A Note: The Natural Curve of Continued Fraction Approximants

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Abstract: Forward recursion generates consecutive normal approximants of continued fractions and other expansions. Backward recursion does the same for reverse approximants. These approximants lie on natural smooth curves. The comments here extend the author's paper [1]

I. Forward Recursion: $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z),\ F_1(z) \to F_2(z) \to \cdots$

A *linear fractional transformation* (LFT) , $f(\zeta) = \dfrac{a\zeta + b}{c\zeta + d}$, can be written in terms of its (one or) two *fixed points (* α and β *)* and its *multiplier (* K *)*, as

$$\frac{f(\zeta)-\alpha}{f(\zeta)-\beta} = K \cdot \frac{z-\alpha}{z-\beta} \quad \Rightarrow \quad \frac{f^{(n)}(\zeta)-\alpha}{f^{(n)}(\zeta)-\beta} = K^n \cdot \frac{z-\alpha}{z-\beta}$$

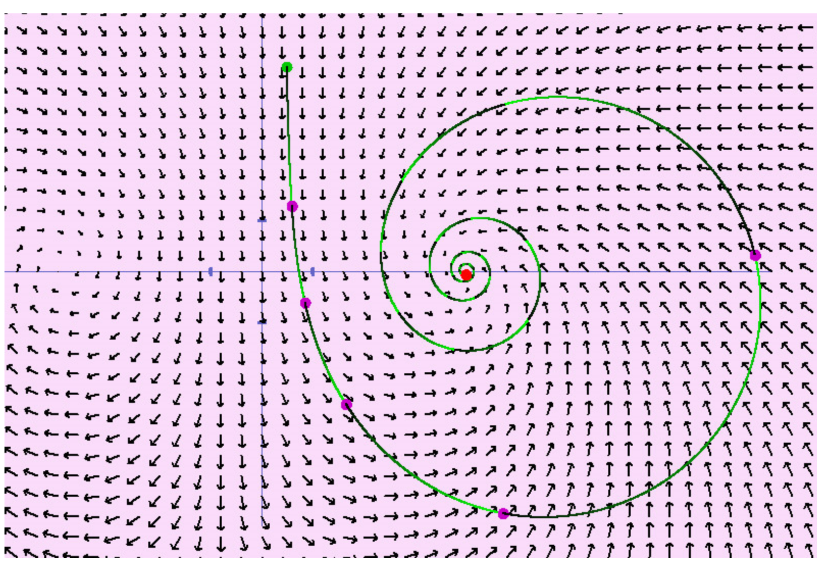
, from which may be deduced

$$f^{(n)}(\zeta) = \frac{\left(\alpha - K^n\beta\right)\zeta + \alpha\beta\left(K^n - 1\right)}{\left(1-K^n\right)\zeta + \left(K^n\alpha - \beta\right)}$$

