# A class of symmetric Laguerre-Hahn Polynomials

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**Abstract**

We show that if $v$ is a regular Laguerre-Hahn form (linear functional), then the symmetric form $u$ defined by the relation $x\sigma u = -\lambda v$ where $\sigma u$ is the even part of $u$, is also regular and Laguerre-Hahn form for every complex $\lambda$ except for a discrete set of numbers depending on $v$. We give explicitly the recurrence coefficients and the structure relation coefficients of the orthogonal polynomials sequence associated with $u$ and the class of the form $u$ knowing that of $v$. Finally, we apply our results to the associated form of the first order for the classical polynomials.

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# 1 Introduction and preliminaries

In many recent papers, different construction processes of Laguerre-Hahn orthogonal polynomials (O.P) grow from well known ones, particularly the associated of classical ones. For instance, we can mention the adjunction of a finite number of Dirac's masses to Laguerre-Hahn forms [1,6,7], the product and the division of a form by a polynomial [3,4,7,10,11]. The whole idea of the following work is to build a new construction process of a Laguerre-Hahn form, which has not yet been treated in the literature of Laguerre-Hahn polynomials. The problem we tackle is as follows:
We study the form $u$, fulfilling $x\sigma u = -\lambda v$, $\lambda \neq 0$, $(u)_{2n+1} = 0$, where $\sigma u$ is the even part of $u$ and $v$ is a given Laguerre-Hahn form.
This paper is arranged in sections : The first provides a focus on the preliminary results and notations used in the sequel. We will also give the regularity condition and the coefficients of the three-term recurrence relation satisfied by the new family of O.P. In the second , we compute the exact class of the Laguerre-Hahn form obtained by the above modification and the structure relation of the O.P. sequence relatively to the form $u$ will follow. In the final section, we apply our results to some examples. The regular forms found in the examples are Laguerre-Hahn of class $\tilde{s} \in \{2, 3\}$.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathcal{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(v)_n := \langle v, x^n\rangle, n \geq 0$ , the moments of $v$. For any form $v$ and any polynomial $h$ let $Dv = v'$, $hv$, $\delta_c$, and $(x-c)^{-1}v$ be the forms defined by: $\langle v', f\rangle := -\langle v, f'\rangle$, $\langle hv, f\rangle := \langle v, hf\rangle$, $\langle \delta_c, f\rangle := f(c)$, and $\langle (x-c)^{-1}v, f\rangle := \langle v, \theta_c f\rangle$ where $\big(\theta_c f\big)(x) = \dfrac{f(x)-f(c)}{x-c}$, $c \in \mathcal{C}$ , $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $f \in \mathcal{P}$ and $v \in \mathcal{P}'$, we have

$$(x-c)^{-1}((x-c)v) = v - (v)_0\delta_c \ , \tag{1}$$

$$(x-c)((x-c)^{-1}v) = v \ , \tag{2}$$

$$(fv)' = f'v + fv' \ . \tag{3}$$

We also define the right-multiplication of a form by a polynomial with

$$(vh)(x) := \Big\langle v, \frac{xh(x) - \xi h(\xi)}{x-\xi} \Big\rangle = \sum_{m=0}^{n} \Big( \sum_{j=m}^{n} a_j (v)_{j-m} \Big) x^m,\ h(x) = \sum_{j=0}^{n} a_j x^j .$$

Next, it is possible to define the product of two forms through

$$\langle uv, f \rangle := \langle u, vf \rangle \ , \quad u, v \in \mathcal{P}',\ f \in \mathcal{P}. \tag{4}$$

Let us define the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ by $(\sigma f)(x) := f(x^2)$ . Then, we define the even part $\sigma v$ of $v$ by $\langle \sigma v, f \rangle := \langle v, \sigma f \rangle$.

Therefore, we have [9]

$$f(x)(\sigma v) = \sigma(f(x^2)v) \ , \tag{5}$$

$$(\sigma v)_n = (v)_{2n} \ , n \geq 0. \tag{6}$$

The form $v$ will be called regular if we can associate with it a sequence $\{S_n\}_{n\geq 0}$ $\big(\deg(S_n) \leq n\ \big)$ such that

$$\langle v, S_n S_m \rangle = r_n \delta_{n,m} \ , \quad n, m \geq 0 \ , \quad r_n \neq 0 \ , \quad n \geq 0 \ .$$

Then, $\deg(S_n) = n$ , $n \geq 0$, and we can always suppose each $S_n$ monic (i.e. $S_n(x) = x^n + \cdots$). The sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal with respect to $v$. It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation $\big($see, for instance, the monograph by Chihara [5]$\big)$

$$\begin{array}{l} S_{n+2}(x) = (x - \xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x) \ , \quad n \geq 0 \ , \\ S_1(x) = x - \xi_0 \ , \qquad S_0(x) = 1 \ . \end{array} \tag{7}$$

with $\big(\xi_n, \rho_{n+1}\big) \in \mathcal{C} \times \mathcal{C} - \{0\}$ , $\quad n \geq 0$ . By convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the associated sequence of first kind for the sequence $\{S_n\}_{n\geq 0}$ satisfying the three-term recurrence relation[5]

$$\begin{array}{l} S_{n+2}^{(1)}(x) = (x - \xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x) \ , \quad n \geq 0, \\ S_1^{(1)}(x) = x - \xi_1, \qquad S_0^{(1)}(x) = 1 \ , \quad \big(S_{-1}^{(1)}(x) = 0\big) \ . \end{array} \tag{8}$$

Also, let $\{S_n(.,\mu)\}_{n\geq 0}$ be co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [5]

$$S_n(x, \mu) = S_n(x) - \mu S_{n-1}^{(1)} \ , \quad n \geq 0. \tag{9}$$

A form $v$ is called symmetric if $(v)_{2n+1} = 0, n \geq 0$ . The conditions $(v)_{2n+1} = 0, n \geq 0$ are equivalent to the fact that the corresponding monic orthogonal polynomials sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (7) with $\xi_n = 0, n \geq 0$ [5,9].

**Proposition 1.1** [9] *When the form $v$ is symmetric, then $v$ is regular if and only if $\sigma v$ and $x\sigma v$ are regular.*

Let $v$ be a regular, normalized form (i.e. $(v)_0 = 1$) and $\{S_n\}_{n\geq 0}$ be its corresponding sequence of polynomials. For a $\lambda \in \mathcal{C}^*$, we can define a new symmetric form $u$ as following:

$$(u)_{2n+2} = -\lambda (v)_n \ , \quad (u)_{2n+1} = 0 \ , \quad (u)_0 = 1 \ , \qquad n \geq 0. \tag{10}$$

Equivalenty,

$$x\sigma u = -\lambda v \ , \qquad \sigma(xu) = 0 \ . \tag{11}$$

From (1) and (11), we have

$$\sigma u = -\lambda x^{-1} v + \delta_0 \ . \tag{12}$$

**Proposition 1.2** *The form $u$ is regular if and only if $\lambda \neq \lambda_n, n \geq 0$ where* $\lambda_n = \dfrac{S_n(0)}{S^{(1)}_{n-1}(0)}$.

