A Space of Siamese Contours in Time-dependent Complex Vector Fields

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Abstract: Siamese Contours – having the same initial points - arise in two or more time-dependent complex vector fields and can be combined into simple sums and products that incorporate the features of the vector fields. These elementary classroom notes illustrate the processes.

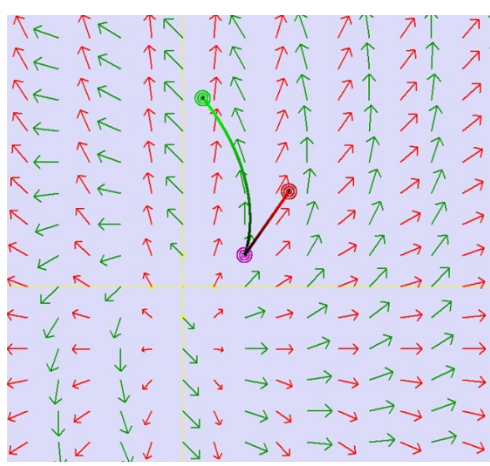
Imagine the following scenario: We know the various paths a particle in the complex plane would follow over one time-unit due to individual underlying vector fields. How can we combine those paths into a single path the particle will actually follow? Answering this simple question, one is led to a certain vector space having sums, scalar products, products, projection products, and compositions easily defined, correlated with a simple norm and metric.

Zeno (or equivalent parametric) contours are defined algorithmically as a distribution of points $\{z_{k,n}\}_{k=0}^{n}$ by the iterative procedure: $z_{k,n} = z_{k-1,n} + \eta_{k,n} \cdot \varphi\left(z_{k-1,n}, \frac{k}{n}\right)$ which arises from the following composition structure:

$$G(z) = \lim_{n\to\infty} G_{n,n}(z) \ , G_{k,n}(z) = g_{k,n}\left(G_{k-1,n}(z)\right) \ , \ g_{k,n}(z) = z + \eta_{k,n} \cdot \varphi(z, \tfrac{k}{n}) \ , G_{1,n}(z) = g_{1,n}(z) \ .$$

Usually $\eta_{k,n} = \dfrac{1}{n}$, providing a partition of the *unit time interval*. $\gamma(z)$ is the continuous arc from z to $G(z)$ that results as $n \to \infty$ (*Euler's Method* is a special case of a Zeno contour).

When $\varphi(z,t)$ is well-behaved an equivalent closed form of the contours , $z(t)$, has the property $\dfrac{dz}{dt} = \varphi(z,t)$, with vector field $f(z,t) = \varphi(z,t) + z$. *Siamese contours* are streamlines or pathlines joined at their origin and arising from different vector fields.



Contours will be abbreviated, using the iterative algorithm, as

(i) $\gamma_1 : z_{k,n} = z_{k-1,n} + \eta_{k,n}\left(f_1(z_{k-1,n},\tfrac{k}{n}) - z_{k-1,n}\right)$ and $\gamma_2 : z_{k,n} = z_{k-1,n} + \eta_{k,n}\left(f_2(z_{k-1,n},\tfrac{k}{n}) - z_{k-1,n}\right)$

Or $\gamma_1 : \dfrac{d\gamma_1}{dt} = \varphi_1(\gamma_1, t)$ and $\gamma_2 : \dfrac{d\gamma_2}{dt} = \varphi_2(\gamma_2, t)$

A parametric form of $\gamma_m(z)$: $z = z(t)$, exists when the equation $\dfrac{dz}{dt} = \varphi_m(z,t)$ admits a closed solution. For example $\gamma_1 : z_1(t) = z_0 e^{t+2it^2} \Rightarrow \varphi_1(z,t) = z(1+4it)$.