, leading to a natural continuization

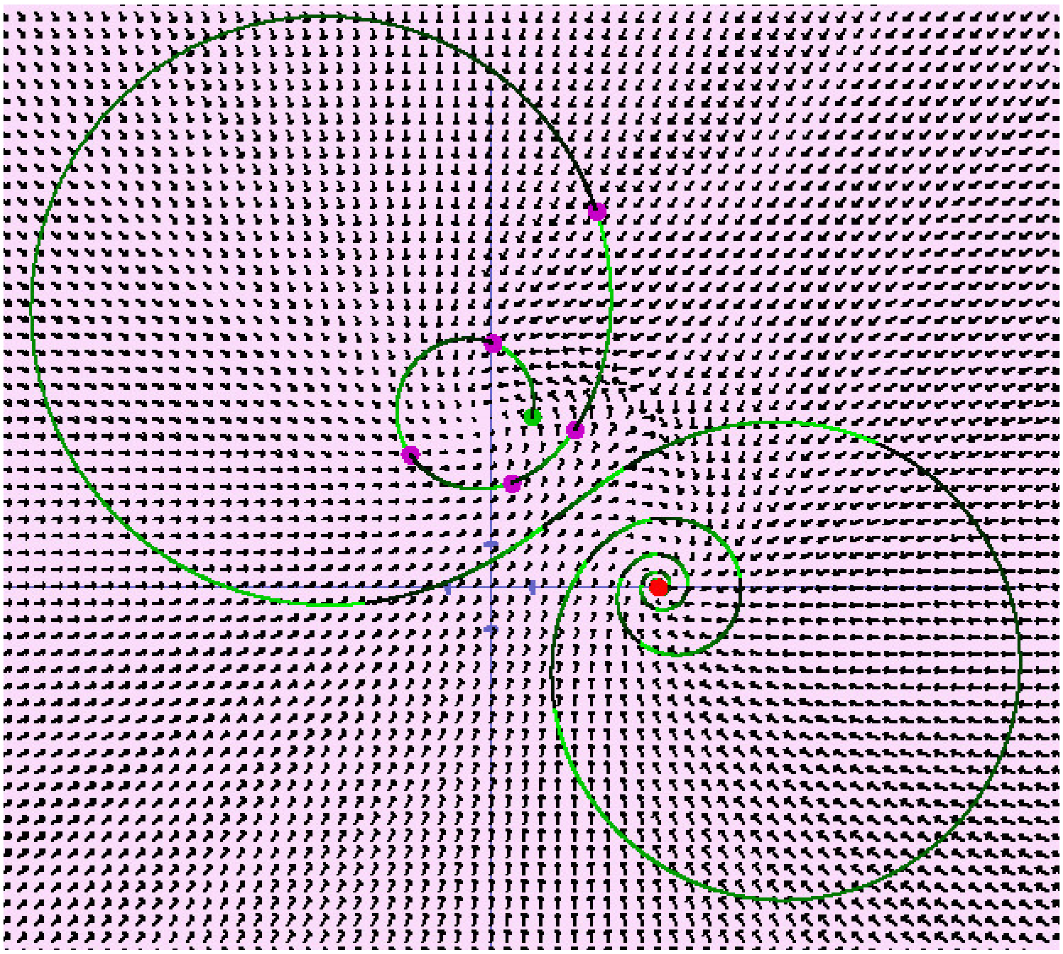
$$f^{(t)}(\zeta) = \frac{\left(\alpha - K^t\beta\right)\zeta + \alpha\beta\left(K^t - 1\right)}{\left(1-K^t\right)\zeta + \left(K^t\alpha - \beta\right)} \quad , \quad t \geq 0 .$$

Example 1 : $\alpha = 4,\ \beta = -4,\ K = .5 + .8i$ with $z = .5 + 4i$, $t : 0 \to 30$



Normal iterates (approximants) occur where the color of the curve changes from green to black. The first five are shown in purple.

Example 1a : $$f^{(t)}(\zeta)=\frac{\left(4-(.8i)^{t}4i\right)\zeta \;+\; 16i\left((8i)^{t}-1\right)}{\left(1-(.8i)^{t}\right)\zeta \;+\; \left(4(.8i)^{t}-4i\right)} \quad , \; \zeta=1+4i \, , \; t:0\rightarrow 30$$



This linear fractional transformation is *loxodromic*, but close to *elliptic*. Hence there is both oscillation about fixed points and convergence to the attractor.

A Strategy for Fixed Point Continued Fractions: $\dfrac{\alpha_1\beta_1}{\alpha_1+\beta_1-}\ \dfrac{\alpha_2\beta_2}{\alpha_2+\beta_2-}\cdots$

From above: Set $\varphi_n(t,\zeta)=\dfrac{\left(\alpha_n-K_n^t\beta_n\right)\zeta+\alpha_n\beta_n\left(K_n^t-1\right)}{\left(1-K_n^t\right)\zeta+\left(K_n^t\alpha_n-\beta_n\right)}$, $K_n=\dfrac{\alpha_n}{\beta_n}$. An adaptation of this:

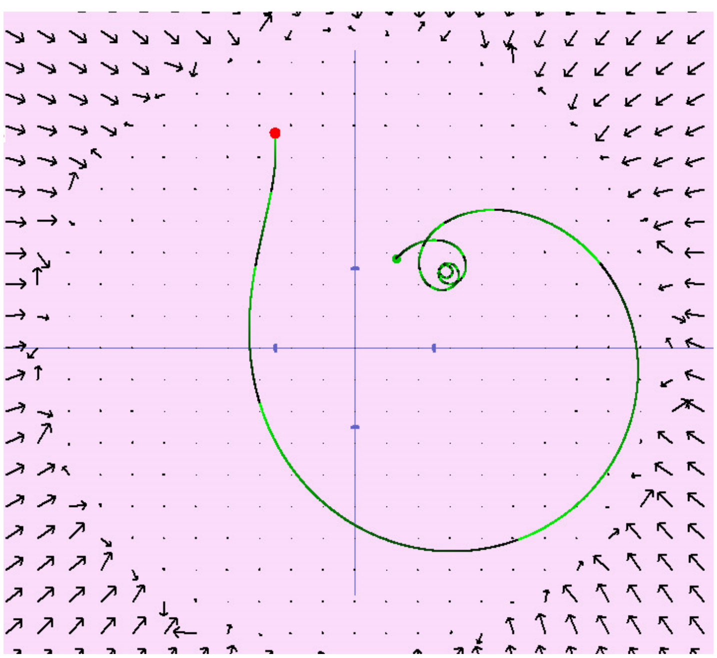
$\varphi_n(t,\zeta)=\dfrac{\alpha_n\beta_n\left(K_n^t-1\right)\mu_n}{\left(K_n^t\alpha_n-\beta_n\right)\mu_n-\zeta}=\dfrac{a_n(t)}{b_n(t)+\zeta}$, with $\mu_n=\dfrac{\beta_n}{\alpha_n-\beta_n}$ a normalizing factor,