**Proof.** We have $u$ is symmetric form. Then, according to Proposition 1.1. $u$ is regular if and only if $x\sigma u$ and $\sigma u$ are regular. But $x\sigma u = -\lambda v$ is regular. So $u$ is regular if and only if $\sigma u = -\lambda x^{-1}\sigma v + \delta_0$ is regular. Or,it was shown in [10] that the form $-\lambda x^{-1}v + \delta_0$ is regular if and only if $\lambda \neq 0$, and $S_n(0,\lambda) \neq 0$ , $n \geq 0$ . Then, we deduce the desired result. □

**Remark.** If $w$ is the symmetrized form associated with the form $v$ $\big($ i.e. $(w)_{2n} = (v)_n$ and $(w)_{2n+1} = 0, n \geq 0$ $\big)$, then (10) is equivalent to $x^2 u = -\lambda w$. Notice that $w$ is not necessarily a regular form in the problem under study. In [3, 8], the authors have solved it only when $w$ is regular.

When $u$ is regular let $\{Z_n\}_{n\geq 0}$ be its corresponding sequence of polynomials satisfying the recurrence relation

$$\begin{aligned} &Z_{n+2}(x) = xZ_{n+1}(x) - \gamma_{n+1}Z_n(x) \ , \quad n \geq 0 \ , \\ &Z_1(x) = x \ , \quad Z_0(x) = 1 \ . \end{aligned} \tag{13}$$

Since $\{Z_n\}_{n\geq 0}$ is symmetric, let us consider its quadratic decomposition [5,9]:

$$Z_{2n}(x) = P_n(x^2) \ , \quad Z_{2n+1}(x) = xR_n(x^2) \ , \tag{14}$$

$$Z^{(1)}_{2n}(x) = R_n\left(x^2, -\gamma_1\right) \ , \quad Z^{(1)}_{2n+1}(x) = xP^{(1)}_n(x^2) \ . \tag{15}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to $\sigma u$ and $x\sigma u$.

From (11), we have

$$R_n(x) = S_n(x) \ , \quad n \geq 0 \ . \tag{16}$$

**Proposition 1.3** *We may write*

$$\gamma_1 = -\lambda \ , \quad \gamma_{2n+2} = a_n \ , \quad \gamma_{2n+3} = \frac{\rho_{n+1}}{a_n} \ , \quad n \geq 0 \tag{17}$$

*where*

$$a_n = -\frac{S_{n+1}(0,\lambda)}{S_n(0,\lambda)} \ , \quad n \geq 0 \ . \tag{18}$$

**Proof.** Using (11) and the condition $\langle u, Z_2\rangle = 0$ , we obtain $\gamma_1 = -\lambda$.
From (8) and (13) where $n \longrightarrow 2n$ and taking (15)-(16) into account, we get

$$S_{n+1}\left(x^2, -\gamma_1\right) = xZ^{(1)}_{2n+1}(x) - \gamma_{2n+2}S_n\left(x^2, -\gamma_1\right) \ .$$

Substituting $x$ by 0 in the above equation, we obtain $\gamma_{2n+2} = a_n$.
From (13), we have

$$\gamma_{2n+2}\gamma_{2n+3} = \frac{\left\langle u, Z^2_{2n+2}\right\rangle}{\left\langle u, Z^2_{2n+1}\right\rangle}\frac{\left\langle u, Z^2_{2n+3}\right\rangle}{\left\langle u, Z^2_{2n+2}\right\rangle} = \frac{\left\langle u, Z^2_{2n+3}\right\rangle}{\left\langle u, Z^2_{2n+1}\right\rangle}. \tag{19}$$

Using (14), (11) and (7), equation (19) becomes

$$\gamma_{2n+2}\gamma_{2n+3} = \rho_{n+1}, \tag{20}$$

then we deduce $\gamma_{2n+3} = \dfrac{\rho_{n+1}}{a_n}$. □

**Corollary 1.4** *When the form $v$ is symmetric, then $u$ is regular for every $\lambda \neq 0$. Moreover,*

$$\begin{cases} \gamma_1 = -\gamma_2 = -\lambda, \\ \gamma_{4n+3} = -\gamma_{4n+4} = \dfrac{1}{\lambda} \displaystyle\prod_{k=0}^{n} \frac{\rho_{2k+1}}{\rho_{2k}}, \\ \gamma_{4n+5} = -\gamma_{4n+6} = -\lambda \rho_{2n+2} \displaystyle\prod_{k=0}^{n} \frac{\rho_{2k}}{\rho_{2k+1}}, n \geq 0 . \end{cases} \tag{21}$$

**Proof.** Taking into account (7)-(8), with $\xi_n = 0$, we get $S_{n+2}(0) = -\rho_{n+1} S_n(0)$ and $S^{(1)}_{n+2}(0) = -\rho_{n+2} S^{(1)}_n(0)$. Then,

$$S_{2n+1}(0) = 0 \ , \quad S_{2n+2}(0) = (-1)^{n+1} \prod_{v=0}^{n} \rho_{2v+1} \ , \quad n \geq 0 \ , \tag{22}$$

$$S^{(1)}_{2n+1}(0) = 0, \quad S^{(1)}_{2n}(0) = (-1)^n \prod_{v=0}^{n} \rho_{2v} \ , \quad n \geq 0 \ . \tag{23}$$

Therefore, $S_{2n+1}(0,\lambda) = -\lambda S^{(1)}_{2n}(0) \neq 0$ and $S_{2n+2}(0,\lambda) = S_{2n+2}(0) \neq 0$. Hence $u$ is regular for every $\lambda \neq 0$ according to proposition 1.2.
From (22)-(23), (17) becomes (21). □

# 2 The Laguerre-Hahn case

**Definition 2.1** [2] *The form $v$ is called Laguerre-Hahn when it is regular and satisfies the Riccati equation*

$$\Phi(z) S'\big(v\big)(z) = B(z) S^2\big(v\big)(z) + C_0(z) S\big(v\big)(z) + D_0(z), \tag{24}$$

