Random Elementary Comments on Series, Virtual Integrals, and Contours

John Gill February 2015

Self-generating series (SGS) may be defined as follows:

$$g_k(z)=z+\eta_k\varphi(z)\ ,\ \sum\eta_k<\infty\ ,\ |\varphi(z)|<R\ \ \&\ \ g_k(z)\in S \text{ for } z\in S$$

$$G_1(z)=g_1(z)\ ,\ G_n(z)=g_n(G_{n-1}(z))\ .$$

$G_n(z)\to G(z)$ under appropriate conditions. For example

Theorem (Gill, 2011*): Let* $\{g_n\}$ *be a sequence of complex functions defined on* $S=(|z|<M)$*.*

Suppose there exists a sequence $\{\rho_n\}$ *such that* $\sum_{k=1}^{\infty}\rho_n<\infty$ *and* $|g_n(z)-z|<C\rho_n$ if $|z|<M$*.*

Set $\sigma=C\sum_1^{\infty}\rho_k$ *and* $R_0=M-\sigma>0$*. Then, for every* $z\in S_0=(|z|<R_0)$*,*

$G_n(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z)\ \to\ G(z)$*, uniformly on compact subsets of* S_0 *.*

Virtual integrals (VI) were described in [1] : Begin with $g_{k,n}(z)=z+\mu_{k,n}\varphi(z)$, $\mu_{k,n}=\frac{1}{n}$

(where in general $0=\mu_{0,n}<\mu_{1,n}<\cdots<\mu_{n,n}=1$ and $\underset{k}{Sup}\left(\mu_{k,n}-\mu_{k-1,n}\right)\to 0$, $n\to\infty$)

Given $\varphi(z)$ continuous on a domain S, and $z\in S\Rightarrow g_{k,n}(z)\in S$. Define

$$G_{k,n}(z)=g_{k,n}(G_{k-1,n}(z))\ ,\ G_{1,n}(z)=g_{1,n}(z)\ ,\quad \text{which may be written}$$

$$G_n(z)\equiv G_{n,n}(z)=z+\frac{1}{n}\varphi(z)+\frac{1}{n}\varphi(G_{1,n}(z))+\frac{1}{n}\varphi(G_{2,n}(z))+\cdots+\frac{1}{n}\varphi(G_{n-1,n}(z))\ ,\ n\to\infty$$

Now, posit a function

$$\psi(z,t)\ ,\ t\in[0,1]\quad \underline{\text{and}}\quad \psi\left(z,\frac{k}{n}\right)\equiv\lim_{m\to\infty}\varphi\left(G_{mk-1,mn}(z)\right),\text{ with }\int_0^1\psi(z,t)\,dt\text{ defined}:$$

$$G_n(z)-z=\frac{1}{n}\psi\left(z,\frac{1}{n}\right)+\frac{1}{n}\psi\left(z,\frac{2}{n}\right)+\frac{1}{n}\psi\left(z,\frac{3}{n}\right)+\cdots+\frac{1}{n}\psi\left(z,\frac{n}{n}\right)\ \approx\int_0^1\psi(z,t)dt$$

Under suitable conditions : $\psi(z,t)=\varphi(z(t))=\dfrac{dz}{dt}$ and $\int_0^1 \psi(z,t)dt = z(1)-z(0)$ for a parametric curve $z(t)$ defined by the process described above. In practice an explicit form of $z(t)$ occurs infrequently and thus an explicit or even implicit form of the integrand cannot be obtained, although the integral's value is easily calculated. Games that can be played with virtual integrals include:

$$g_{k,n}(z)=z+\frac{k}{n^2}\varphi(z) \Rightarrow G_{n,n}(z)=z+\frac{1}{n^2}\varphi(z)+\frac{2}{n^2}\varphi(G_{1,n}(z))+\cdots+\frac{n}{n^2}\varphi(G_{n-1,n}(z))$$