may be used in the well-known forward recursion scheme

$A_n=b_nA_{n-1}+a_nA_{n-2}$, $B_n=b_nB_{n-1}+a_nB_{n-2}$, where the nth approximant is $\dfrac{A_n}{B_n}$.

Note that $$\varphi_n(t,\zeta)=\begin{cases}0 & \text{if } t=0\\ \dfrac{\alpha_n\beta_n}{(\alpha_n+\beta_n)-\zeta} & \text{if } t=1\end{cases}$$

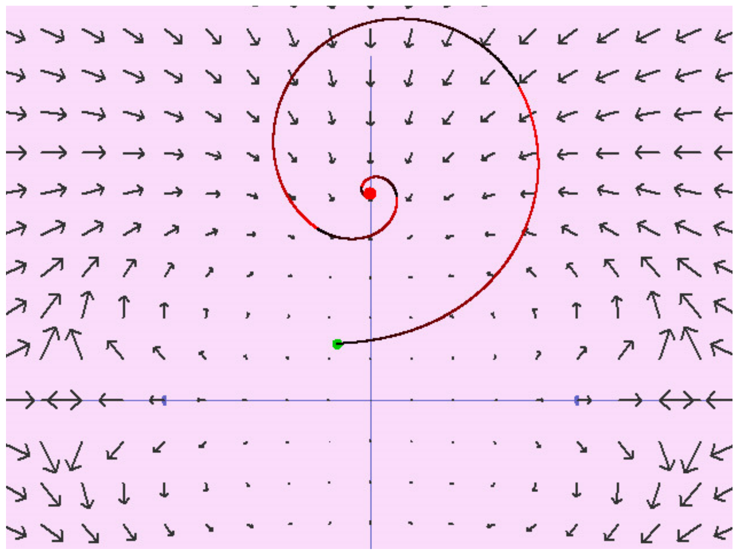
Example 2: $C_n(\zeta)=\dfrac{(1+i\sqrt{n})z}{1+i\sqrt{n}+z-\zeta}$, CF$=\lim\limits_{n\to\infty}C_1\circ C_2\circ\cdots\circ C_n(0)$, $z=-2+3i$. n=40



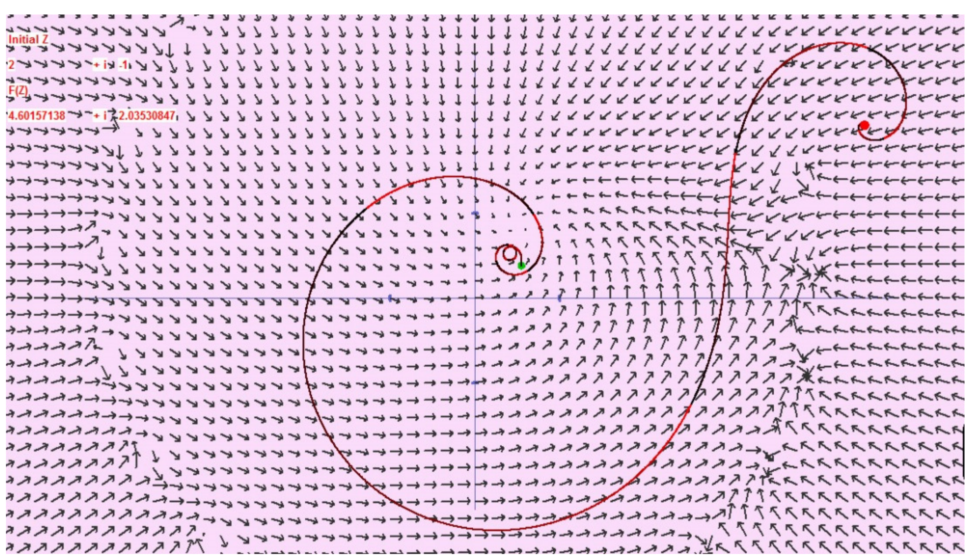
The initial point (green) on the curve is the value of the *first approximant*, not z.

The algorithms for continuization described in this note work best for expansions that are *limit-periodic* and do not have alternating signs.