*where $\Phi$ monic , $B$ , $C_0$ and $D_0$ are polynomials and $S(v)(z)$ designes the formal Stieltjes function of the form $v$ defined by:*

$$S(v)(z) = -\sum_{n \geq 0} \frac{(v)_n}{z^{n+1}}. \tag{25}$$

It was shown in [6] that equation (24) is equivalent to

$$\big(\Phi(x) v\big)' + \Psi(x) v + B\big(x^{-1} v^2\big) = 0 \tag{26}$$

with

$$\Psi(x) = -\Phi'(x) - C_0(x). \tag{27}$$

**Remark.**[2] When $B = 0$ in (24) or (26), the form $v$ is semiclassical.

**Proposition 2.2** [2] *Define $d = \max\big(\deg(\Phi), \deg(B)\big)$ and $p = \deg(\Psi)$.*
*The Laguerre-Hahn form $v$ satisfying (26) is of class $s = \max(d-2, p-1)$ if and only if*

$$\prod_{c \in \mathcal{Z}} \big\{ \big|\Phi'(c) + \Psi(c)\big| + \big|B(c)\big| + \big|\langle v, \theta_c^2 \Phi + \theta_c \Psi + v \theta_0 \theta_c B \rangle\big| \big\} \neq 0, \tag{28}$$

*where $\mathcal{Z}$ denotes the set of zeros of $\Phi$.*

**Corollary 2.3** [2] *The form $v$ satisfying (24) is of class $s$ if and only if*

$$\prod_{c\in\mathcal{Z}}(|C_0(c)|+|B(c)|+|D_0(c)|)\neq 0. \tag{29}$$

**Proposition 2.4** *If $v$ is a Laguerre-Hahn form and satisfies (24), then for every $\lambda\in\mathcal{C}-\{0\}$ such that $\lambda\neq\lambda_n, n\geq 0$, the form $u$ defined by (10) is regular and Laguerre-Hahn. It satisfies*

$$\tilde{\Phi}(z)S'\big(u\big)(z)=\tilde{B}(z)S^2\big(u\big)(z)+\tilde{C}_0(z)S\big(u\big)(z)+\tilde{D}_0(z), \tag{30}$$

*where*

$$\begin{cases}\tilde{\Phi}(z)=z\Phi(z^2), \quad \tilde{B}(z)=-2\lambda^{-1}z^3B(z^2),\\ \tilde{C}_0(z)=2z^2C_0(z^2)-\Phi(z^2)-4\lambda^{-1}z^2B(z^2),\\ \tilde{D}_0(z)=-2z\big(\lambda D_0(z^2)-C_0(z^2)+\lambda^{-1}B(z^2)\big),\end{cases} \tag{31}$$

*and $u$ is of class $\tilde{s}$ such that $\tilde{s}\leq 2s+5$.*

**Proof.** From (11) and (25), we have

$$S\big(v\big)(z^2)=-z\lambda^{-1}S\big(u\big)(z)-\lambda^{-1}. \tag{32}$$

Make a change of variable $z\longrightarrow z^2$ in (24), multiply by $-2\lambda z$ and substitute (32) in the obtained equation, we get (30)-(31).
The form $u$ satisfies the distributional equation

$$\big(\tilde{\Phi}(x)u\big)'+\tilde{\Psi}u(x)+\tilde{B}\big(x^{-1}u^2\big)=0, \tag{33}$$

where $\tilde{\Phi}$ and $\tilde{B}$ are the polynomials defined by (31) and

$$\tilde{\Psi}(x)=-\tilde{\Phi}'(x)-\tilde{C}_0(x)=2x^2\Psi(x^2)+4\lambda^{-1}x^2B(x^2). \tag{34}$$

Then, $\deg(\tilde{\Phi})\leq 2s+5$ , $\deg(\tilde{B})\leq 2s+7$ and $\deg(\tilde{\Psi})=\tilde{p}\leq 2s+6$.
Thus, $\tilde{d}=\max\big(\deg(\tilde{\Phi}),\deg(\tilde{B})\big)\leq 2s+7$ and $\tilde{s}=\max(\tilde{d}-2,\tilde{p}-1)\leq 2s+5$. □

**Proposition 2.5** *The class of $u$ depends only on the zero $x=0$ of $\Phi$.*

**Proof.** Since $v$ is a Laguerre-Hahn form of class $s$, $S(v)(z)$ satisfies (24), where the polynomials $\Phi$, $B$, $C_0$ and $D_0$ are coprime. Let $\tilde{\Phi}$, $\tilde{B}$, $\tilde{C}_0$ and $\tilde{D}_0$ be as in Proposition 2.4.
Let $c$ be a zero of $\tilde{\Phi}$ different from 0, this implies that $\Phi(c^2)=0$. We know that $|B(c^2)|+|C_0(c^2)|+|D_0(c^2)|\neq 0$
(1) if $B(c^2)\neq 0$, then $\tilde{B}(c)\neq 0$,
(2) if $B(c^2)=0$ and $C_0(c^2)\neq 0$, then $\tilde{C}_0(c)\neq 0$,
(3) if $B(c^2)=C_0(c^2)=0$, then $\tilde{D}_0(c)\neq 0$, whence $|\tilde{B}(c)|+|\tilde{C}_0(c)|+|\tilde{D}_0(c)|\neq 0$ . □

Concerning the class of $u$, we have the following result.

**Proposition 2.6** *Let $t$ =deg($\Phi$), $r$ =deg($B$) and $p$ =deg($\Psi$). Under the conditions of Proposition 2.4., for the class of $u$, the following four different cases hold:*
*1) If $\Phi(0)\neq 0$, then*

$$\tilde{s}=\begin{cases}2s+5 & if\, r>p,\ t<r+1\ ,\\ 2s+3 & otherwise\ .\end{cases} \tag{35}$$

*2) If $\Phi(0)=0$ and $X=\lambda D_0(0)-C_0(0)+\lambda^{-1}B(0)\neq 0$, then*

$$\tilde{s}=\begin{cases}2s+4 & if\, r>p,\ t<r+1\ ,\\ 2s+2 & otherwise\ .\end{cases} \tag{36}$$

*3) If* $\Phi(0) = X = 0$ *and* $Y = 2C_0(0) - 4\lambda^{-1}B(0) - \Phi'(0) \neq 0$, *then*

$$\tilde{s} = \begin{cases} 2s+3 & if\, r > p,\, t < r+1\ . \\ 2s+1 & otherwise\ . \end{cases} \tag{37}$$

*4) If* $\Phi(0) = X = Y = 0$, *then*

$$\tilde{s} = \begin{cases} 2s+2 & if\, r > p,\ t < r+1\ , \\ 2s & \quad otherwise\ . \end{cases} \tag{38}$$

*where the polynomials* $\Phi$, $B$, $C_0$ *and* $D_0$ *are defined in (24)*

**Proof.** 1) If $\Phi(0) \neq 0$, then from (31) we obtain $\tilde{C}_0(0) \neq 0$. Therefore, it is not possible to simplify. From (31) and (34), we have

$\deg(\tilde{\Phi}) = 2t+1$, $\deg(\tilde{B}) = 2r+3$ and $\tilde{p} := \deg(\tilde{\Psi}) \leq \max(2p+2, 2r+2)$.
We will distinguish two cases:

(a) $p < r$, then $\tilde{p} = 2r+2$ and $\tilde{s} = \max(2t-1, 2r+1)$. If $t < r+1$, then $\tilde{s} = 2r+1 = 2s+5$. If $t \geq r+1$, then $\tilde{s} = 2t-1 = 2s+3$.

