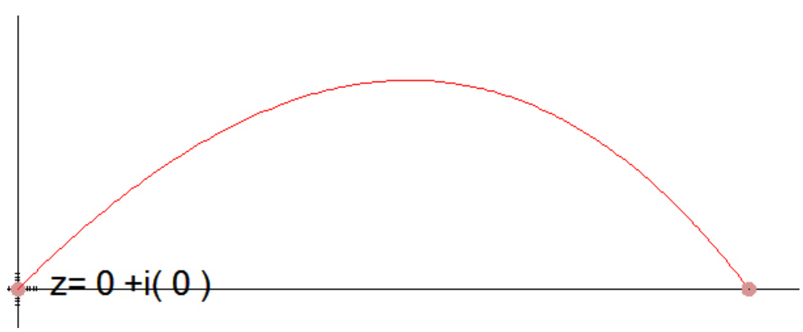

Figure 1: Zeno's Arrow flight, computed using $G(z)=\lim\limits_{n\to\infty}G_n(z)$. Resistance due to air proportional to velocity.

Observe that $G_{n,n}(z) = z + \varphi_{1,n}(z) + \varphi_{2,n}(G_{1,n}(z)) + \varphi_{3,n}(G_{2,n}(z)) + \cdots + \varphi_{n,n}(G_{n-1,n}(z))$, if we set

$\varphi_{k,n}(z) = \eta_{k,n}\varphi(z)$. This is a *Tannery series* that does not conform to the hypotheses of Tannery's Theorem [1], and thus its convergence is a matter of more diligent and delicate investigative techniques.

Write $\mathbb{F}(z)$ for the vector or force field defined by $f(z) := \varphi(z) + z$. The graph of the passage of z to $G(z)$ is a contour $\gamma(z)$ whose length depends to some extent on the value of C . The contour – which is self-generating - follows the flow of the vector field $\mathbb{F}(z)$, and may terminate at some unpredictable point or at a "well" (attractive fixed point of $f(z)$). If the path of the contour closely approaches a singularity the result is chaotic behavior.

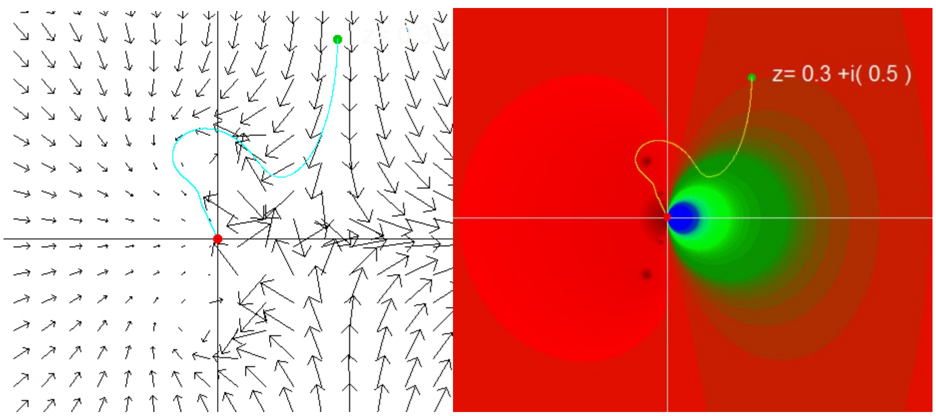

Figure 2: $f(z) = e^{\frac{1}{z}}$, n=50,000 , $C = 10$. The contour is drawn to an apparent attractor, but If C is boosted to 13, the tip of the contour enters the black hole of the essential singularity.

In the figures, vectors are scaled, with shorter lengths corresponding to smaller displacements and longer lengths corresponding to greater displacements. The self-generating contour begins at the green point and ends at the red point. C_n can be thought of as an "extender". Increasing its value lengthens the contour. The color images are flux graphs showing displacements.

These self-generating contours are somewhat related to "active contours" or "snakes" used in image processing [4]. In that context a virtual continuum of points constituting a snake are attracted to edges of images. The journey of each initial point is similar to the subject of this note. A force field surrounding the edge must exist to compel each point forward. The mathematical constructs of snakes are formulated in Euclidean 2-space, rather than the complex plane, and involve finite rather than infinite iteration theory.

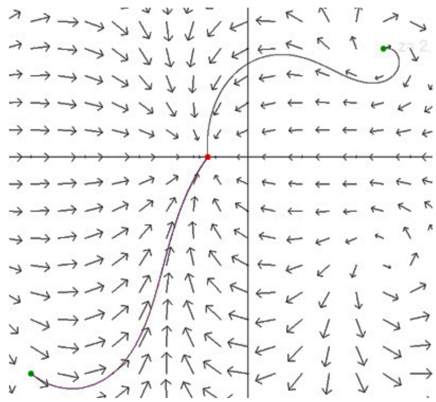

Figure 3: f(z) = - Cos(z). C = 10 , n=10,000. Contours "converge" to an attractor of f(z).