$$\begin{aligned} G_{n,n}(z)&=z+\frac{1}{n}\cdot\frac{1}{n}\varphi(z)+\frac{1}{n}\cdot\frac{2}{n}\varphi(G_{1,n}(z))+\cdots+\frac{1}{n}\cdot\frac{n}{n}\varphi(G_{n-1,n}(z)) \\ &=z+\frac{1}{n}\cdot\frac{1}{n}\psi\left(z,\frac{0}{n}\right)+\frac{1}{n}\cdot\frac{2}{n}\psi\left(z,\frac{1}{n}\right)+\cdots+\frac{1}{n}\cdot\frac{n}{n}\psi\left(z,\frac{n-1}{n}\right) \end{aligned}$$

$$\Rightarrow \quad G_{n,n}(z)-z \quad \approx \quad \int_0^1 t\psi(z,t)dt$$

and

$$g_{k,n}(z)=z+\frac{1}{k+n}\varphi(z) \Rightarrow G_{n,n}(z)=z+\frac{1}{1+n}\varphi(z)+\frac{1}{2+n}\varphi(G_{1,n}(z))+\cdots+\frac{1}{n+n}\varphi(G_{n-1,n}(z))$$

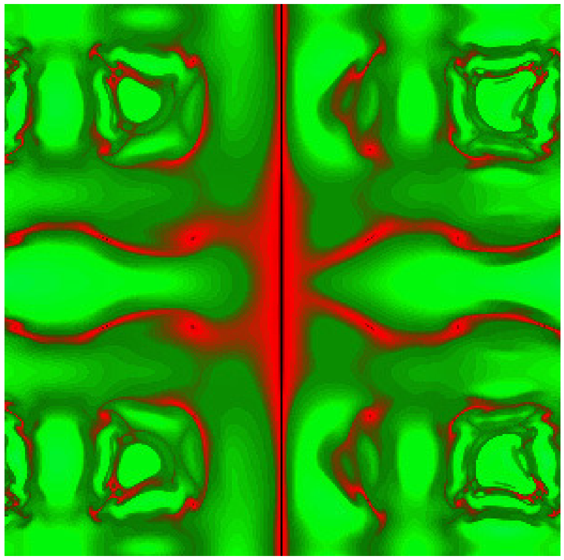
$$\begin{aligned} G_{n,n}(z)&=z+\frac{1}{n}\cdot\frac{1}{\frac{1}{n}+1}\varphi(z)+\frac{1}{n}\cdot\frac{1}{\frac{2}{n}+1}\varphi(G_{1,n}(z))+\cdots+\frac{1}{n}\cdot\frac{1}{\frac{n}{n}+1}\varphi(G_{n-1,n}(z)) \\ &=z+\frac{1}{n}\cdot\frac{1}{\frac{1}{n}+1}\psi\left(z,\frac{0}{n}\right)+\frac{1}{n}\cdot\frac{1}{\frac{2}{n}+1}\psi\left(z,\frac{1}{n}\right)+\cdots+\frac{1}{n}\cdot\frac{1}{\frac{n}{n}+1}\psi\left(z,\frac{n-1}{n}\right) \end{aligned}$$

$$\Rightarrow \quad G_{n,n}(z)-z \quad \approx \quad \int_0^1 \frac{1}{t+1}\psi(z,t)dt$$

Images that follow are topographical, colors corresponding to moduli.

Example 1a: (SGS) $g_k(x+iy)=x+iy+\frac{1}{k^2}\left(xCos(y)+iySin(x)\right)$ or

$x_k+iy_k=x_{k-1}+iy_{k-1}+\frac{1}{k^2}\left(x_{k-1}Cos(y_{k-1})+iy_{k-1}Sin(x_{k-1})\right)$. $S_n(z)=G_n(z)-z$. -10<x,y<10



Hybrid SGS+VI:

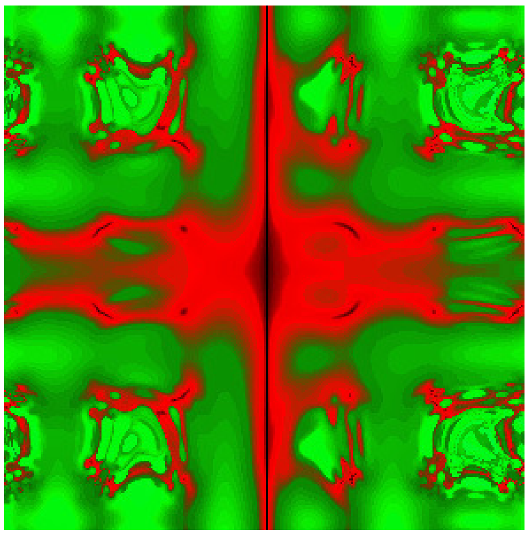
Define $g^*_{k,n}(z)=z+\left(\eta_k-\mu_{k,n}\right)\cdot\varphi(z)$; $G^*_{1,n}(z)=z+\left(\eta_1-\mu_{1,n}\right)\cdot\varphi(z)$, $G^*_{k,n}(z)=g^*_{k,n}\left(G^*_{k-1,n}(z)\right)$

Thus

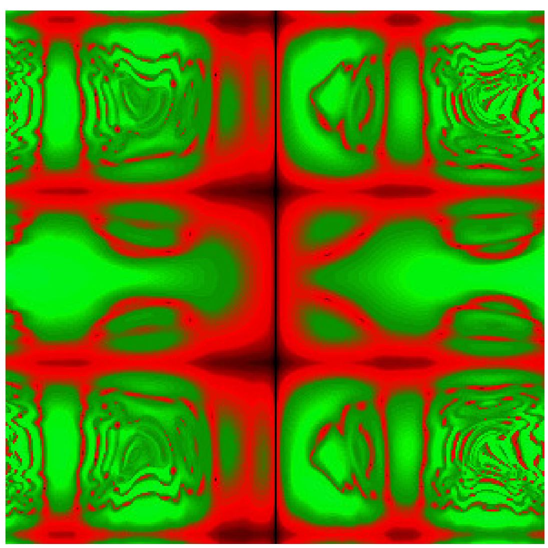
$$
\begin{aligned}
G^*_{n,n}(z) &= z+\eta_1\varphi(z)+\eta_2\varphi(G^*_{1,n}(z))+\cdots+\eta_n\varphi(G^*_{n-1,n}(z)) \\
&\quad -\left[\mu_{1,n}\varphi(z)+\mu_{2,n}\varphi(G^*_{1,n}(z)+\cdots+\mu_{n,n}\varphi(G^*_{n,n}(z)\right] \\
&= z \ + \ S^*_n(z) \ - \ V^*_n(z)
\end{aligned}
$$

Example 1b : $\varphi(x+iy)=xCos(y)+iySin(x)$

$$G^*_{n,n}(z)=z+\frac{1}{1^2}\varphi(z)+\frac{1}{2^2}\varphi(G^*_{1,n}(z))+\cdots+\frac{1}{n^2}\varphi(G^*_{n-1,n}(z))$$
$$-\left[\frac{1}{n}\varphi(z)+\frac{1}{n}\varphi(G^*_{1,n}(z)+\cdots+\frac{1}{n}\varphi(G^*_{n,n}(z)\right] \quad , \quad S^*_n(z)-V^*_n(z)=G^*_{n,n}(z)-z$$
$$= z + S^*_n(z) - V^*_n(z)$$

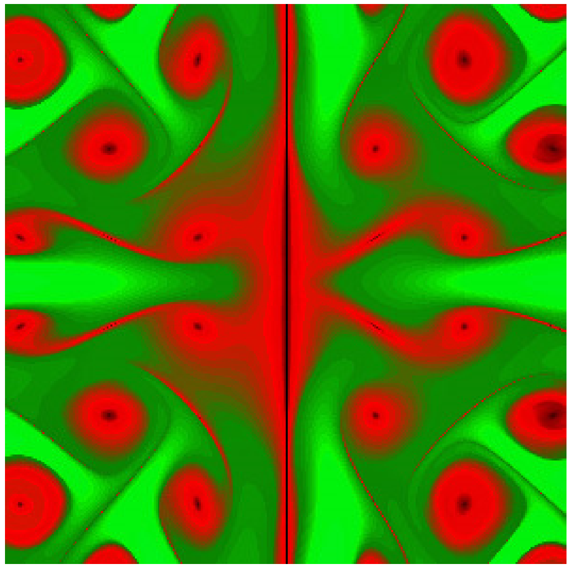


Example 1c : $V^*_n(z)=\frac{1}{n}\varphi(z)+\frac{1}{n}\varphi(G^*_{1,n}(z))+\cdots+\frac{1}{n}\varphi(G^*_{n,n}(z))$, $\varphi(z)=xCos(y)+iySin(x)$