Example 3: $$Tan(z) = \frac{z}{1-}\ \frac{\frac{1}{3}z^2}{1-}\ \frac{\frac{1}{15}z^2}{1-\cdots}\ \frac{\frac{1}{4k^2-1}z^2}{1-\cdots}\ , \quad Tan(5+8i) \approx 0+i$$



Example 4: $$\frac{\alpha_1\beta_1}{\alpha_1+\beta_1-}\ \frac{\alpha_2\beta_2}{\alpha_2+\beta_2-}\ldots, \text{ where } \quad \alpha_n = \left(x+\tfrac{1}{n}\right)+i\left(y-\tfrac{1}{n}\right),\ \ \beta_n = \left(\tfrac{n}{n+1}\right)+i\left(\tfrac{n}{2}\right),\ \ z = 2-i$$



A general composition of LFTs:

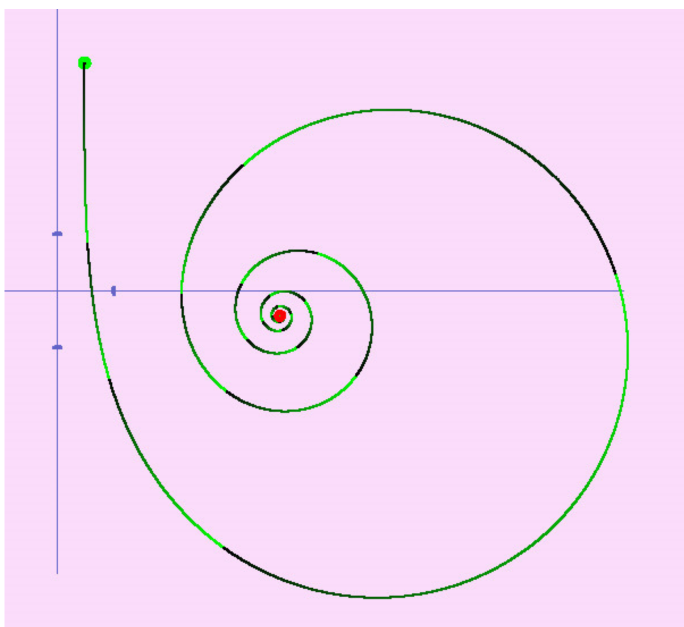
$$f_n(\zeta)=\frac{a_n\zeta+b_n}{c_n\zeta+d_n}\ ,\quad F_n(\zeta)=f_1\circ f_2\circ\cdots\circ f_n(\zeta)\ \to\ F(\zeta).$$

We have $F_n(\zeta)=\dfrac{A_n\zeta+B_n}{C_n\zeta+D_n}$ with the forward recursion scheme

$$A_n=a_nA_{n-1}+c_nB_{n-1},B_n=b_nA_{n-1}+d_nB_{n-1},C_n=a_nC_{n-1}+c_nD_{n-1},D_n=b_nC_{n-1}+d_nD_{n-1}\,.$$

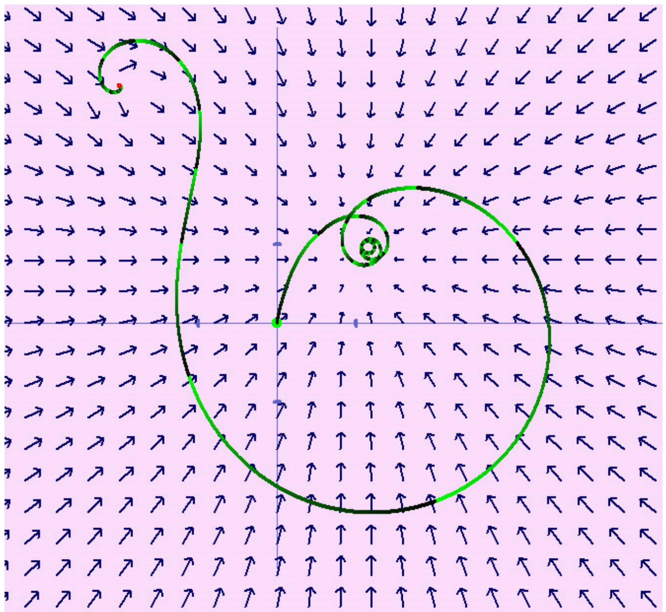
Defining $f_n^{(t)}(\zeta)=\dfrac{\left(\alpha_n-K_n^t\beta_n\right)\zeta+\alpha_n\beta_n\left(K_n^t-1\right)}{\left(1-K_n^t\right)\zeta+\left(K_n^t\alpha_n-\beta_n\right)}=\dfrac{a_n^{(t)}\zeta+b_n^{(t)}}{c_n^{(t)}\zeta+d_n^{(t)}}$ generates an algorithm for continuizing the composition.