General Formulae:

It is a simple matter to verify the following :

(1) $$G(z) = z + \lim_{n\to\infty} \frac{C_n}{n} \sum_{k=1}^{n} \varphi\left(G_{k-1,n}(z)\right) \; = \oint_{\gamma(z)} d\zeta$$

(2) $$\mathrm{L}\left(\gamma(z)\right) = \oint_{\gamma(z)} |d\zeta| = \lim_{n\to\infty} \frac{C_n}{n} \sum_{k=1}^{n} \left|\varphi\left(G_{k-1,n}(z)\right)\right|$$ and

(3) $$\oint_{\gamma(z)} \varphi(\zeta) d\zeta = \lim_{n\to\infty} \frac{C_n}{n} \sum_{k=1}^{n} \varphi^2\left(G_{k-1,n}(z)\right),$$ provided these limits exist.

Previously discussed convergence behavior of $\{G_n(z)\}$, includes the case $\varphi(z) = c$, where the sums above are trivial Riemann sums and describe Riemann integrals, and $\varphi(z) = \alpha z + \beta$, $\alpha > 0$, where $G(z) = e^{\alpha C} z + \frac{\beta}{\alpha}\left(e^{\alpha c} - 1\right)$ [2]. In other scenarios $\{G_n(z)\}$ *resembles* a Riemann integral, but is a more convoluted concept.

In Figure 2, $L\left(\gamma(.3+.5i)\right) \approx 1.017$

In Figure 3, $L\left(\gamma(-4-4i)\right) \approx 6.1119$ and $\oint_{\gamma} \varphi(z) dz \approx 21.208 + 33.275i$.

Additional elementary theory . . .

The Simple Attractor Case: Suppose $|f(z)-\alpha| < \rho|z-\alpha|$, $0 \le \rho < 1$ in a convex region S .

(Boundedness of derivative might lead to this, using $|f(z)-f(\alpha)| \le \int_{\alpha}^{z} |f'(\zeta)||d\zeta|$) By drawing a pair of concentric circles

$$c_1(z) = \{\zeta : |\zeta-\alpha| = |z-\alpha|\} \quad \text{and} \quad c_2(z) = \{\zeta : |\zeta-\alpha| = \rho|z-\alpha|\}$$

it is not difficult to establish the following: $|g_n(z)-\alpha| \le |z-\alpha| - \frac{C}{n}|f(z)-z|$, leading to

$|g_n(z)-\alpha| \le \left(1-\frac{C}{n}(1-\rho)\right)\cdot|z-\alpha|$. Thus $|G_n(z)-\alpha| \le \left(1-\frac{C}{n}(1-\rho)\right)^n \cdot |z-\alpha|$.

Writing , for large n ,

$$\left(1-\frac{C}{n}(1-\rho)\right)^n = \left[\left(1-\frac{1}{n\left(\frac{1}{C(1-\rho)}\right)}\right)^{n\left(\frac{1}{C(1-\rho)}\right)}\right]^{C(1-\rho)} \sim \left(\frac{1}{e}\right)^{C(1-\rho)}$$

For large values of C , $G_n(z) \approx \alpha$. In Figure 1, $|G_n(z)-\alpha| \le .000045$, but the origin is a "false" attractor - too close and all is lost. If $C = C_n = \sqrt{n}$ then the last expression approximates $\left(\frac{1}{e}\right)^{(1-\rho)\sqrt{n}} \to 0$ and convergence occurs. However, the analogy between these infinite compositions and the simple Riemann integral collapses. Nevertheless, general expansions of this nature can be interesting:

Example: $g_n(z) = z + \frac{1}{\sqrt{n}} Sin(z) \Rightarrow G_n(z) \to \pi$, for $0 < z < 9$. And $G_n(10) \to 3\pi$.

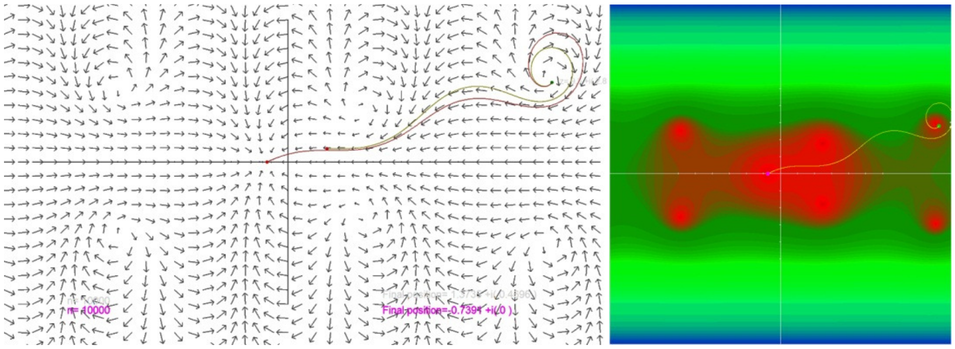

Figure 5: Vector field is $F(z) = -Cos(z)$ with (1) $g(z) = z + \frac{1}{n}\left(-Cos(z) - z\right)$ generating the shorter contour and (2) $g(z) = z + \frac{1}{\sqrt{n}}\left(-Cos(z) - z\right)$ generating the longer contour.

N = 20,000. Contour (2) terminates at the attractive fixed point -.7390.