Compare this (VI*) virtual integral with a "standard" virtual integral:

Example 1d : (VI) $V_n(z)=\frac{1}{n}\varphi(z)+\frac{1}{n}\varphi(G_{1,n}(z))+\cdots+\frac{1}{n}\varphi(G_{n-1,n}(z))$, $\varphi(z)=xCos(y)+iySin(x)$,



Integrating a SGS into a virtual integral. Start with a SGS (and a more appropriate notation):

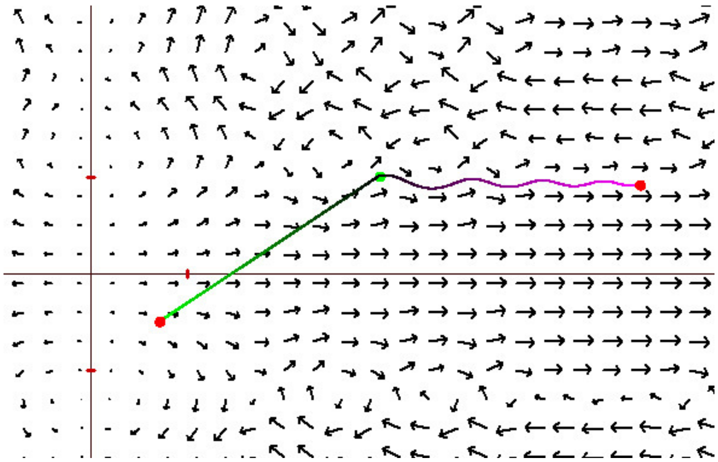
$S_n(z)=z+\eta_1\varphi(z)+\eta_2\varphi(S_1(z))+\cdots+\eta_n\varphi(S_{n-1}(z)) \;\rightarrow\; \alpha(z)$. Define

$V_n^S(z)=\frac{1}{n}\varphi(z)+\frac{1}{n}\varphi(S_1(z))+\cdots+\frac{1}{n}\varphi(S_n(z))$, and

$Z_n^S(z)=\frac{1}{n}\varphi(\alpha(z))+\frac{1}{n}\varphi(\alpha(z))+\cdots+\frac{1}{n}\varphi(\alpha(z)) \;=\; \varphi(\alpha(z))$.

Then a routine ε-argument shows that $V_n^S(z)\rightarrow\varphi(\alpha(z))$ as $n\rightarrow\infty$. All that's required is continuity of φ at appropriate values.

Example 2 : $\varphi(z) = xCos(y^2) + iySin(x^2)$, $\eta_k = \dfrac{1}{k^2}$, $z = 3 + i$, $n = 1000$

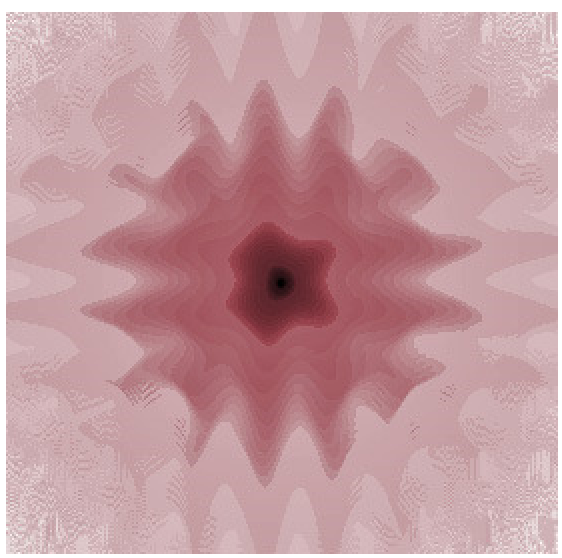


The standard Zeno contour is in purple and the contour $V_n^S(z) + z$ is in green.