Example 5 : (A variation of example 1 above) $\alpha_n=3+\frac{1}{n},\beta_n=-3-\frac{1}{n},K_n=\frac{1}{2+n}+.8i$



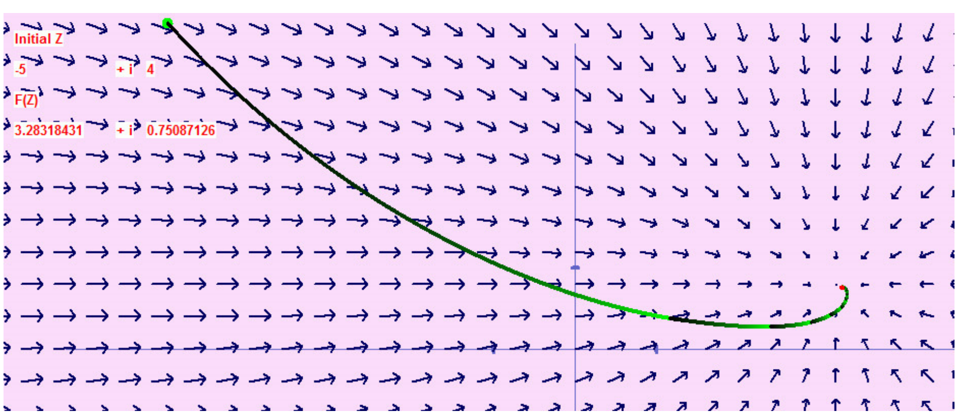
The initial point (green) is the initial value of z.

(Example 2) :(again) The continued fraction using the LFT algorithm of this section:

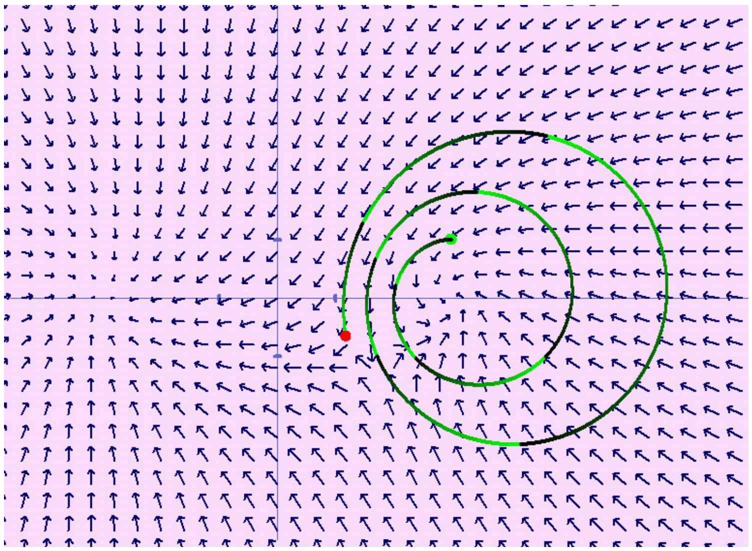


Example 6 : $CF_n(\zeta)=\frac{\left(3+\frac{1}{1}i\right)\left(4+\frac{1}{1}-2i\right)}{\left(7+\frac{1}{1}\right)+\left(\frac{1}{1}-2\right)i-}\;\frac{\left(3+\frac{1}{2}i\right)\left(4+\frac{1}{2}-2i\right)}{\left(7+\frac{1}{2}\right)+\left(\frac{1}{2}-2\right)i-}\;\cdots\;\frac{\left(3+\frac{1}{n}i\right)\left(4+\frac{1}{n}-2i\right)}{\left(7+\frac{1}{n}\right)+\left(\frac{1}{n}-2\right)i-\zeta}$,