Pictures like this reinforce the conjecture that (1) implies that $G_n(z) \to \beta$, a point on the generated contour that extended would culminate at the attractive fixed point α the ultimate destination of the contour (2). The contours are of course not exactly alike. More about $G_n(z) \to \beta$ a little later.

Generalization of the Simple Attractor case: Consider generating functions of the form $g_{k,n}(z) = z + \eta_{k,n}\varphi(z)$ with $f(z) := \varphi(z) + z$ and $\lim_{n\to\infty} \eta_{k,n} = 0$ for $k = 1, 2, ..., n$. Write $G_n(z) = g_{n,n} \circ g_{n-1,n} \circ \cdots \circ g_{1,n}(z)$. The condition $\left|f(z) - \alpha\right| < \rho\left|z - \alpha\right|$, $0 \le \rho < 1$ is probably a bit unrealistic and should be generalized by some variant of the inverse-square law of attraction. One possibility is: $\left|f(z) - \alpha\right| \le \kappa\left|z - \alpha\right|$ where $\kappa = \kappa\left(z, \alpha\right) = \left[\frac{r\left|z - \alpha\right|}{1 + \left|z - \alpha\right|}\right]^2$, $0 \le r \le 1$.

Then , $\left|z - \alpha\right| < R \;\Rightarrow\; \kappa < r^2\left(\frac{R}{1+R}\right)^2 = \rho(R) := \rho < 1$. Of course, if $\left|f\,'(\alpha)\right| < 1$ there is a neighborhood of α in which $\left|f(z) - \alpha\right| < \rho\left|z - \alpha\right|$, $0 \le \rho < 1$. The previous discussion gives

$\left|G_n(z) - \alpha\right| \le \left|z - \alpha\right| \cdot \prod_{k=1}^{n}\left(1 - \mu(k,n)\right) = \left|z - \alpha\right| \cdot P_n$ with $\mu(k,n) = \left(1 - \rho\right)\eta_{k,n}$. Now, write

$$\left[(1-\mu(k,n))^{\frac{1}{\mu(k,n)}}\right]^{\mu(k,n)} \sim \left(\frac{1}{e}\right)^{\mu(k,n)}$$ for large values of n. Hence,

$P_n \sim \left(\frac{1}{e}\right)^{\sigma(n)}, \sigma(n)=\sum_{k=1}^{n}\mu(k,n)$. Then $\sigma(n)\to\infty \Rightarrow G_n(z)\to\alpha$.

Example: $\eta_{k,n}=\frac{C_n k}{n^2}$ where $C_n=\sqrt{n}$.

The Superposition of Two Vector Fields: Suppose a particle z is subject to two fields at once. Write $g_n(z)=z+\frac{c_n}{n}\varphi(z)=z+\frac{c_n}{n}\left(f_1(z)-z+f_2(z)-z\right)$ or $g_n(z)=z+\frac{c_n}{n}\left(F(z)-z\right)$, with

$F(z)=\left(f_1(z)+f_2(z)-z\right)$. Frequently $f_1(\alpha)=\alpha\in R$ and $f_1(\beta)=\beta\in R$ $,\alpha<\beta$.

Assume that $f_1, f_2 \in C[\alpha,\beta]$. It is easily verified that $f_1(x)\neq f_2(x)$ over this interval implies the existence of $\mu\in[\alpha,\beta]$ with $F(\mu)=\mu$, possibly an attractor. Then for z's belonging to a neighborhood of μ it is possible that $G_n(z)\to\mu$ provided $c_n=\sqrt{n}$ or a similar factor.

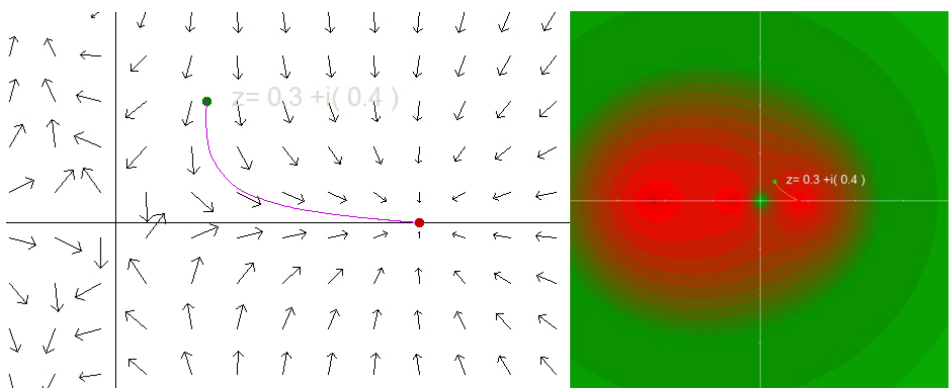

Figure 6: Superposition of $f_1=1-z^2$ and $f_2=\frac{1}{z^2}$. $\alpha\approx .618, \beta=1$ implies $\mu=1+0i$. The contour (c(n)=square root of n) reaches .99999 . . .+0i . n=100,000.

Derivatives . . .?

Although motion comes to a stand-still at "instants" on the time axis, Instantaneous velocity does exist as it does when Z is parametrically defined as a differentiable function of time t: Z(t).