Next, reverse the roles in the previous description:

$$V_n(z) = \mu_{1,n}\varphi(z) + \mu_{2,n}\varphi(G_{1,n}(z)) + \mu_{3,n}\varphi(G_{2,n}(z)) + \cdots + \mu_{n-2,n}\varphi(G_{n-1,n}(z))$$
$$S_n^V(z) = z + \eta_1\varphi(z) + \eta_2\varphi(V_1(z)) + \eta_3\varphi(V_2(z)) + \cdots + \eta_{n-2}\varphi(V_{n-3}(z))$$

Example 3 : $\mu_{k,n} = \dfrac{1}{n}$, $\eta_k = \dfrac{1}{k^2}$, $\varphi(z) = xCos(y) + iySin(x)$. $S_n^V(z)$



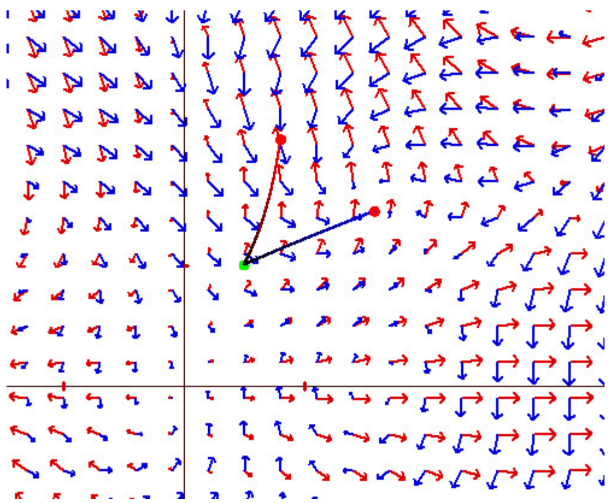
Interlacing contours: "Blend" two contours as follows:

$$z_{k,n} = z_{k-1,n} + \mu_n \varphi_1(\zeta_{k-1,n}) \quad \text{and} \quad \zeta_{k,n} = \zeta_{k-1,n} + \mu_n \varphi_2(z_{k-1,n}) \quad ,$$

analogous to the system

$$\frac{dz}{dt} = \varphi_1(\zeta) \ \text{ and } \ \frac{d\zeta}{dt} = \varphi_2(z) \ .$$

Example 3 : $\varphi_1(z) = xCos(y) + iySin(x)$ and $\varphi_2(z) = y\sin(x+y) + ixCos(x-y)$ [R,B]



A system of three contours follows the same pattern:

$$\begin{aligned} z_{k,n} &= z_{k-1,n} + \mu_n \varphi_1(\zeta_{k-1,n}) \\ \zeta_{k,n} &= \zeta_{k-1,n} + \mu_n \varphi_2(\omega_{k-1,n}) \\ \omega_{k,n} &= \omega_{k-1,n} + \mu_n \varphi_3(z_{k-1,n}) \end{aligned}$$

Analogous to $\quad \dfrac{dz}{dt} = \varphi_1(\zeta) \ , \ \dfrac{d\zeta}{dt} = \varphi_2(\omega) \ , \ \dfrac{d\omega}{dt} = \varphi_3(z)$

Interlacing algorithms:

(1) $z_{k,n}=z_{k-1,n}+\mu_n\varphi_1(\zeta_{k-1,n})$ and $\zeta_{k,n}=\zeta_{k-1,n}+\mu_n\varphi_2(z_{k-1,n})$, $\dfrac{dz}{dt}=\varphi_1(\zeta)$ and $\dfrac{d\zeta}{dt}=\varphi_2(z)$.

Suppose the contour to be graphed is the first. Then it is not difficult to ascertain

$$z_{n,n}-\alpha=\mu_n\sum_{k=0}^{n-1}\varphi_1(\zeta_{k,n}) \sim \int_0^1\psi_1(\alpha,t)dt \sim \int_0^1\varphi_1(\zeta(t))dt \quad \text{where} \quad \zeta_{k,n}=\alpha+\mu_n\sum_{j=1}^{k}\varphi_2(z_{j-1,n})$$

Of course, $\varphi_1(\zeta(t))$ is vague at best, and implies what I like to call a *virtual integral*.