(b) $p \geq r$, then $\tilde{p} = 2p+2$ and $\tilde{s} = \max(2t-1, 2p+1) = 2s+3$.

Thus, from the above situation, we deduce (35).

2) If $\Phi(0) = 0$, then from (31) we have $\tilde{B}(0) = \tilde{C}_0(0) = \tilde{D}_0(0) = 0$, therefore (30)-(31) is divisible by $z$. Thus, $u$ fulfils (30) with

$$\begin{cases} \tilde{\Phi}(z) = \Phi(z^2)\ , \quad \tilde{B}(z) = -2\lambda^{-1}z^2B(z^2)\ , \\ \tilde{C}_0(z) = z\big(2C_0(z^2) - 4\lambda^{-1}B(z^2) - (\theta_0\Phi)(z^2)\big)\ , \\ \tilde{D}_0(z) = -2\lambda D_0(z^2) + 2C_0(z^2) - 2\lambda^{-1}B(z^2)\ . \end{cases} \tag{39}$$

Whether $\lambda D_0(0) - C_0(0) + \lambda^{-1}B(0)\big) = X \neq 0$, it is not possible to simplify, which means that the class of $u$ verifies (36).

3) If $\Phi(0) = X = 0$, then it is possible to simplify (30)-(39) by $z$. Thus, $u$ fulfils (30) with

$$\begin{cases} \tilde{\Phi}(z) = z(\theta_0\Phi)(z^2)\ , \quad \tilde{B}(z) = -2\lambda^{-1}zB(z^2)\ , \\ \tilde{C}_0(z) = 2C_0(z^2) - 4\lambda^{-1}B(z^2) - (\theta_0\Phi)(z^2)\ , \\ \tilde{D}_0(z) = -2z\big(\theta_0(\lambda D_0 - C_0 + \lambda^{-1}B)\big)(z^2)\ . \end{cases} \tag{40}$$

Therefore, if $2C_0(0) - 4\lambda^{-1}B(0) - \Phi'(0) = Y \neq 0$, it is not possible to simplify, which means that the class of $u$ verifies (37).

4) If $\Phi(0) = X = Y = 0$, then it is possible to simplify (30)-(40) by $z$. Thus $u$ fulfils (30) with

$$\begin{cases} \tilde{\Phi}(z) = (\theta_0\Phi)(z^2)\ , \quad \tilde{B}(z) = -2\lambda^{-1}B(z^2)\ , \\ \tilde{C}_0(z) = z\big(\theta_0(2C_0 - 4\lambda^{-1}B - \theta_0\Phi)\big)(z^2)\ , \\ \tilde{D}_0(z) = -2\big(\theta_0(\lambda D_0 - C_0 + \lambda^{-1}B)\big)(z^2)\ . \end{cases} \tag{41}$$

Assuming that $\tilde{\Phi}(0) = \Phi'(0) = 0$, then from the condition $Y = 0$ we obtain $C_0(0) = 2\lambda^{-1}B(0)$. Thus from the last result and the condition $X = 0$, we get $D_0(0) = \lambda^{-2}B(0)$. So that, in this case we have $B(0) \neq 0$, since $v$ is Laguerre-Hahn of class $s$ and so satisfies (29).

Then, it is not possible to simplify (30)-(41) since $\tilde{B}(0) \neq 0$, which means the class of $u$ is (38). $\square$

**Remarks. 1.** As a consequence of Proposition 2.6, we have:

(i) The symmetric Laguerre-Hahn linear functional $u$ of class $\tilde{s} = 1$ appears if and only if the linear functional $v$ is Laguerre-Hahn of class $s = 0$ with $\Phi(0) = \lambda D_0(0) - C_0(0) + \lambda^{-1}B(0) = 0$ and $2C_0(0) - 4\lambda^{-1}B(0) - \Phi^{'}(0) \neq 0$.

(ii) The symmetric Laguerre-Hahn linear functionals $u$ of class $\tilde{s} = 2$ and $\tilde{s} = 3$ appear not only when the linear functional $v$ is Laguerre-Hahn of class $\tilde{s} = 0$ but also when $s = 1$ under some conditions satisfied by $\Phi$, $B$ and $C$.

**2.** Unfortunately, we are not able to describe all the symmetric Laguerre-Hahn linear functionals of classes 2 and 3, especially because we still don't know the nonsymmetric Laguerre-Hahn linear functionals of class $\tilde{s} = 1$ [2] (Some examples considered in Section 3).

Note that the sequence of orthogonal polynomials (OPS) relatively to a Laguerre-Hahn form has a structure relation [2,6]. Then, if we consider that the form $v$ is Laguerre-Hahn, its OPS $\{S_n\}_{n\geq 0}$ fulfils the following structure relation

$$\begin{aligned}&\Phi(x)S'_{n+1}(x) - B(x)S_n^{(1)}(x) = \\ &\qquad \frac{1}{2}\big(C_{n+1}(x) - C_0(x)\big)S_{n+1}(x) - \rho_{n+1}D_{n+1}(x)S_n(x)\ ,\ n \geq 0,\end{aligned} \tag{42}$$

with

$$\left\{\begin{array}{l} C_{n+1}(x) = -C_n(x) + 2(x-\xi_n)D_n(x) \\ \rho_{n+1}D_{n+1}(x) = -\Phi(x) + \rho_n D_{n-1}(x) \\ \qquad\qquad -(x-\xi_n)C_n(x) + (x-\rho_n)^2 D_n(x) \end{array}\right. ,n \geq 0, \tag{43}$$

where $C_0(x)$ and $D_0(x)$ are the same polynomials as in (24) .
Notice that $\delta_0 D_{-1}(x) = B(x), \deg C_n \leq s+1$ and $\deg D_n \leq s, n \geq 0$ [2].

