Boundary Observations of Continued Fractions

[1]

Haakon Waadeland

Norwegian University of Science and Technology, Trondheim

1. Introduction

In complex analysis boundary values are essential. Analytic or harmonic functions are completely defined in proper regions from the values on the boundary. Do we have related properties for continued fractions

$$\mathrm{K}(a_n/1), \tag{1.1}$$

where all a_n are in a given region E? In the present paper we shall see some cases where we have affirmative answers. In one of the cases we shall go deeper into the topic to dig for more information.

2. Background material

As a start we state three basic results on convergence and values of continued fraction.

i) Parabola Theorem:

Let P be the parabolic region defined by

$$|w| - \Re(w) \leq 1/2 \tag{2.1}$$

and H the half plane defined by

$$\Re(w) \geq -1/2. \tag{2.2}$$

Then, any continued fraction (1.1) with all a_n in (2.1), $a_n \neq 0$, $\quad a_n < M$ for some $M > 0$ converges, and the value is in (2.2).

The above theorem is a very simplified version of the Parabola Theorem, as presented in e.g. [2,p.98–107], [4,p.130-31], [5]. Another property, not included in these versions is the following:

Any point in the half plane (2.2), except for the origin, is the value of some continued fraction with all a_n in (2.1).

---

[1]Revised version 15 Oct 2010

In the following C is a complex number with $\Re(C) > -1/2$, and R is a positive number with $0 < R < |1+C|$.

ii) Oval Theorem:

Let O be the region bounded by a cartesian oval, defined by

$$|w(1+\overline{C}) - C(|1+C|^2 - R^2)| + R|w| \le R(|1+C|^2 - R^2), \tag{2.3}$$

let D be the disk, defined by

$$|w - C| < R. \tag{2.4}$$

Then any continued fraction (1.1) with all a_n in (2.3) converges, and the value is in (2.4). All points z in (2.4), $z \neq 0$, are values of such a continued fraction.

This is a simplified version of the Oval Theorem, and moreover the Oval Sequence Theorem, as presented in [1], [2,p.67], [4,p.141 and p.145] .

iii) Worpitzky's Theorem:

Let W be the disk, defined by

$$|w| \le 1/4, \tag{2.5}$$

and V the disk, defined by

$$|w| \le 1/2. \tag{2.6}$$

Then any continued fraction with all a_n in W converges, and the value is in V. All points in V, except for the origin, are values of some continued fraction (1.1) with all a_n in W. [2,p.94], [4,p.35], [9].

The last part was not contained in Worpitzky's original version.

3. Over to the boundary

In the papers [6], [7] is raised the question about what will the set of values be, if we restrict the set of a_n- values to the boundary of respectively P, O and W. The question is solved and illustrated in all three cases:

In the (parabolic region/half plane)-case we get the same half plane as before, i. e. the restriction to the boundary does not make any difference as far as set of values is concerned.

In the Worpitzky case we get an annulus $1/6 \le |w| \le 1/2$, and in the oval case we get a disk with a hole (not concentric, as in the Worpitzky case).

In [7] it is also seen and stated that the oval case in a certain sense acts as a bridge between the parabola case and the Worpitzky case. Take C to be ≥ 0, and study the variation in the picture for varying values of C: For $C = 0$ and $R \leq 1/2$ the oval is a circle, and we have the Worpitzky case, where the picture is the annulus mentioned above. For $C \to \infty$ the hole shrinks towards the origin, and the disk approaches the half plane .

We shall in the rest of the paper restrict ourselves to the disk case/ circle case. We state the result from [8] as follows:

The set of values for the family of all continued fractions (1.1) with $|a_n| = 1/4$ is the annulus given by

$$1/6 \leq |w| \leq 1/2. \tag{3.1}$$

More generally, also from [8]:

Let ρ be a positive number, $\leq 1/2$. Then the set of values for the family of all continued fractions (1.1) with $|a_n| = \rho(1-\rho)$ is the annulus given by

$$\frac{\rho(1-\rho)}{1+\rho} \leq |w| \leq \rho. \tag{3.2}$$

As described in [8] an attempt to illustrate (3.1) gave an annulus, centered around the origin, but a more narrow one, in particular because of lack of points w where $|w|$ is near 1/2. A closer investigation, including probabilistic considerations, offered an explanation: For any $N \geq 4$ it was proved that

$$prob\{(1/2 - 1/(2N+3)) \leq |f| \leq 1/2)\} \leq .0021 \cdot (.1430)^{N-4}, \tag{3.3}$$

or, for $N = 9$

$$prob\{(1/2 - 1/21) \leq |f| \leq 1/2)\} \leq 1.25 \cdot 10^{-7}.$$

This explains the narrow annulus. Moreover, as mentioned in [8], this is a rather rough result. The inequality would still hold for a much smaller value on the right hand side. We shall not go into that, only refer to the experiment, leading to Fig. 1, which is produced by MATLAB: There are 500 points, equally spaced on the unit circle. Plotted are the approximants of order 10^6 of a large number of continued fractions with $|a_n| = 1/4$, picked at random from the 500 ones, multiplied by 1/4.