(2)
$$\begin{aligned} z_{k,n}&=z_{k-1,n}+\mu_n\varphi_1(\zeta_{k-1,n})\\ \zeta_{k,n}&=\zeta_{k-1,n}+\mu_n\varphi_2(\omega_{k-1,n})\\ \omega_{k,n}&=\omega_{k-1,n}+\mu_n\varphi_3(z_{k-1,n})\end{aligned} \quad \text{, or} \quad \frac{dz}{dt}=\varphi_1(\zeta) \ , \ \frac{d\zeta}{dt}=\varphi_2(\omega) \ , \ \frac{d\omega}{dt}=\varphi_3(z).$$

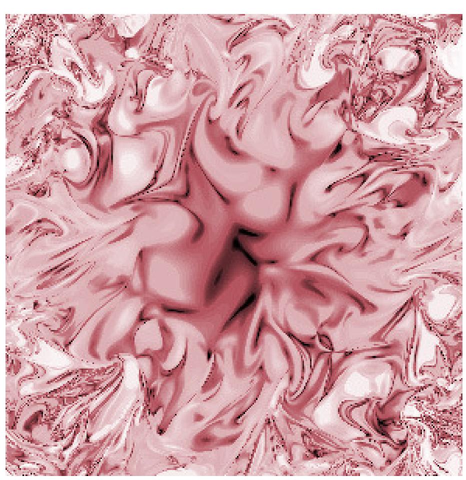
Again, supposing the first contour is our target:

$$z_{n,n}-\alpha=\mu_n\sum_{k=0}^{n-1}\varphi_1(\zeta_{k,n}) \sim \int_0^1\psi_1(\alpha,t)dt \sim \int_0^1\varphi_1(\zeta(t))dt \quad \text{and}$$

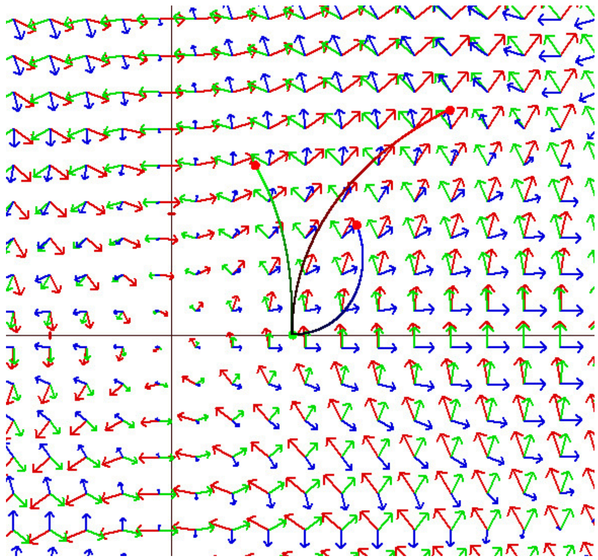
$$\zeta_{k,n}=v_{2,n}(\alpha)+\mu_n\sum_{l=0}^{k-1}\varphi_2\left(v_{3,n}(\alpha)+\mu_n\sum_{j=0}^{l}\varphi_3(z_{j,n})\right) \ , \ v_{m,n}(\alpha)=\alpha+\mu_n\varphi_m(\alpha)$$

Example 4 :

$$\begin{aligned}\varphi_1&=\left(xCos(x+y)+ySin(x-y)\right)+i\left(xSin(x-y)+yCos(x+y)\right)\\ \varphi_2&=xCos(y)+iySin(x)\\ \varphi_3&=2yCos(x)+i2xSin(y)\end{aligned} \qquad \lambda(\alpha)=\int_0^1\psi_1(\alpha,t)dt$$