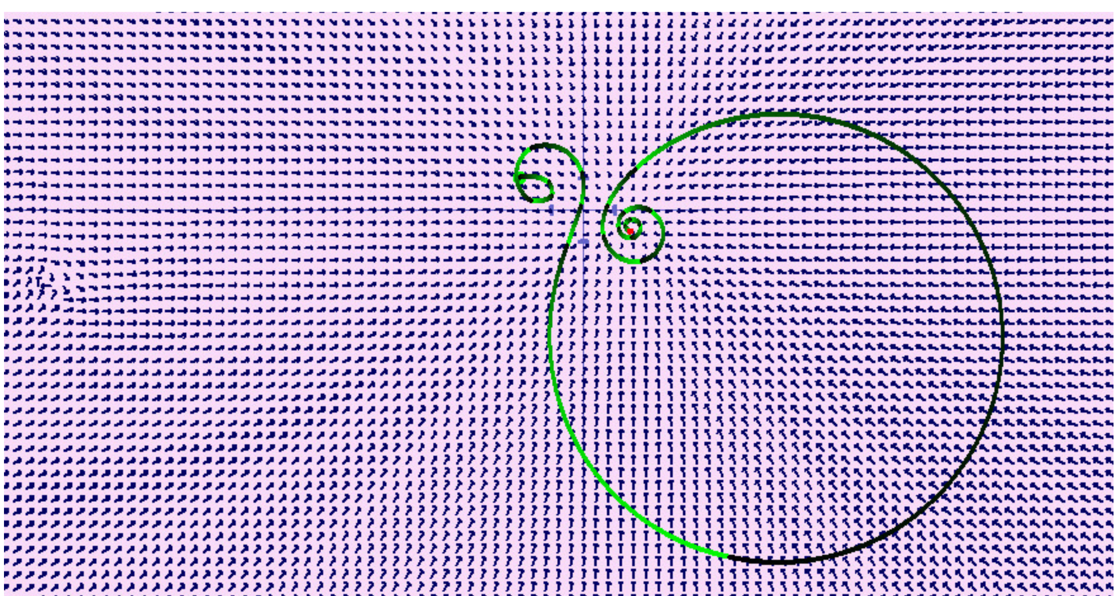
$\zeta=-5+4i,\quad n=10$



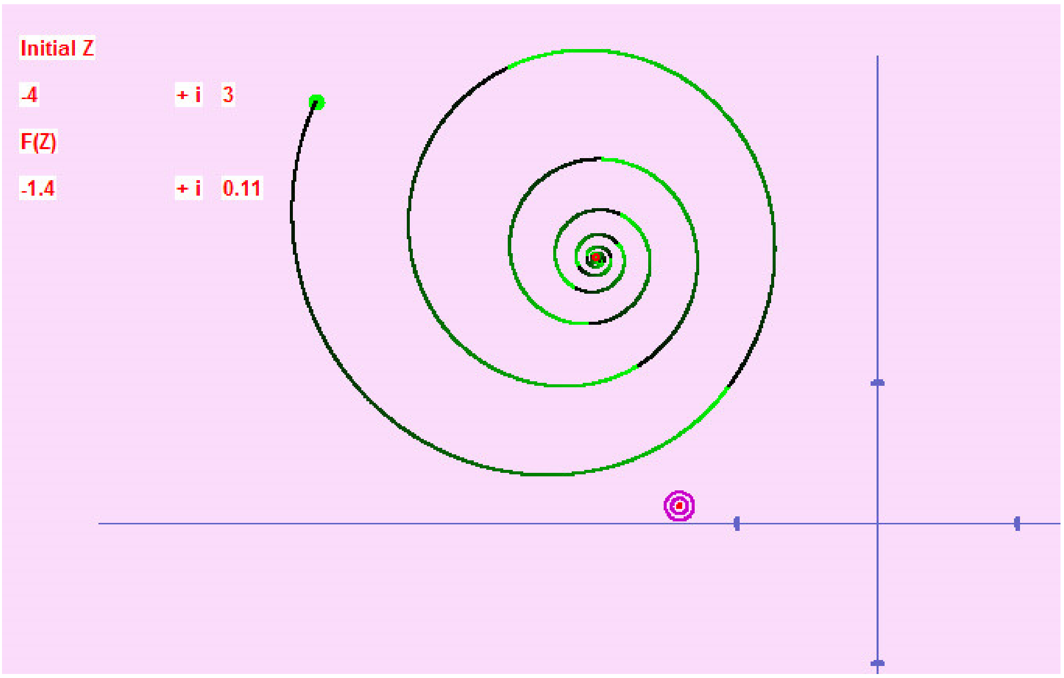
Example 7 : $\alpha_n = 3,\ \beta_n = -3,\ K_n = \frac{1}{10} + \left(1 + \frac{1}{5+n}\right)i$, $z = 3+i,\ n = 10$. These LFTs are close to *elliptic*, whose iterations oscillate and do not converge; loose spiraling occurs:



Example 8 : $\alpha_n = 1 + \frac{1}{n},\ \beta_n = -n,\ K_n = \frac{1}{n} + \frac{4}{5}i,\ z = -2+i,\ n = 20$



Example 9 : $C(z)=\frac{\frac{1}{3}z^2}{1+}\ \frac{\frac{4}{15}z^2}{1+}\ \cdots\ \frac{\frac{k^2}{4k^2-1}z^2}{1+}\ \cdots\ ,\quad F(z)=ArcTan(z)=\frac{z}{1+C(z)}$



The value of the arc tangent is shown as a purple bullseye. Initial value of z in green.