If a point in the plane is moved simultaneously by infinitesimal actions of two vector fields - such as two force fields acting on a particle - the combined action is given by the following, in which contours *represent* vector fields:

A commutative ***Contour Sum***: $\gamma = \gamma_1 \overset{\leftrightarrow}{\oplus} \gamma_2$ where

(ii) $\gamma : z_{k,n} = z_{k-1,n} + \eta_{k,n}\left(F(z_{k-1,n},\tfrac{k}{n}) - z_{k-1,n}\right)$, $F(z,t) = f_1(z,t) + f_2(z,t) - z$.

Or $\gamma : \dfrac{d\gamma}{dt} = \varphi(\gamma,t) = \varphi_1(\gamma,t) + \varphi_2(\gamma,t)$. It is assumed that contours begin at the same point of origin: z_0, defining a *Siamese contour space*. In the following images vector clusters represent vector fields over the unit time interval, from black (t=0) to light color (t=1).

Suppose that we are given two parametrically-defined contours:

<u>*Example 1*</u> (One VF is time-dependent) $\gamma_1 : z_1(t) = z_0(t+1) + it^2$ and $\gamma_2 : z_2(t) = \dfrac{z_0}{1 - tz_0}$.

How do we find $\gamma = \gamma_1 \overset{\leftrightarrow}{\oplus} \gamma_2$? Not by simple addition: $z_3(t) = z_0(t+1) + it^2 + \dfrac{z_0}{1-tz_0}$.

Rather, we first find the vector fields f_1, f_2 in which these contours are embedded:

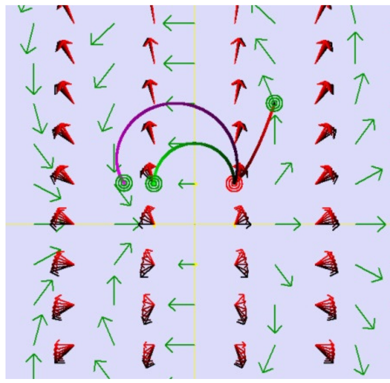
$\dfrac{dz_1}{dt} = \dfrac{z + it(t+2)}{t+1} = \varphi_1(z,t) = f_1(z,t) - z$ and $\dfrac{dz_2}{dt} = z^2 = \varphi_2(z,t) = f_2(z,t) - z$,

then solve $\dfrac{dz}{dt} = \varphi_1(z,t) + \varphi_2(z,t) = \dfrac{(t+1)z^2 + z + it(t+2)}{t+1}$, or apply the algorithm:

$$\gamma: z_{k,n} = z_{k-1,n} + \eta_{k,n}\left(\varphi_1(z_{k-1,n}, \tfrac{k}{n}) + \varphi_2(z_{k-1,n}, \tfrac{k}{n})\right)$$

$$= z_{k-1,n} + \eta_{k,n}\left(\frac{{z_{k-1,n}}^2(\tfrac{k}{n}+1) + z + i\tfrac{k}{n}(\tfrac{k}{n}+2)}{\tfrac{k}{n}+1}\right)$$

$$= z_{k-1,n} + \eta_{k,n}\left(\frac{{z_{k-1,n}}^2(k+n) + z\cdot n + i\tfrac{k}{n}(k+2n)}{k+n}\right)$$

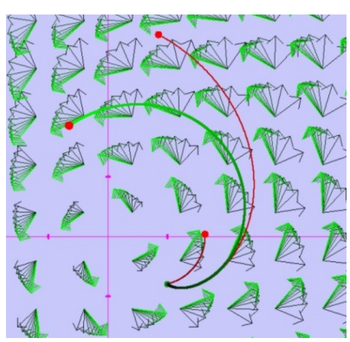


The two vector fields are shown in red (time-dependent VF) and green (normal VF), and the contour sum is in purple: $z_0 = 1+i$.

As seen above, it may be that $\frac{dz}{dt} = \varphi_1(z,t)$ and $\frac{dz}{dt} = \varphi_2(z,t)$ each have simple closed solutions $z_1(t)$ and $z_2(t)$, whereas $\frac{dz}{dt} = \varphi_1(z,t) + \varphi_2(z,t)$ is intractable.