According to proposition 2.4, the form $u$ is also Laguerre-Hahn and its OPS $\{Z_n\}_{n\geq 0}$ satisfies a structure relation. In general, $\{Z_n\}_{n\geq 0}$ fulfils

$$\begin{aligned}&\tilde{\Phi}(x)Z'_{n+1}(x) - \tilde{B}(x)Z_n^{(1)}(x) = \\ &\qquad \frac{1}{2}\big(\tilde{C}_{n+1}(x) - \tilde{C}_0(x)\big)Z_{n+1}(x) - \gamma_{n+1}\tilde{D}_{n+1}(x)Z_n(x)\ \ ,n \geq 0,\end{aligned} \tag{44}$$

with

$$\left\{\begin{array}{l} \gamma_{n+1}\tilde{D}_{n+1}(x) = -\tilde{\Phi}(x) + \gamma_n\tilde{D}_{n-1}(x) - x\tilde{C}_n(x) + x^2\tilde{D}_n(x), \\ \tilde{C}_{n+1}(x) = -\tilde{C}_n(x) + 2x\tilde{D}_n(x), \quad ,n \geq 0, \end{array}\right. \tag{45}$$

where $\tilde{C}_0(x)$ , $\tilde{D}_0(x)$ are given by (31) and $\gamma_0\tilde{D}_{-1}(x) = \tilde{B}(x)$.

We are going to establish the expression of $\tilde{C}_n$ and $\tilde{D}_n$ , $n \geq 0$ in terms of those of the sequence $\{S_n\}_{n\geq 0}$.

**Proposition 2.7** *The sequence $\{Z_n\}_{n\geq 0}$ fulfils (44) with (for $n \geq 0$)*

$$\left\{\begin{array}{l} \tilde{C}_{2n+1}(x) = \Phi(x^2) + 2x^2C_n(x^2) + 4\gamma_{2n+1}x^2D_n(x^2)\ , \\ \tilde{D}_{2n+1}(x) = 2x^3D_n(x^2). \end{array}\right. \tag{46}$$

$$\left\{\begin{array}{l} \tilde{C}_{2n+2}(x) = -\Phi(x^2) + 2x^2C_{n+1}(x^2) + 4x^2\gamma_{2n+2}D_n(x^2), \\ \tilde{D}_{2n+2}(x) = 2x\gamma_{2n+2}D_n(x^2) + 2x\gamma_{2n+3}D_{n+1}(x^2) + 2xC_{n+1}(x^2)\ . \end{array}\right. \tag{47}$$

*$\tilde{C}_0(x)$ and $\tilde{D}_0(x)$ are given by (31) and $\gamma_{n+1}$ by (17).*

**Proof.** Change $x \longrightarrow x^2$, $n \longrightarrow n-1$ in (42) and multiply by $2x^3$, we obtain by taking (14)-(16) and (31) into account,

$$\tilde{\Phi}(x)Z'_{2n+1}(x) - \tilde{B}(x)Z^{(1)}_{2n}(x) = \big\{x^2\big(C_n(x^2) - C_0(x^2) + 2\lambda^{-1}B(x^2)\big) + \Phi(x^2)\big\}Z_{2n+1}(x) - 2\rho_n x^2 D_n(x^2)Z_{2n-1}(x), \quad n \geq 1.$$

Using (13) and (17) where $n \longrightarrow 2n-1$, the last equation becomes

$$\tilde{\Phi}(x)Z'_{2n+1}(x) - \tilde{B}(x)Z^{(1)}_{2n}(x) = \big\{x^2\big(C_n(x^2) - C_0(x^2) + 2\lambda^{-1}B(x^2) + 2\gamma_{2n+1}D_n(x^2)\big) + \Phi(x^2)\big\}Z_{2n+1}(x) - 2\gamma_{2n+1}x^3 D_n(x^2)Z_{2n}(x), \; n \geq 0.$$

From (44) and the above equation, we have for $n \geq 0$

$$\left\{\frac{\tilde{C}_{2n+1}(x) - \tilde{C}_0(x)}{2} - X_n(x)\right\} Z_{2n+1}(x) = \gamma_{2n+1}\left\{\tilde{D}_{2n+1} - Y_n(x)\right\} Z_{2n}(x)$$

with

$$X_n(x) = \big(C_n(x^2) - C_0(x^2) + 2\gamma_{2n+1}D_n(x^2) - 2\lambda^{-1}B(x^2)\big)\, x^2 + \Phi(x^2)$$

and $Y_n(x) = 2x^3 D_n(x^2)$.

$Z_{2n+1}$ and $Z_{2n}$ have no common zeros, then $Z_{2n+1}$ divides $Y_n(x) - \tilde{D}_{2n+1}(x)$, which is a polynomial of degree at most equal to $2s+5$.

Then, we have necessarily $Y_n(x) - \tilde{D}_{2n+1}(x) = 0$ for $n > s+2$, and also $X_n(x) = \dfrac{\tilde{C}_{2n+1}(x) - \tilde{C}_0(x)}{2}$, $n > s+2$. Therefore, $\tilde{C}_{2n+1}(x) = 2X_n(x) + \tilde{C}_0(x)$ and $\tilde{D}_{2n+1} = Y_n(x)$, $n > s+2$. Then, by (31), we get (46) for $n > s+2$. By virtue of the recurrence relation (45) and (31), we can easily prove by induction that the system (46) is valid for $0 \leq n \leq s+2$. Hence (46) is valid for $n \geq 0$. Finally, from (45) and (46), we give (47). □

**Remark.** In the Laguerre-Hahn case, the polynomials $\tilde{C}_n$ and $\tilde{D}_n$ of (46)-(47) enable to obtain the coefficients of the fourth-order differential equation satisfied by each $Z_n$ , $n \geq 0$. See, for instance [6, page 90].

# 3 Examples

In the next examples, we apply our results to the associated form of the first order for the classical polynomials.

**Example 3.1** *Let $v$ be the associated form of the first order of Hermite. Here [6,7]*

$$\rho_{2n+1} = n+1,\; \rho_{2n+2} = \frac{2n+3}{2}, \quad n \geq 0, \tag{48}$$

$$\begin{cases} \Phi(x) = 1\,, \quad \psi(x) = 2x\,, \quad B(x) = -1\,, \\ D_n(x) = -2\,, \quad C_n(x) = -2x\,. \end{cases} \tag{49}$$

*In this case, the form $v$ is a Laguerre-Hahn form of class $s = 0$ and from the Corollary 1.4. $u$ is regular for every $\lambda \neq 0$.*
*From* (18), (21) − (23) *and* (48), *we obtain*

$$a_{2n} = \frac{2\lambda\Gamma(n+\frac{3}{2})}{\sqrt{\Pi}\Gamma(n+1)}, \quad n \geq 0 \tag{50}$$

*and (21) becomes*

$$\begin{cases} \gamma_{4n+1} = -\gamma_{4n+2} = -a_{2n} \ , \\ \gamma_{4n+3} = -\gamma_{4n+4} = \dfrac{n+1}{a_{2n}} \ , \quad n \geq 0 \ . \end{cases} \tag{51}$$