II. Backward Recursion: $F_{k,n}(\zeta)=f_{n-k}\circ f_{n-k+1}\circ f_{n-k+2}\circ\cdots\circ f_{n-k+k}(\zeta)\to F_n(\zeta)$ as $k\to n-1$

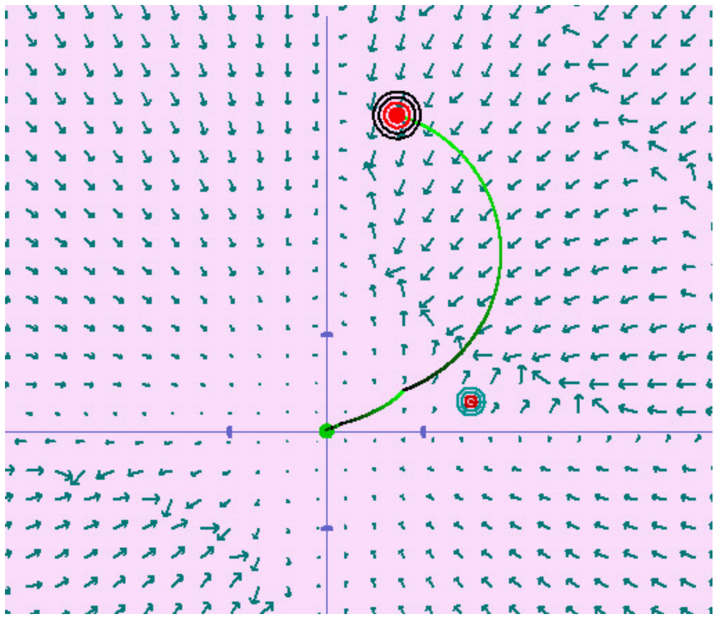
The idea is that *backward recursion* – a very efficient evaluation protocol for CFs – starts at some low level, say $f_n(\zeta)$, and progresses upward to $f_{n-1}\circ f_n(\zeta)$, $f_{n-2}\circ f_{n-1}\circ f_n(\zeta)$, etc. At each step a point in the plane is determined, a sort of *reverse approximant*. The continuization process simply links these discrete points together in a natural continuum.

For LFTs let $t:0\to 1$ in the following

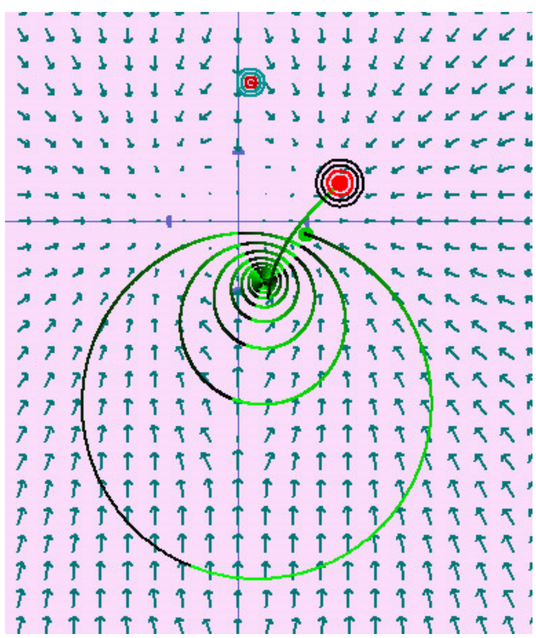
$$F_{k+1,n}^{(t)}(\zeta)=\frac{\left(\alpha_m-K_m^t\beta_m\right)F_{k,n}(\zeta)+\alpha_m\beta_m\left(K_m^t-1\right)}{\left(1-K_m^t\right)F_{k,n}(\zeta)+\left(K_m^t\alpha_m-\beta_m\right)}\quad,\ m=n-k-1\quad,k:0,1,2,3,\ldots,n-1$$

That is to say $$F_{k+1,n}^{(t)}(\zeta)=f_{n-k-1}^{(t)}(F_{k,n}(\zeta))=\begin{cases}F_{k,n}(\zeta)\ if\ t=0\\ F_{k+1,n}(\zeta)\ if\ t=1\end{cases}$$

Example 10: $CF = \frac{z}{1-}\ \frac{\frac{1}{3}z^2}{1-}\ \frac{\frac{1}{15}z^2}{1-\cdots}\ \frac{\frac{1}{4k^2-1}z^2}{1-\cdots} = Tan(z)$, $Tan(1.5+.3i) \approx .722+3.257i$.