Fig 1 illustrates (3.3) as well as the roughness.

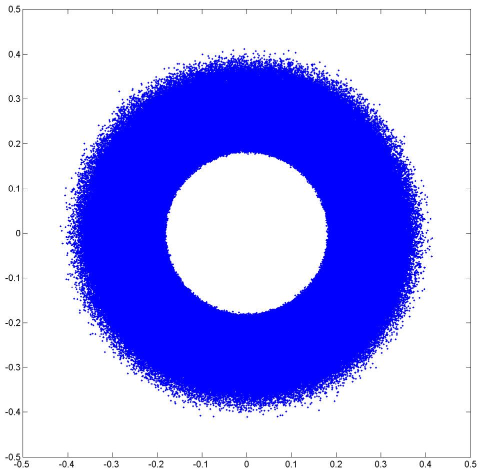

Fig.1

The picture is somewhat different for ρ-values $< 1/2$, meaning that $\rho = 1/2$ represents an exceptional case. Using the same argument and method as in [8] we find, after some computation, that the left hand side of (3.3) has to be replaced by

$$prob\{\rho - \frac{2\rho(1-2\rho)}{((1-\rho)/\rho)^{(n+1)} - 2\rho}, \rho\}. \tag{3.4}$$

For $\rho \to 1/2$ (3.4) approaches the left hand side of (3.3). The right hand side is not explicitely known, except for knowing that it has the same or a larger order of magnitude as the one in (3.3).

Fig. 2 is produced by MATLAB in exactly the same way as Fig. 1, except for having $|a_n| = \rho(1-\rho)$ with $\rho = 1/3$, i. e. $|a_n| = 2/9$. The interval in (3.3) will for $\rho = 1/3, N = 9$ in (3.4) be replaced by

$$\left[\frac{1}{3} - \frac{1}{4605}, \frac{1}{3}\right].$$

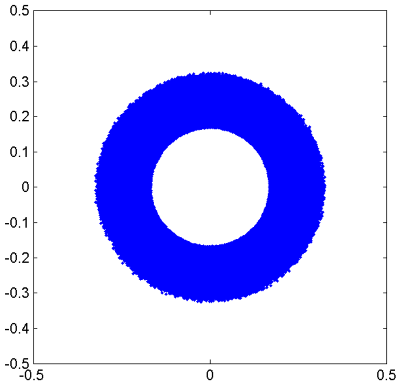

Fig.2

There is no visible "empty" annulus $1/3 - \varepsilon < |z| < 1/3$ and almost no "fringes".

4. Two points on the boundary

Rather than having the whole circle as element region proper subsets may also be of interest. The "opposite" case to the whole circle is, next to the trivial one-point case, two point sets. We omit a general discussion and restrict the element set to the following:

$$\{\frac{-i}{4}, \frac{i}{4}\}$$

In order to pick at random in the two-point set we use from Maple $(-1)^{rand}$. Then $+1$ and -1 come at random, both with probability 1/2. The continued fractions to be studied here are (1.1) with

$$a_n = \frac{i}{4}(-1)^{rand} \qquad (4.1).$$

In the illustration below, Fig. 3, all approximant of order 128 from 128 continued fractions with (4.1) are plotted as small circles. Further experiments indicate strongly that the picture is close to the set of values we want.

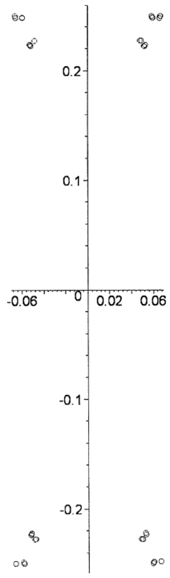

Fig.3

We shall show the first steps in the process leading towards the set of values of all continued fractions with a_n as in (4.1). In the process we go from one continued fraction (z) with all a_n as in (4.1) to another one (w) in the same family by

$$w = (i/4)/(1+z) \quad or \quad w = (-i/4)/(1+z). \tag{4.2}$$

Here $1/6 < |z| < 1/2$, i. e.

$$1/6 < |i/(4w) - 1| < 1/2, \tag{4.3}$$

leading to

$$|w - i/3| < 1/6 \qquad |w - 9i/35| > 3/70, \tag{4.3}$$

and the same with i replaced by $-i$. An illustration in the complex plane follows. The continued fraction values are in the sheded part of the illustration.