*Taking into account that the form v is Laguerre-Hahn and by virtue of Proposition 2.4 and Proposition 2.6, the form u is Laguerre-Hahn of class* $\tilde{s} = 3$*. It satisfies* (30) *and* (33) *with*

$$\begin{cases} \tilde{\Phi}(x) = x \ , \quad \tilde{\Psi}(x) = 4x^4 - 4\lambda^{-1}x^2 \ , \quad \tilde{B}(x) = 2\lambda^{-1}x^3 \ , \\ \tilde{C}_0(x) = -4x^4 + 4\lambda^{-1}x^2 - 1 \ , \quad \tilde{D}_0(x) = -4x^3 + 2(2\lambda + \lambda^{-1})x \ . \end{cases} \tag{52}$$

*Finally, from* (49), (51) *and Proposition 2.7 we give the element of the structure relation of the sequence* $\{Z_n\}_{n\geq 0}$ *for* $n \geq 0$

$$\begin{cases} \tilde{C}_0(x) = -4x^4 + 4\lambda^{-1}x^2 - 1 \ , \\ \tilde{C}_{4n+1}(x) = -4x^4 + 8a_{2n}x^2 + 1 \ , \\ \tilde{C}_{4n+2}(x) = -4x^4 - 8a_{2n}x^2 - 1 \ , \\ \tilde{C}_{4n+3}(x) = -4x^4 - \dfrac{8(n+1)}{a_{2n}}x^2 + 1 \ , \\ \tilde{C}_{4n+4}(x) = -4x^4 + \dfrac{8(n+1)}{a_{2n}}x^2 - 1 \ , \\ \tilde{D}_0(x) = -4x^3 + 2(2\lambda + \lambda^{-1})x \ , \\ \tilde{D}_{4n+2} = -4x^3 - 4(a_{2n} + \dfrac{n+1}{a_{2n}})x \ , \\ \tilde{D}_{4n+1}(x) = \tilde{D}_{4n+3}(x) = -4x^3 \ , \\ \tilde{D}_{4n+4}(x) = -4x^3 + 4(a_{2n+2} + \dfrac{n+1}{a_{2n}})x \ . \end{cases}$$

**Example 3.2** *Let v be the associated form of the first order of Laguerre. Here [6,7]*

$$\rho_{n+1} = (n+2)(\alpha+n+2), \xi_n = 2n + \alpha + 3, \quad n \geq 0 \tag{53}$$

$$\begin{cases} \Phi(x) = x \ , \quad \psi(x) = x - (\alpha+3) \ , \quad B(x) = -(\alpha+1) \ , \\ D_n(x) = -1 \ , \quad C_n(x) = -x + \alpha + 2n + 2 \ , \quad n \geq 0 \ . \end{cases} \tag{54}$$

*We assume* $\alpha(\alpha+1) \neq 0$ *then v is a Laguerre-Hahn form of class* $s = 0$*.*
*From* (7), (8) *and* (53)*, we have by induction*

$$S_n(0) = (-1)^n \frac{\Gamma(\alpha+n+2) - \Gamma(n+2)\Gamma(\alpha+1)}{\alpha\Gamma(\alpha+1)}, \quad n \geq 0 \ . \tag{55}$$

*and*

$$S_n^{(1)}(0) = (-1)^n \frac{\Gamma(\alpha+n+3) - \Gamma(n+3)\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+2)}, \quad n \geq 0 \ . \tag{56}$$

(9) *and* (55) − (56)*, give*

$$S_n(0,\lambda) = \frac{(-1)^n b_n(\alpha,\lambda)}{\alpha\Gamma(\alpha+2)}, \quad n \geq 0 \tag{57}$$

*where*

$$b_n(\alpha,\lambda) = (\alpha+\lambda+1)\Gamma(\alpha+n+2) - (1+\lambda)\Gamma(n+2)\Gamma(\alpha+2), \quad n \geq 0. \tag{58}$$

*Then, $u$ is regular if and only if*

$\lambda \neq 0$ *and*

$$\lambda \neq -1 - \frac{\alpha\Gamma(\alpha+n+2)}{\Gamma(\alpha+n+2)-\Gamma(n+2)\Gamma(\alpha+2)}, \quad n \geq 0\,. \tag{59}$$

*From* (18) *and* (57), *we have*

$$a_n = \frac{b_{n+1}(\alpha,\lambda)}{b_n(\alpha,\lambda)}, \quad n \geq 0\,. \tag{60}$$

*Using* (17) *and* (60), *we obtain*

$$\begin{cases} \gamma_1 = -\lambda\,, \\ \gamma_{2n+2} = \dfrac{b_{n+1}(\alpha,\lambda)}{b_n(\alpha,\lambda)}\,, \\ \gamma_{2n+3} = (n+2)(\alpha+n+2)\dfrac{b_n(\alpha,\lambda)}{b_{n+1}(\alpha,\lambda)}\,, \quad n \geq 0\,. \end{cases} \tag{61}$$

*According to Proposition 2.4, the form $u$ is Laguerre-Hahn . It satisfies* (30) *and* (33) *with*

$$\begin{cases} \tilde{\Phi}(x) = x^2\,, \quad \tilde{\Psi}(x) = 2x^3 - \Big(3 + 2(2\lambda^{-1}+1)(\alpha+1)\Big)x\,, \\ \tilde{B}(x) = 2\lambda^{-1}(\alpha+1)x^2\,, \\ \tilde{C}_0(x) = -2x^3 + \Big(2\alpha+3+4\lambda^{-1}(\alpha+1)\Big)x\,, \\ \tilde{D}_0(x) = -2x^2 + 2\Big(\lambda+1+(\alpha+1)(1+\lambda^{-1})\Big)\,. \end{cases} \tag{62}$$

*From* (54), *we have*

$$\begin{cases} \Phi(0) = 0\,, \\ X = -\lambda^{-1}(\lambda+1)(\lambda+\alpha+1)\,, \\ Y = \lambda^{-1}\Big((2\alpha+3)\lambda + 4(\alpha+1)\Big)\,. \end{cases}$$

*Now it is enough to use Proposition 2.6 to obtain the following*

*(1) If $\lambda$ satisfies* (59) *and $\lambda \notin \{-1, -\alpha-1\}$, then the class of $u$ is $\tilde{s} = 2$.*

*(2) If $\lambda \in \{-1, -\alpha-1\}$ and $\lambda \neq -4\dfrac{\alpha+1}{2\alpha+3}$, then the class of $u$ is $\tilde{s} = 1$.*