Example 2 (Both VFs are time-dependent) $\gamma_1 : z_1(t) = z_0 e^{t+2it^2} \Rightarrow \varphi_1(z,t) = z(1+4it)$ and

$\gamma_2 : z_2(t) = \frac{2z_0}{2 - z_0 it^2} \Rightarrow \varphi_2(z,t) = z^2 it$. Here $\frac{dz}{dt} = \varphi(z,t) = z(1+4it) + z^2 it$ admits no easy solution. However, as seen in *Example 1*, the corresponding Zeno contour provides a constructive approach: $\gamma: z_{k,n} = z_{k-1,n} + \frac{1}{n}\left(z_{k-1,n}(1+4i\tfrac{k}{n}) + z_{k-1,n}^2 i\tfrac{k}{n}\right)$,



The TDVF of the *sum* (shown) is $F(z,t)=2z+4zit+z^2it$. It is tempting to assume the resultant contour (in green) is somehow an average or a sum of the two parametric curves (in red), but that is not, in general, the case. The sum $\gamma=\gamma_1 \ddot{\oplus} \gamma_2$ represents a path wherein each point is moved forward by a simultaneous infinitesimal application of the two (or more) vector fields and depends less on the original contours and more on the VFs underlying them. But the sum of contours notation is useful.

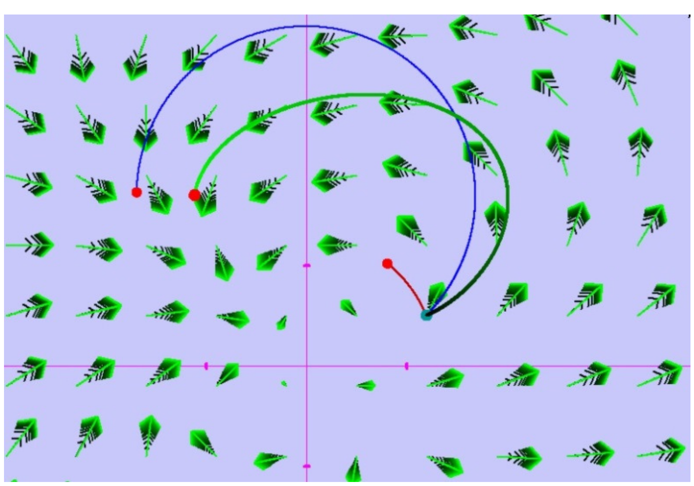
If two (or more) TDVFs admit parametric formulations of their contours, and $\frac{dz}{dt}=\varphi(z,t)=\varphi_1(z,t)+\varphi_2(z,t)$ is solvable in closed form $z(t)$, then it may be possible to demonstrate how $z(t)$ and $z_1(t)$, $z_2(t)$ are related, as seen in this next example:

<u>*Example 3*</u> $f_1(z,t)=z+izt$, $f_2(z,t)=z+2z^2t \Rightarrow \frac{dz}{dt}=\varphi(z,t)=2z^2t+izt$. Thus $z_1(t)=z_0e^{it^2/2}$ and $z_2(t)=\frac{z_0}{1-t^2z_0}$, with $z(t)=\frac{i\omega_0 e^{it^2/2}}{1-2\omega_0 e^{it^2/2}}$, $\omega_0=\frac{z_0}{2z_0+i}$.

Tedious algebra reveals:

$$z(t)=\frac{iz_1(t)\left(1+t^2z_2(t)\right)}{z_2(t)\left(2+it^2-2t^2z_1(t)\right)+i-2z_1(t)}.$$

$z_1(t)$ is red, $z_2(t)$ is blue, and the contour $z(t)$ is in green.