Suppose $g_{k,n}(z) = z + \eta_{k,n}\varphi(z)$, continuous, with $\lim_{n\to\infty}\eta_{k,n} = 0$. Let $t \in [0,1]$ with a finite (or truncated) decimal expansion. For convenience, set $Z\left(\frac{k}{n}\right) = G_{k,n}(z)$. Then

$$Z\left(\tfrac{k}{n}\right) = Z\left(\tfrac{k-1}{n}\right) + \eta_{k,n}\cdot\varphi\left(Z\left(\tfrac{k-1}{n}\right)\right) \quad \text{and} \quad \frac{\Delta Z}{\Delta t} = \frac{Z\left(\frac{nt_0}{n}\right) - Z\left(\frac{nt_0-1}{n}\right)}{1/n} = n\cdot\eta_{nt_0,n}\cdot\varphi\left(Z\left(\tfrac{nt_0-1}{n}\right)\right).$$

If n is restricted to powers of 10, then, for sufficiently large n, nt_0 is an integer and $Z(t_0)$ may be approximated by iterating from 1 to nt_0 rather than n. Therefore

$$\left(\frac{dZ}{dt}\right)_{t=t_0} = \varphi\left(Z(t_0)\right)\cdot\lim_{n\to\infty}\left[n\cdot\eta_{nt_0,n}\right]$$

Note that $g_{k,n}(z) = z + \eta_{k,n}\varphi(z)$ does not have to be analytic.

Example: $\eta_{k,n} = \dfrac{C}{n} \Rightarrow Z'(t_0) = C\varphi\left(Z(t_0)\right)$

Example: $\eta_{k,n} = C\dfrac{k}{n^2} \Rightarrow Z'(t_0) = Ct_0\varphi\left(Z(t_0)\right)$.

Example: $\eta_{k,n} = \dfrac{C}{n}$, $f(z) = z + \dfrac{1}{z}$, $\varphi(z) = \dfrac{1}{z}$, $C = 100$, $z = 1+5i$, $t_0 = .235$

$$\Rightarrow \quad \left(\frac{dZ}{dt}\right)_{t=t_0} \approx 19.6 - 4i \quad (n = 100{,}000).$$

More on integrals . . .

To evaluate a contour integral of a function T(z) along the generated contour, there is the following: $\oint_{\gamma(z)} T(\zeta)d\zeta = \lim_{n\to\infty}\dfrac{C}{n}\sum_{k=1}^{n}\left[T\left(Z_{k-1,n}\right)\varphi\left(Z_{k-1,n}\right)\right]$.

Also, $\oint_{\gamma(z)} \varphi(\zeta)d\zeta = C\int_0^1 \varphi^2\left(Z(t)\right)dt$. Both with $\eta_{k,n} = \dfrac{C}{n}$.

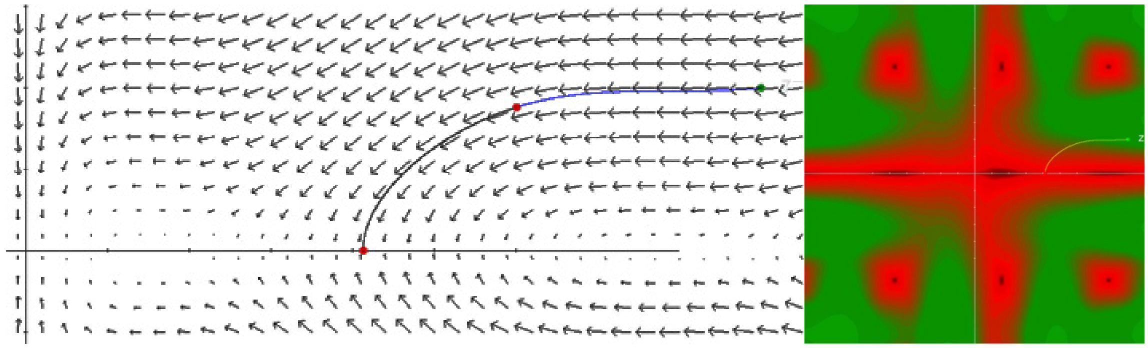

Figure 7: $f(x+iy) = xCosy + iySinx$ (non-analytic). The contour from 9+2i , terminates at the attractor $Z(1)=4.1282 + 0i$, $\eta_{k,n} = \dfrac{C}{n}$ using C=10. The shorter contour terminates at $Z(.03) = 6.0191 + 1.7677i$, at which point $\dfrac{dy}{dx} \approx .3097$ and $\left(\dfrac{dZ}{dt}\right)_{t=.3} \approx -71.9664 - 22.2912i$. The length and integral value for the longer contour are: $L(\gamma(9+2i)) \approx 5.7341$ and $\oint_{\gamma} \varphi(z)dz \approx 34.334 + 12.2916i$. n=100,000

A Continuum of Attractors . . .

For this discussion, consider an interval on the positive real axis, composed entirely of attractive fixed points of $f(z)$, (continuous but not analytic – if it were the identity function would result!).