The process is to repeat this. The next step is to replace w by z in (4.3) (also (4.3) with i replaced by $-i$. Next replace z by $i/(4w) - 1$ and $-1//4w) - 1$.

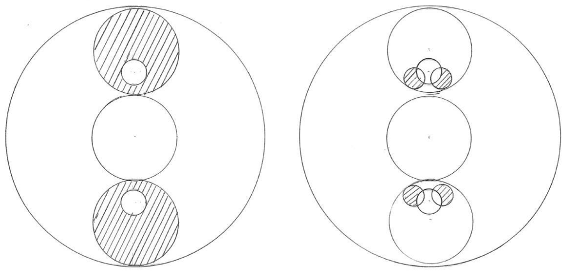

Fig.4

Illustration is presented above. In each of the four sheded disks there should have been a hole. It is skipped here for practical reasons.

Let D_0 denote the domain in Fig.1, i.e. the annulus $1/6 \leq |z| \leq 1/2$, and D_1 the domain in Fig.4, left part, D_2 the domain in Fig.4, right part . We then have:

$$D_1 = ((i/4)/(1+D_0)) \cup (-i/4)/(1+D_0)), D_2 = ((i/4)/(1+D_1)) \cup (-i/4)/(1+D_1)), \tag{4.4}$$

and generally

$$D_{k+1} = ((i/4)/(1+D_k)) \cup (-i/4)/(1+D_k)). \tag{4.5}$$

Already $k = 1$, i.e. the illustration Fig.4, right part , supports the conjecture that the set in Fig.3 is the set of values for the family of continued fractions with a_n as in (4.1), and more so higher values of k in (4.5) .

5. A final example

The cases discussed in Sections 3 and 4 are in a way extreme, the first one with a large number of points on the circle, simulating a continuous uniform distribution on the circle itself, the other one with only two points. The cases with an arbitrary number > 2, may also be of interest. Out of the many experiments carried out in the process of writing this paper we show here only one, in which we have 7 points. We shall not discuss this or other cases, only refer to the illustration in Fig. 5, included merely to show an illustration of a case between the two extreme ones.

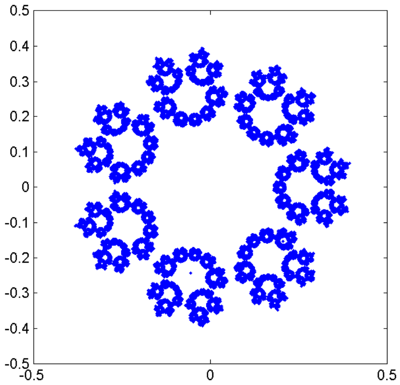

Fig.5

REFERENCES

1. L. Jacobsen and W.J. Thron, *Oval convergence regions and circular limit regions for continued fractions $K(a_n/1)$, Analytic theory of continued fraction II* (W.J. Thron. ed), Lecture Notes in Mathematics **1199**, Springer-Verlag, Berlin (1986), 90–126.

2. W.B. Jones and W.J. Thron, *Continued fractions: Analytic theory and applications.* Encyclopedia of Mathematics and its Applications 11, Addison-Wesley (now available from Cambridge University Press),1980.

3. W.B. Jones, W.J. Thron and H. Waadeland, *Value regions for continued fractions $K(a_n/1)$ whose elements lie in parabolic regions*, Math.Scand.**56**(1985), 5–14.

4. L. Lorentzen and H. Waadeland, *Continued fractions with applications*, in C. Brezinski, L, Wuytack (Eds.), Studies in Computational Mathematics, vol 3, North-Holland. Amsterdam, London, New York, Tokyo, 1992.

5. W.J. Thron, *On parabolic convergence regions for continued fractions*, Math. Z. **69** (1958), 173–182.

6. H. Waadeland, *Boundary versions of Worpitzky's theorem and of parabola theorems, Analytic theory of continued fraction III* (L.Jacobsen, ed), Lecture Notes in Mathematics **1406**, Springer-Verlag, Berlin, (1989), 135–142.

7. H. Waadeland, *On continued fractions $K(a_n/1)$, where all a_n are lying on a cartesian oval*, Rocky Mountain Journal of Mathematics, Volume 22, Number 3, Summer 1992.

8. H. Waadeland, *Some probabilistic remarks on the boundary version of Worpitzky's Theorem*, Orthogonal Functions, Moment Theory and Continued Fractions, Lecture Notes in Pure and Applied Mathematics, volume 199, (W.B. Jones, A.Sri Ranga eds.), Marcel Dekker Inc., New York, Basel, Hong Kong, 1998, 409–416.

9. J.D. Worpitzky, *Untersuchung über die Entwickelung der monodromen und monogenen Funktionen durch Kettenbrüche*, Friedrichs-gymnasium und Realschule Jahresbericht, Berlin 3–39 (1865).