Continuing with algebraic analogies, we define a ***Scalar Product***:

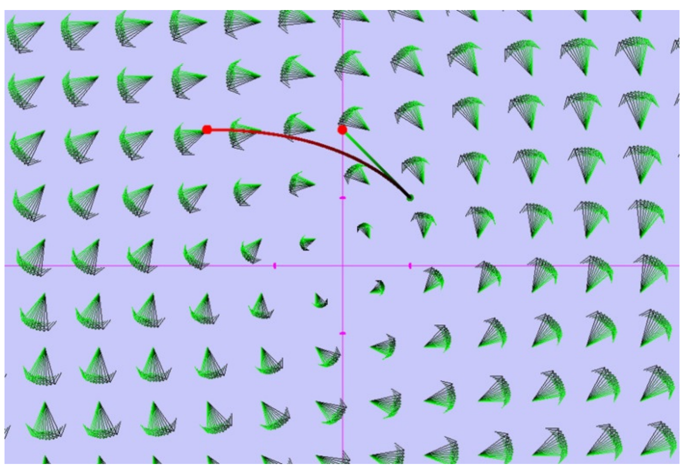
$$\gamma = \alpha \odot \gamma_1 : \ \frac{d\gamma}{dt} = \alpha \cdot \varphi_1(\gamma, t) ,$$

That provides a distributive feature:

$$\gamma = \alpha \odot (\gamma_1 \ddot{\oplus} \gamma_2) = (\alpha \odot \gamma_1) \ddot{\oplus} (\alpha \odot \gamma_2) \ : \ \frac{d\gamma}{dt} = \alpha(\varphi_1(\gamma,t) + \varphi_2(\gamma,t)) = \alpha\varphi_1(\gamma,t) + \alpha\varphi_2(\gamma,t)$$

<u>*Example 4*</u> $\gamma : z(t) = z_0(1+\mathrm{it}) \ \Rightarrow \ \varphi_1(z,t) = \frac{iz}{1+it}$. Thus $\alpha\gamma : z(t) = z_0(1+\mathrm{it})^{\alpha}$

For α=2, $z_0 = 1+i$. The γ- TDVF is $f(z,t) = z \cdot \frac{1+i(1+t)}{1+it}$. γ is green and $\alpha\gamma$ is red.



A ***Contour Product*** is defined: $\gamma = \gamma_1 \ddot{\otimes} \gamma_2 : \frac{d\gamma}{dt} = \varphi_1(\gamma,t) \cdot \varphi_2(\gamma,t)$, from which one derives

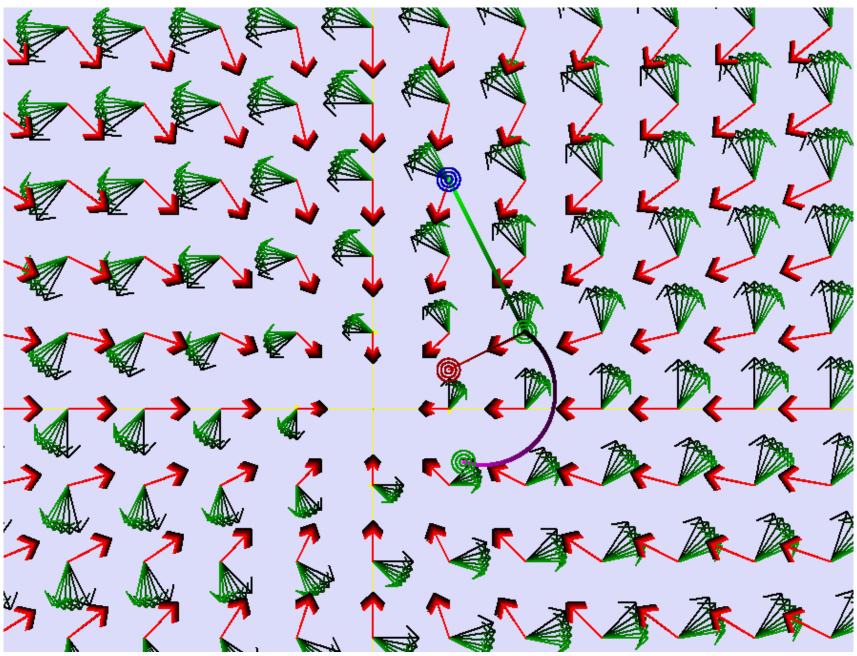
$$\gamma = (\alpha \odot \gamma_1) \ddot{\otimes} \gamma_2 = (\alpha \odot \gamma_2) \ddot{\otimes} \gamma_1$$