Furthermore, assume a uniform measure of attractive force (at least close to the x-axis):

$$\left|f(x+iy) - x\right| \le \rho \left|(x+iy) - x\right| = \rho \left|y\right| .$$

In addition, we specify that x and y are in the first quadrant of the complex plane $\mathbb{C}$.

Set $U_{k,n} + iV_{k,n} = G_{k,n}(z)$, and $U_0 + iV_0 = x + iy$. Figure 7 shows such a field $\mathbb{F}$, with streamlines flowing diagonally down towards the right . A Zeno contour will follow the force field - so the only question is whether the contour terminates early, or leads to a point on the axis, or travels beyond the axis. An open question is whether different generators $\eta_{k,n}$ produce slightly different contours, terminating at slightly different attractors.

To this end, we use the previous result concerning a single attractor:

Thus, with $\mu_{k,n} = \mu(k,n)$, $\left|G_{1,n}(z) - U_0\right| \le (1-\mu_{1,n})\cdot V_0$,

$\left|G_{2,n}(z) - U_{1,n}\right| \le (1-\mu_{2,n})\left|G_{1,n}(z) - U_{1,n}\right| < (1-\mu_{2,n})\left|G_{1,n}(z) - U_0\right| \le (1-\mu_{2,n})(1-\mu_{1,n})\cdot V_0$ . .

Leading to $\left|G_{n,n}(z) - U_{n,n}\right| < V_0 \cdot \prod_{k=1}^{n}(1-\mu_{k,n}) \sim \left(\frac{1}{e}\right)^{\sigma_n} \to 0, \text{ if } \sigma_n = \sum_{k=1}^{n}\mu_{k,n} \to \infty$.

It is an easy matter to verify that the following criteria establish the basic inequality mentioned above:

Let $f(z) = f(x+iy) = U(x,y) + iV(x,y)$ and specify that for some interval on the positive real axis and for values of $|y|$ sufficiently small the following hold:

$$|U - x| \le c_1|y| \text{ and } |V| \le c_2|y|, \text{ with } c_1^{\,2} + c_2^{\,2} < 1.$$

Then $\quad |f(z) - x| \le \rho|z - x|$

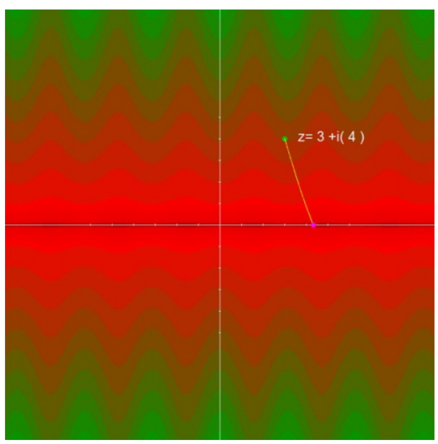

Figure 7: $f(x+iy) = (x+.3y) + i\left(.3ySin^2(x)\right)$, a non-conservative vector field, with each point on the positive real axis an attractor. $\eta_{k,n} = \frac{1}{\sqrt{n}}$ with $n = 10^6$ produces $G(2+4i) = 3.315$.

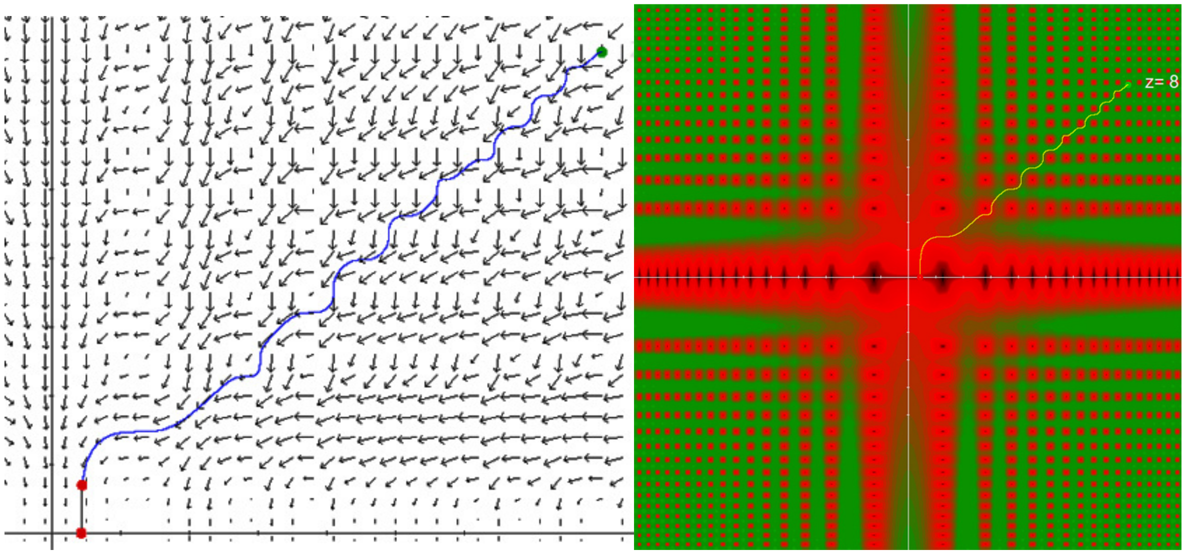

Figure 8: $f(z) = xCos(y^2) + iySin(x^2)$, $\eta_{k,n} = \frac{10}{n}$, $n = 100{,}000$. Z=8+7i. Blue contour ends at $Z(.3)$. Continuum of attractors on the real axis. Here, $\alpha = .4259 + 0i$ with $L(\gamma) \approx 11.66$ and $\oint_\gamma \varphi dz \approx 11.48 + 56.05i$.