**Remark.** *The symmetric Laguerre-Hahn forms of class $s = 1$ have been described in [2] .*

*Finally, from* (54), (61) *and Proposition 2.7 we have for $n \geq 0$*

$$\begin{cases} \tilde{C}_0(x) = -2x^3 + \Big(2\alpha+3+4\lambda^{-1}(\alpha+1)\Big)x\,, \\ \tilde{C}_{2n+1}(x) = -2x^3 + \left(2\alpha+1+4\alpha\dfrac{(\lambda+1)\Gamma(n+2)\Gamma(\alpha+2)}{b_n(\alpha,\lambda)}\right)x\,, \\ \tilde{C}_{2n+2}(x) = -2x^3 - \left(2\alpha+1+4\alpha\dfrac{(\lambda+1)\Gamma(n+2)\Gamma(\alpha+2)}{b_n(\alpha,\lambda)}\right)x\,, \\ \tilde{D}_0(x) = -2x^2 + 2\Big(\lambda+1+(\alpha+1)(1+\lambda^{-1})\Big), \\ \tilde{D}_{2n+1}(x) = -2x^2\,, \\ \tilde{D}_{2n+2} = -2x^2 + 2\alpha(\lambda+1)\Gamma(n+2)\Gamma(\alpha+2)\left(\dfrac{n+2}{b_{n+1}(\alpha,\lambda)} - \dfrac{1}{b_n(\alpha,\lambda)}\right)\,. \end{cases}$$

**Example 3.3** *Let $v$ be the associated form of the first order of Bessel. Here[6,7]*

$$\begin{cases} \xi_n = \dfrac{1-\theta}{(n+\theta)(n+\theta+1)} \ , \\ \rho_{n+1} = -\dfrac{(n+2)(n+2\theta)}{(2(n+\theta)+3)(2(n+\theta)+1)(n+\theta+1)^2} , \ n \geq 0. \end{cases} \tag{63}$$

$$\begin{cases} \Phi(x) = x^2 \ , \quad \Psi(x) = -2(\theta+1)x - 2\dfrac{\theta-1}{\theta} \ , \\ B(x) = -\dfrac{2\theta-1}{\theta^2(2\theta+1)} \ , \\ D_n(x) = 2(\theta+n)+1 \ , \\ C_n(x) = 2(n+\theta)x + 2\dfrac{\theta-1}{n+\theta} \ , \quad n \geq 0 \ . \end{cases} \tag{64}$$

*We assume $\theta(2\theta-1) \neq 0$, then $v$ is a Laguerre-Hahn form of class $s=0$.*
*By applying the same process as we did to obtain (57)-(58) and using the above results, we can get*

$$S_{n,\lambda}(0) = \frac{2^n c_n(\theta,\lambda)}{\Gamma(2\theta+2n+1)}, \quad n \geq 0 \tag{65}$$

*where for $n \geq 0$*

$$c_n(\theta,\lambda) = \big(2\theta-1-\lambda\theta(2\theta+1)\big)\Gamma(2\theta+n) + (-1)^n\big(1+\lambda\theta(2\theta+1)\Gamma(n+2)\Gamma(2\theta)\big). \tag{66}$$

*Then, $u$ is regular for every $\lambda \neq 0$ such that*

$$\lambda \neq -\frac{1}{\theta(2\theta+1)} + 2\frac{\Gamma(2\theta+n)}{(2\theta+1)\Big(\Gamma(2\theta+n) - (-1)^n\Gamma(n+2)\Gamma(2\theta)\Big)} \ . \tag{67}$$

*Using* (17) *and* (65)*, we obtain*

$$\begin{cases} \gamma_1 = -\lambda \ , \\ \gamma_{2n+3} = \dfrac{(n+2\theta)(n+2)c_n(\theta,\lambda)}{(\theta+n+1)(2\theta+2n+3)c_{n+1}(\theta,\lambda)} \ , \\ \gamma_{2n+2} = -\dfrac{c_{n+1}(\theta,\lambda)}{(n+\theta+1)(2n+2\theta+1)c_n(\theta,\lambda)} \ , \quad n \geq 0 \ . \end{cases} \tag{68}$$

*By virtue of Proposition 2.4., the form $u$ is also Laguerre-Hahn . It satisfies* (30) *and* (33) *with*

$$\begin{cases} \tilde{\Phi}(x) = x^4 \ , \quad \tilde{\Psi}(x) = -(4\theta+3)x^3 - 4\Big(\dfrac{\theta-1}{\theta} + \dfrac{2\theta-1}{\lambda\theta^2(2\theta+1)}\Big)x \ , \\ \tilde{B}(x) = 2\dfrac{2\theta-1}{\lambda\theta^2(2\theta+1)}x^2 \ , \\ \tilde{C}_0(x) = (4\theta-1)x^3 + 4\Big(\dfrac{\theta-1}{\theta} + \dfrac{2\theta-1}{\lambda\theta^2(2\theta+1)}\Big)x \ , \\ \tilde{D}_0(x) = 4\theta x^2 + 2\Big(\dfrac{2\theta-1}{\lambda\theta^2(2\theta+1)} + 2\dfrac{\theta-1}{\theta} - \lambda(2\theta+1)\Big) \ . \end{cases} \tag{69}$$

*From* (64)*, we have*

$$\begin{cases} \Phi(0) = 0 \ , \\ X = \lambda^{-1}(2\theta+1)\Big(\lambda + \dfrac{1}{\theta(2\theta+1)}\Big)\Big(\lambda - \dfrac{2\theta-1}{\theta(2\theta+1)}\Big) \ , \\ Y = 4\dfrac{\theta-1}{\lambda\theta}\Big(\lambda + \dfrac{2\theta-1}{\theta(\theta-1)(2\theta+1)}\Big) \ . \end{cases}$$

*Now, it is enough to use Proposition 2.6 to obtain the following:*
*(1) If* $\lambda$ *satisfies* (67) *and* $\lambda \notin \left\{-\dfrac{1}{\theta(2\theta+1)}, \dfrac{2\theta-1}{\theta(2\theta+1)}\right\}$, *then the class of* $u$ *is* $\tilde{s}=2$. *(2) If* $\lambda \in \left\{-\dfrac{1}{\theta(2\theta+1)}, \dfrac{2\theta-1}{\theta(2\theta+1)}\right\}$ *and* $\lambda \neq -\dfrac{2\theta-1}{\theta(\theta-1)(2\theta+1)}$, *then the class of* $u$ *is* $\tilde{s}=1$.
*Again, from Proposition 2.7 we have for* $n \geq 0$

$$\tilde{C}_0(x) = (4\theta-1)x^3 + 4\left(\frac{\theta-1}{\theta} + \frac{2\theta-1}{\lambda\theta^2(2\theta+1)}\right)x\ ,$$