Example 5 $\gamma_1 : z(t) = z_0(1+\text{it}) \;\Rightarrow\; \varphi_1(z,t) = \dfrac{iz}{1+it} \Rightarrow f_1(z,t) = z\left(\dfrac{1+i(1+t)}{1+it}\right)$,

$\gamma_2 : z(t) = \dfrac{z_0}{1+t} \;\Rightarrow\; \varphi_2(z,t) = \dfrac{-z}{1+it} \Rightarrow f_2(z,t) = \dfrac{zt}{1+t}$. Therefore

$\gamma : \dfrac{dz}{dt} = \dfrac{-iz^2}{(1+it)(1+t)} = \varphi(z,t)$ and $\gamma : z(t) = \dfrac{z_0}{1+\omega(t)}$, $\omega(t) = z_0 \cdot \dfrac{1-i}{2} Ln\left(\dfrac{1+it}{1+t}\right)$.

$z_0 = 2+i$. Contour (1) is green, contour (2) is red, the product contour is purple.



Another concept is the ***Projection Product*** of two contours: Suppose a point is moving along a particular pathline arising from a TDVF $f_1(z,t)$ and a secondary vector or force field $f_2(z,t)$ is applied to the point in such a way that only the *projection* of infinitesimal vectors from the secondary field apply to the point. Let γ_1 and γ_2 be two contours (primary and secondary):

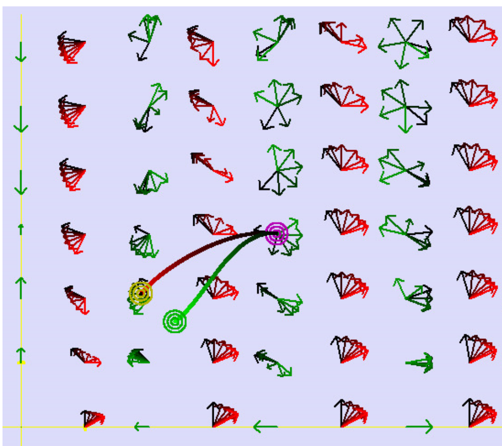
$$\gamma_1 : z_{k+1,n} = z_{k,n} + \eta_{k,n}\varphi_1(z_{k,n}, \tfrac{k}{n}) \quad , \quad \gamma_2 : z_{k+1,n} = z_{k,n} + \eta_{k,n}\varphi_2(z_{k,n}, \tfrac{k}{n})$$

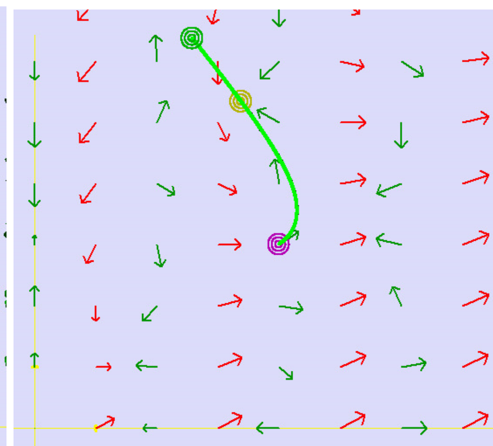
Define a *non-commutative* ***Projection Product:***