Generalization of a continuum of attractors . . .

If one wishes to construct a vector field having curves other than portions of the real axis as attractors life can become a bit more complicated. Two conditions underlie much of what follows: (a) zeno contours must (ultimately) be normal to the curve of attractors, and (b) the condition $|f(z) - \alpha(z)| \le \rho |z - \alpha(z)|, 0 \le \rho < 1$ holds, where $\alpha(z)$ is the attractor for each z.

First, consider two simple cases:

(1) $C = \{z : x = y\}$ and (2) $C = \{z : |z| = 1\}$.

(1) Beginning with a point $z = x + iy$ off C, one easily finds that the vector normal to C terminates at $\alpha(z) = \left(\frac{x+y}{2}\right)(1+i) \in C$. Then, setting $f(z) = U + iV$, the slope of a vector perpendicular to C is -1 , leading to $V = -U + (x + y)$. Requiring the inequality in (b) be replaced by equality leads to $U = \frac{1}{2}\left((1+\rho)x + (1-\rho)y\right)$ and $V = \frac{1}{2}\left((1-\rho)x + (1+\rho)y\right)$.

(2) Setting $x = \cos t, y = \sin t$ and using simple geometry gives $U = x\left(1-\rho\left(1-\frac{1}{\sqrt{x^2+y^2}}\right)\right)$ and $V = y \cdot \frac{U}{x}$.

For other curvilinear contours of attractors, three equations should be satisfied. Write $C = \{Z(t) = X(t) + iY(t) : t \in I\}$, where it is assumed the curve C has tangent lines at all points. Thus for each $z \notin C$, $\alpha(z) = Z(t)$ for some t.

(i) $X'(t)\left(X(t)-x\right)+Y'(t)\left(Y(t)-y\right)=0$ (vector normal to C)

(ii) $X'(t)\left(U-x\right)+Y'(t)\left(V-y\right)=0$ (vector normal to C)

(iii) $\left(U-X(t)\right)^2+\left(V-Y(t)\right)^2 \le \rho^2\left[\left(X(t)-x\right)^2+\left(Y(t)-y\right)^2\right]$ (attractive force)

For even very simple attractor curves, like $C = \{Z(t) : X(t) = t, Y(t) = t^2\}$, the usefulness of these three equations might be questionable. However, a simple modification might prove just as appropriate: instead of having each vector approach its attractor normally, simply require the vector field to have a uniform orientation. In this particular instance, allow the vectors to approach the parabola along horizontal streamlines. Then the resulting function is $U = \sqrt{y} + \rho\left(x - \sqrt{y}\right)$ and $V = y$ in the first quadrant.

Behavior near a repelling fixed point . . .

In Figure 8 the contour snakes around a number of repelling fixed points on its way to an attractor. Under limited conditions this behavior can be anticipated. Assume that β is such a point for the function $f(z)$ and that, in a neighborhood of this point,

$|f(z)-\beta| \geq \rho|z-\beta|$ for $\rho > 1$. Furthermore assume that the repeller pushes the value of $f(z)$ into the half-plane defined by $|\theta| < \frac{\pi}{2}$ where $\theta = \arg\left(\frac{f(z)-\beta}{z-\beta}\right)$. Then if $\rho > Sec(\theta)$ we have $|g_{k,n}(z)-\beta| > |z-\beta|$. Therefore $|G_{n,n}(z)-\beta| > |z-\beta|$. This follows from drawing a simple picture in which the various vectors in the equation

$g_{k,n}(z)-\beta = (1-\eta_{k,n})(z-\beta)+\eta_{k,n}(f(z)-\beta)$ are examined. For analytic functions, if $|f'(\beta)| > 1$, writing $\left|\frac{f(z)-f(\beta)}{z-\beta}\right| = |f'(\beta)+\delta(z)| \geq |f'(\beta)|-|\delta(z)|$, $\delta(z) \to 0$ as $z \to \beta$, one sees that $|f(z)-\beta| \geq \rho|z-\beta|$, $\rho > 1$ if $z \in N_\varepsilon(\beta)$ for small enough ε.

Winding contour . . .?