$$\tilde{C}_{2n+1}(x) = \big(4(\theta+n)+1\big)x^3$$

$$+\,4\left(\frac{\Big(2\theta-1-\lambda\theta(2\theta+1)\Big)\Gamma(2\theta+n) - (-1)^n\Big(1+\lambda\theta(2\theta+1)\Big)\Gamma(n+2)\Gamma(2\theta)}{c_n(\theta,\lambda)}\right)x\ ,$$

$$\tilde{C}_{2n+2}(x) = (4\theta+4n+3)x^3$$

$$-\,4\left(\frac{\Big(2\theta-1-\lambda\theta(2\theta+1)\Big)\Gamma(2\theta+n) - (-1)^n\Big(1+\lambda\theta(2\theta+1)\Big)\Gamma(n+2)\Gamma(2\theta)}{c_n(\theta,\lambda)}\right)x\ ,$$

$$\tilde{D}_0(x) = 4\theta x^2 + 2\left(\frac{2\theta-1}{\lambda\theta^2(2\theta+1)} + 2\frac{\theta-1}{\theta} - \lambda(2\theta+1)\right)\ ,$$

$$\tilde{D}_{2n+1}(x) = 2\big(2(\theta+n)+1\big)x^2\ ,$$

$$\tilde{D}_{2n+2}(x) = 4(\theta+n+1)x^2 + 2\Big(2\theta-1-\lambda\theta(2\theta+1)\Big)\Gamma(2\theta+n)\left(\frac{2\theta+n}{c_{n+1}(\theta,\lambda)} - \frac{1}{c_n(\theta,\lambda)}\right)$$
$$+\,2(-1)^n\Big(1+\lambda\theta(2\theta+1)\Big)\Gamma(n+2)\Gamma(2\theta)\left(\frac{n+2}{c_{n+1}(\theta,\lambda)} + \frac{1}{c_n(\theta,\lambda)}\right)\ .$$

**Example 3.4** *Let* $v$ *be the associated form of the first order of Gegenbauer. Here[6,7]*

$$\begin{cases} \rho_{2n+1} = \dfrac{4(n+1)(n+\alpha+1)}{(4n+2\alpha+3)(4n+2\alpha+5)}, \\ \rho_{2n+2} = \dfrac{(2n+3)(2n+2\alpha+3)}{(4n+2\alpha+5)(4n+2\alpha+7)}, \quad n \geq 0. \end{cases} \tag{70}$$

$$\begin{cases} \Phi(x) = x^2-1, \quad \Psi(x) = -2(\alpha+2)x, \quad B(x) = \dfrac{(2\alpha+1)}{(2\alpha+3)}, \\ C_n(x) = 2(n+\alpha+1)x, \quad D_n(x) = 2n+2\alpha+3, \quad n \geq 0. \end{cases} \tag{71}$$

*We assume* $2\alpha+1 \neq 0$, *then* $v$ *is a Laguerre-Hahn form of class* $s=0$. *From the Corollary 1.4., u is regular for every* $\lambda \neq 0$.

*From* (18), (21) *and* (70), *we obtain*

$$a_{2n} = \frac{2\lambda(2\alpha+3)\Gamma(n+\frac{3}{2})\Gamma(\alpha+n+\frac{3}{2})\Gamma(\alpha+1)}{\sqrt{\Pi}(4n+2\alpha+3)\Gamma(n+1)\Gamma(\alpha+n+1)\Gamma(\alpha+\frac{3}{2})}, \quad n \geq 0 \tag{72}$$

$$\begin{cases} \gamma_{4n+1} = -\gamma_{4n+2} = -a_{2n}\ , \\ \gamma_{4n+3} = -\gamma_{4n+4} = \dfrac{4(n+1)(n+\alpha+1)}{(4n+2\alpha+3)(4n+2\alpha+5)a_{2n}}\ , \quad n \geq 0\ . \end{cases} \tag{73}$$

*According to Proposition 2.4. and Proposition 2.6., the form u is Laguerre-Hahn of class* $\tilde{s}=3$. *It satisfies* (30) *and* (33) *with*

$$\begin{cases} \tilde{\Phi}(x) = x^5 - x\ , \quad \Psi(x) = -4(\alpha+2)x^4 + \dfrac{4(2\alpha+1)}{\lambda(2\alpha+3)}x^2\ , \\ \tilde{B}(x) = -2\dfrac{2\alpha+1}{\lambda(2\alpha+3)}\ , \\ \tilde{C}_0(x) = (4\alpha+3)x^4 - \dfrac{4(2\alpha+1)}{\lambda(2\alpha+3)}x^2 + 1\ , \\ \tilde{D}_0(x) = 4(\alpha+1)x^3 - 2\left(\lambda(2\alpha+3) + \dfrac{2\alpha+1}{\lambda(2\alpha+3)}\right)x\ . \end{cases} \tag{74}$$

*Finally, from Proposition 2.7, we have for* $n \geq 0$

$$\tilde{C}_0(x) = (4\alpha+3)x^4 - \frac{4(2\alpha+1)}{\lambda(2\alpha+3)}x^2 + 1\ ,$$

$$\tilde{C}_{4n+1}(x) = (8n+4\alpha+5)x^4 - 4(4n+2\alpha+3)a_{2n}x^2 - 1\ ,$$

$$\tilde{C}_{4n+2}(x) = (8n+4\alpha+7)x^4 + 4(4n+2\alpha+3)a_{2n}x^2 + 1\ ,$$

$$\tilde{C}_{4n+3}(x) = (8n+4\alpha+9)x^4 + 16\frac{(n+1)(\alpha+n+1)}{(4n+2\alpha+3)a_{2n}}x^2 - 1\ ,$$

$$\tilde{C}_{4n+4}(x) = (8n+4\alpha+11)x^4 - 16\frac{(n+1)(n+\alpha+1)}{(4n+2\alpha+3)a_{2n}}x^2 + 1\ ,$$

$$\tilde{D}_0(x) = 4(\alpha+1)x^3 - 2\left(\lambda(2\alpha+3) + \frac{2\alpha+1}{\lambda(2\alpha+3)}\right)x\ ,$$

$$\tilde{D}_{4n+1}(x) = 2(4n+2\alpha+3)x^3\ ,$$

$$\tilde{D}_{4n+2}(x) = 4(2n+\alpha+2)x^3 - 2\left(4\frac{(n+1)(\alpha+n+1)}{(4n+2\alpha+3)a_{2n}} - (4n+2\alpha+3)a_{2n}\right)x\ ,$$

$$\tilde{D}_{4n+3}(x) = 2(4n+2\alpha+5)x^3\ ,$$

$$\tilde{D}_{4n+4}(x) = 4(2n+\alpha+1)x^3 - 2\left(\frac{4(n+1)(n+\alpha+1)}{(4n+2\alpha+3)a_{2n}} + (4n+2\alpha+7)a_{2n+2}\right)x\ .$$

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