$\gamma = \gamma_1 \overleftarrow{\otimes} \gamma_2 : z_{k+1,n} = z_{k,n} + \eta_{k,n} \cdot \varphi_1(z_{k,n}, \tfrac{k}{n}) \cdot \left(1 + \lambda(z_{k,n}, \tfrac{k}{n})\right)$,where $\lambda(z_{k,n}, \tfrac{k}{n}) = \dfrac{\varphi_1 \bullet \varphi_2}{\varphi_2 \bullet \varphi_2}$. Thus

$$\frac{dz}{dt} = \varphi_1(z,t)\left(1 + \lambda(z,t)\right) \quad \text{. . . rarely solvable in closed form.}$$

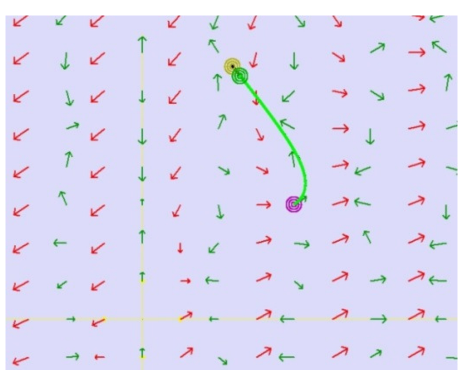
Example 6 Suppose $\gamma_1 : z(t) : f_1(z,t) = x(Cos(x+yt)+1) + iy(Sin(y+xt)+1)$ and $\gamma_2 : z(t) : f_2(z,t) = (2xt + x - y) + i(x - yt + y)$, $z_0 = 4+3i$, defined by Zeno contours. (Purple rings around initial point, green contour is primary, red contour is the *projection product contour* ending in yellow rings. The green vector clusters are primary VF, red clusters are secondary VF.)





The first image is of the paths in the TDVFs The second image is of the stable VFs, t=1

In a stable VF the projection product contour lies on a streamline, but the pathlines diverge in a TDVF. If one has control over the secondary VF - an on-off switch, say – then a simple adjustment assures only a positive boost: Set $\lambda(z_{k,n}, \tfrac{k}{n}) = 0$ if $\varphi_1(z_{k,n}, \tfrac{k}{n}) \circ \varphi_2(z_{k,n}, \tfrac{k}{n}) < 0$:

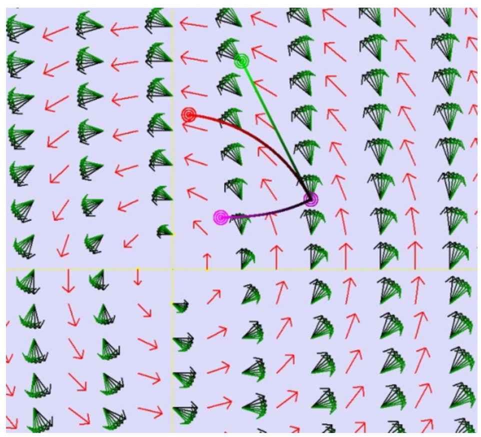


Define ***Contour Composition:*** $\gamma=\gamma_1\circ\gamma_2:\quad z_{k+1,n}=z_{k,n}+\eta_{k,n}\cdot\varphi_1(\varphi_2(z,t),t)$ or $\dfrac{dz}{dt}=\varphi_1\circ\varphi_2$

Example 7 $\quad\gamma_1:z(t)=z_0(1+it)\ \Rightarrow\ \varphi_1=\dfrac{iz}{1+it},\quad \gamma_2:z(t)=z_0e^{it}\ \Rightarrow\ \varphi_2=iz$,

$\varphi=\varphi_1\circ\varphi_2=\dfrac{-z}{1+it}$. Therefore

$$\gamma=\gamma_1\circ\gamma_2:z(t)=z_o\left(e^{-Arc\tan(t)}Cos(Ln\sqrt{1+t^2})+ie^{-Arc\tan(t)}Sin(Ln\sqrt{1+t^2})\right)$$



γ_1 is green, γ_2 is red, and $\gamma=\gamma_1\circ\gamma_2$ is purple. $z_o=4+2i$

The relation between a contour $z=z(t)$ and its vector field $\mathbb{F}:f(z,t)$ has been discussed. Here are a few simple correspondences. $\alpha(0)=0$ and $\beta(0)=0$.