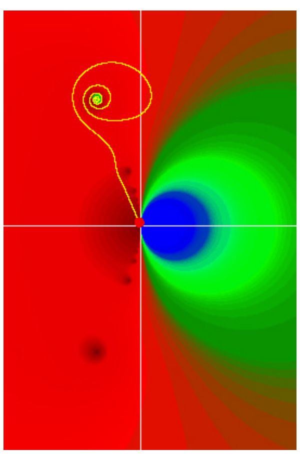

Figure 9: A phenomenon that occurs at some repelling fixed points. The following discussion concerns only one possibility. From $g_n(z) = z+\eta_n(f(z)-z) = z+\eta_n((f(z)-\beta)-(z-\beta))$ it can be seen that circular or moderate outward spiraling motion results from $\frac{\pi}{4} \leq \arg(f(z)-z)-\arg(z-\beta) \leq \frac{\pi}{2}$ or $\frac{\pi}{4} \leq \arg(f'(\beta)-1) \leq \frac{\pi}{2}$ for $z \approx \beta$. This condition is met if $1 \leq \operatorname{Re} f'(\beta) \leq 1+\operatorname{Im} f'(\beta)$. In the figure to the left $f(z) = e^{\frac{1}{z}}$ and a repelling fixed point exists at $\beta \approx -.072+.2i$, where $f'(\beta) \approx 1.39+4.40i$.

Locating an attractor using a Zeno contour . . .

Since the contours terminate at attractors under certain conditions, they can be used to find such attractors.

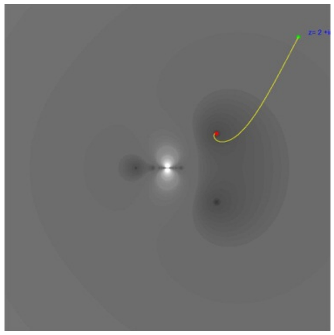

Figure 10: $f(z)=Cos\left(\frac{1}{z}\right)$, $\eta_{k,n}=\frac{1}{\sqrt{n}}$, $n=100,000$. [-2.5,2.5] $\alpha\approx.7493+.5224i$.

Locating a repeller using a reverse Zeno contour . . .

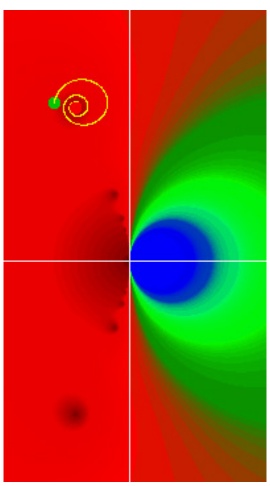

Figure 11: Sometimes it is possible to backtrack to locate a repelling fixed point by using a kind of reverse Zeno contour. That is to say, employ

$g_n(z)=z+\eta_n\left(z-f(z)\right)$ and compute $G_n(z)$ with $z\approx\beta$.

In the figure to the left ($f(z)=e^{\frac{1}{z}}$) the starting point is $z=-.1+.25i$ and the reverse contour spirals in toward the fixed point which is $\beta\approx-.071+.204i$. However, if one starts just a tad too far away this approach sends the contour out on a wild ride and nothing is gained.

Extending convergence to an attractor for vector fields $f_{k,n} \to f$ . . .

Suppose $\left|f_{k,n}(z) - f(z)\right| \le \sigma_{k,n} \to 0$ for relevant values of the variable ,$1 \le k \le n$, $n \to \infty$.

And, as before, $\left|f(z) - \alpha\right| \le \rho\left|z - \alpha\right|$ or its more general form. Start with

$$g_{k,n}(z) = z + \eta_{k,n}\varphi_{k,n}(z) = z + \eta_{k,n}\left(f_{k,n}(z) - z\right)$$, which can be expressed as

$$g_{k,n}(z) - \alpha = \left(1 - \eta_{k,n}\right)\left(z - \alpha\right) + \eta_{k,n}\left(f_{k,n}(z) - \alpha\right)$$

The vector/geometric argument advanced before gives

$$\left|g_{k,n}(z) - \alpha\right| \le \left(1 - \eta_{k,n}\left(1 - \rho\right)\right)\left|z - \alpha\right| + \eta_{k,n}\left|f_{k,n}(z) - f(z)\right|$$. Hence, setting

$G_n(z) = g_{n,n} \circ g_{n-1,n} \circ \cdots \circ g_{1,n}(z)$, $\left|g_{k,n}(z) - \alpha\right| \le \left(1 - \mu_{k,n}\right)\left|z - \alpha\right| + \eta_{k,n}\sigma_{k,n}$, $\mu_{k,n} = \eta_{k,n}\left(1 - \rho\right)$
gives

$$\left|G_n(z) - \alpha\right| \le \prod_{k=1}^{n}\left(1 - \mu_{k,n}\right) \cdot \left|z - \alpha\right| + 1 \cdot \sum_{k=1}^{n}\eta_{k,n}\sigma_{k,n} ,$$ Provided $\eta_{k,n} < \dfrac{1}{1 - \rho}$.

As before, $\prod_{k=1}^{n}\left(1 - \mu_{k,n}\right) \sim \left(\dfrac{1}{e}\right)^{\sum_{k=1}^{n}\mu_{k,n}}$. Therefore, requiring $\sum_{k=1}^{n}\eta_{k,n}\sigma_{k,n} \to 0$ and $\sum_{k=1}^{n}\mu_{k,n} \to \infty$ leads to convergence (at least heuristically!)

Example: $\eta_{k,n} = c_n \dfrac{k}{n^2}$, $c_n = \sqrt{n}$, $\sigma_{k,n} = \dfrac{1}{n}$.

Inner vs outer iterations . . .