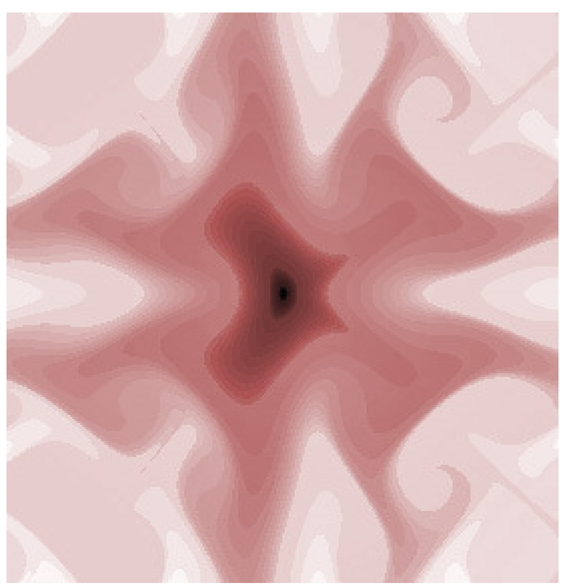
Example 5 : $\varphi_1(z) = 2y + 2xi$, $\varphi_2(z) = -y + xi$, $\varphi_3(z) = xCos(y) + iySin(x)$ [R,G,B]



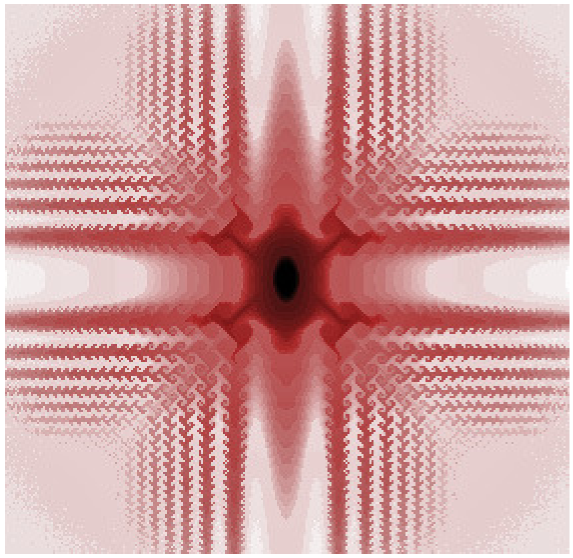
The image of $\nu(\zeta) = \int_0^1 z(\zeta,t)dt$ where $\dfrac{dz}{dt} = \varphi(z,t)$ and $z(\zeta,0) = \zeta$:

From $z_{k,n} = z_{k-1,n} + \mu_{k,n}\varphi(z_{k-1,n})$, $z_{0,n} = \zeta$, we have $\nu \approx \mu_{1,n}z_{1,n} + \mu_{2,n}z_{2,n} + \cdots + \mu_{n,n}z_{n,n}$

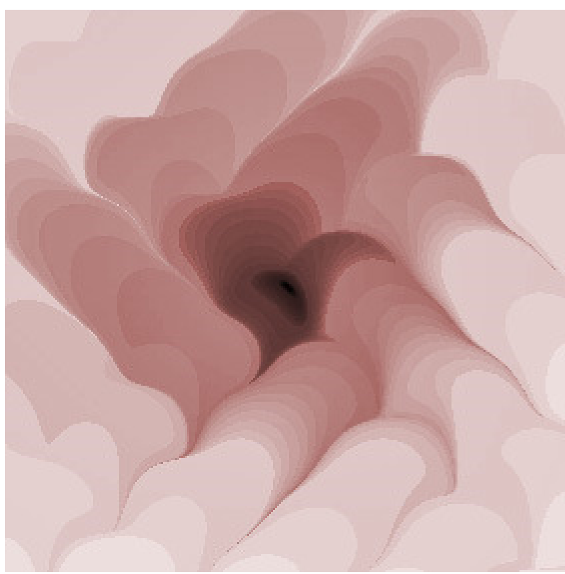
Example 6 : $\nu(\zeta)$ in $\zeta - plane$ for $\varphi(z) = xCos(y) + iySin(x)$



Example 7 : $\nu(\zeta)$ in $\zeta - plane$ for $\varphi(z) = xCos(y^2) + iySin(x^2)$



Example 8 : $\nu(\zeta)$ in $\zeta - plane$ for $\varphi(z) = \left(xSin(x+y) + yCos(x-y)\right) + i\left(xCos(x+y) - ySin(x-y)\right)$



[1] J. Gill, *Informal Notes:Zeno Contours, Parametric Forms, & Integrals* , Comm. Anal. Th. Cont. Frac., Vol XX (2014)