(1) $\quad z(t)=z_0+\alpha(t)\quad\leftrightarrow\quad f(z,t)=z+\alpha'(t)$

(2) $\quad z(t)=z_0e^{\alpha(t)}\quad\leftrightarrow\quad f(z,t)=z\left(1+\alpha'(t)\right)$

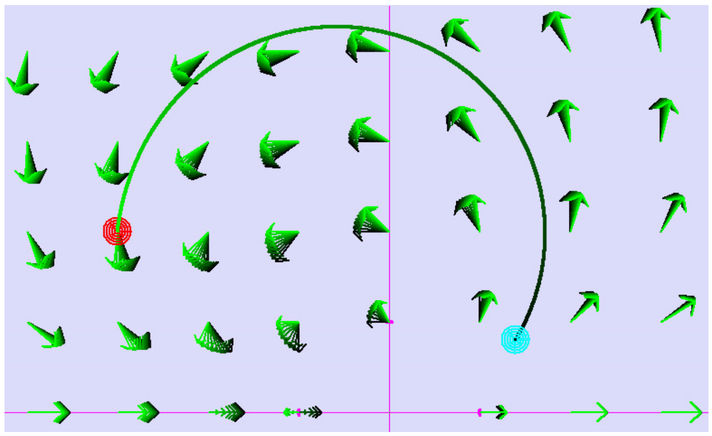
(3) $\quad z(t)=z_0\left(1+\alpha(t)\right)\quad\leftrightarrow\quad f(z,t)=\dfrac{z}{1+\alpha(t)}\left(1+\alpha(t)+z\alpha'(t)\right)$

(4) $\quad z(t)=\dfrac{z_0+\alpha(t)}{1-z_0\beta(t)}\quad\leftrightarrow\quad f(z,t)=z+\dfrac{z^2\beta'(t)+z\left(\alpha'(t)\beta(t)-\alpha(t)\beta'(t)\right)+\alpha'(t)}{1+\alpha(t)\beta(t)}$

(5) $\quad z(t)=\dfrac{z_0\alpha(t)}{1-z_0\beta(t)}\quad\leftrightarrow\quad f(z,t)=\dfrac{z^2\alpha(t)\beta'(t)+z\left(\alpha^2(t)+\alpha(t)\alpha'(t)\right)}{\alpha^2(t)}$, $\alpha(0)=1$

Example 8 $f(z,t)=\frac{z}{1+t^2}\left(z+(t+1)^2\right) \quad \leftrightarrow \quad z(t)=\frac{z_0(1+t^2)}{1-z_0 t}$.

The *floating vessel problem*: given a target point $z(1)$, find an initial point $z(0)$ for $z(1)=-3+2i$. Thus $z(0)=\frac{7}{5}+\frac{4}{5}i$ from (5)



A *norm* $\|\gamma\|$ of a contour in a z_0- based space can be formulated as $\|\gamma\|=\underset{t\in[0,1]}{Sup}|\varphi(z_0,t)|$,

giving rise to a metric: $d(\gamma_1,\gamma_2)=\|\gamma_1-\gamma_2\|=\|\gamma_1 \ddot{\oplus}(-1)\cdot\gamma_2\|$.

Example 9 $\gamma_1 : f_1(z,t)=(2x-yt)+i(2y+xt), z_0=-1.5+i$,

$$\gamma_2 : f_2(z,t)=x(1+Cos(t^2))+i(y(1+Cos(t^2))+t), z_0=-1.5+i$$