Up to this point the compositional structure of the Zeno contour or sequence has followed the pattern $G_{k,n}(z) = g_{k,n} \circ g_{k-1,n} \circ \cdots \circ g_{1,n}(z)$, an *outer* or *backward* composition. This follows the Zeno's Arrow illustration by moving from a point to the next point by adding a small increment. However, in applications in the analytic theory of continued fractions, going the other way is more appropriate. That is to say, consider $F_{k,n}(z) = g_{1,n} \circ g_{2,n} \circ \cdots \circ g_{k,n}(z)$, a *forward* or *inner* composition [6].

When $g_{k,n}(z) = z + \frac{c_n}{n}\varphi(z)$ there is no difference in the two schemes, but in more complicated scenarios there may be; for instance $g_{k,n}(z) = z + \frac{c_n k}{n^2}\varphi(z)$ produces differing expansions. Nevertheless, the arguments advanced before for outer iteration suffice for inner iteration. That is to say, when $|f(z) - \alpha| \le \rho |z - \alpha|$ then both $G_{n,n}(z) \to \alpha$ and $F_{n,n}(z) \to \alpha$. The two generated contours separate for small values of n, but coalesce as n becomes larger.

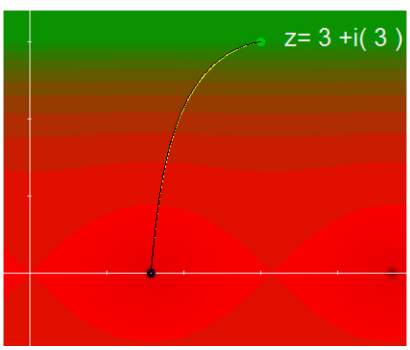

Figure 12: $f(z) = z + Cos(z)$, $g_{k,n}(z) = z + \frac{c_n k}{n^2} Cos(z)$

Both the yellow contour $G_{n,n}(z) \to \alpha$ and the black contour $F_{n,n}(z) \to \alpha$ where $\alpha = 1.57079... = \frac{\pi}{2}$.

$n = 100,000$ and $c_n = \sqrt{n}$.

Premature termination of Zeno contours . . .

It has been nominally established that $\eta_{k,n} = \frac{1}{\sqrt{n}}$ forces the contour to terminate at an attractor, while $\eta_{k,n} = \frac{C}{n}$ *seems* to cause the contour to terminate early. The following informal discussion supports this conclusion: Suppose $f(z) - \alpha = \rho(z)(z - \alpha)$,where $|\rho(z)| < 1$.

From $g_n(z) - \alpha = \left(1 - \frac{C}{n}\right)(z - \alpha) + \frac{C}{n}\left(f(z) - \alpha\right)$ one obtains $g_n(z) - \alpha = \left(1 - \sigma_n(z)\right)(z - \alpha)$ with $\sigma_n(z) = \frac{C}{n}\left(1 - \rho(z)\right)$. Repeated application, using $z_k = z_{k-1} + \frac{C}{n}\left(f(z_{k-1}) - z_{k-1}\right)$, provides

$z_n - \alpha = (z - \alpha)\prod_{k=0}^{n-1}\left(1 - \sigma_n(z_k)\right)$. Write

$$\left(1 - \sigma_n(z_k)\right) = \left[\left(1 - \frac{1}{1/\sigma_n(z_k)}\right)^{1/\sigma_n(z_k)}\right]^{\frac{C}{n}(1-\rho(z_k))} \sim \left(\frac{1}{e}\right)^{\frac{C}{n}(1-\rho(z_k))} \qquad \text{so that}$$

$G_n(z) = z_n \approx \alpha + (z - \alpha)\cdot\left(\frac{1}{e}\right)^{C(1-s_n)}$ where $s_n = \frac{1}{n}\sum_{k=0}^{n-1}\rho(z_k) \sim \rho(z)$, an average of sorts.

Then $G_n(z) = z_n \approx \alpha + (z - \alpha)\cdot\left(\frac{1}{e}\right)^{C(1-\rho(z))}$ and the Zeno contour stops short of the attractor.

These are merely imprecise ramblings, of course.

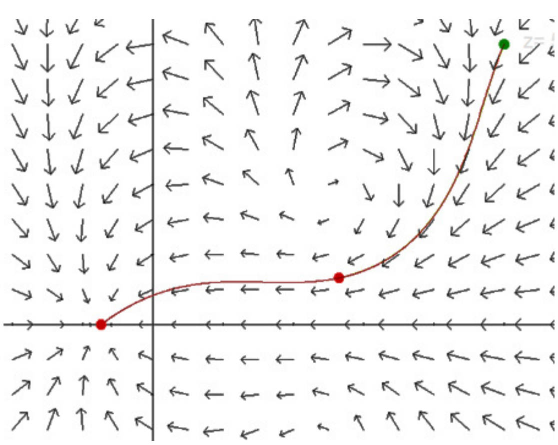

Figure 13: $f(z) = -Cos(z)$. One contour, $\eta_{k,n} = \frac{1}{\sqrt{n}}$, terminates at the attractor $\alpha = -.7391...$, while the second, $\eta_{k,n} = \frac{C}{n}$, C=1, ends prematurely. Both contours follow a streamline closely.

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