Example 11 : $CF = \frac{z}{1+}\ \frac{\frac{1}{3}z^2}{1+}\ \frac{\frac{4}{15}z^2}{1+\cdots}\ \frac{\frac{k^2}{4k^2-1}z^2}{1+\cdots}$, $ArcTan(.2+2i) \approx 1.51+.53i$



Additional Examples:

The procedure described for CFs can be used elsewhere. For example, suppose

(6) $f_n(z) = K_n(z)\cdot(z-\alpha_n)+\alpha_n$ in an appropriate region in the complex plane. Then

a fairly simple theory evolves if $|K_n(z)| < \rho_n < 1$. This condition is not essential, however, and

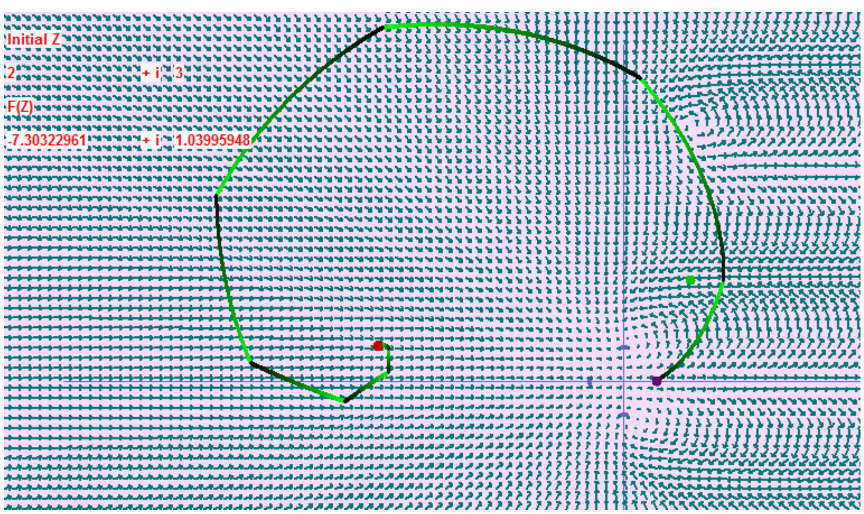
$K_n^t(z)$ might mimic the multiplier for CFs, with $t: 0 \to 1$. If so,

(7) $f_n^{(t)}(z) = K_n^t(z-\alpha_n)+\alpha_n \;=\; \begin{cases} z & if \;\; t=0 \\ f_n(z) & if \;\; t=1 \end{cases}$, and allowing t to increase more or less

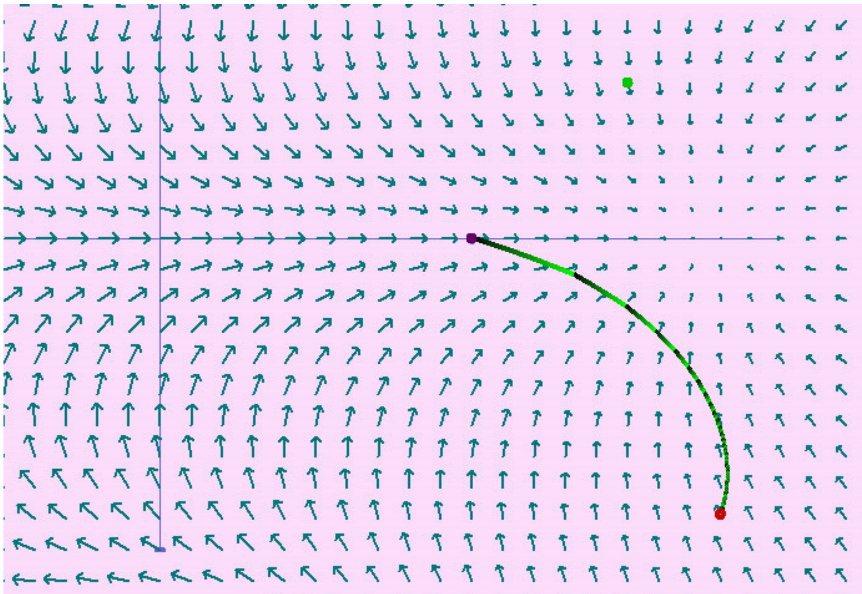
continuously from 0 to 1 may fill in the spaces between either regular or reverse approximants with smooth curves.

Example 12: $e^z = 1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots,\;\; f_n^{(t)}(\zeta) = \left(1+\frac{1}{\zeta}\cdot\frac{z^n}{n!}\right)^t\cdot\zeta\;,\; G_n(\zeta) = f_n \circ f_{n-1} \circ \cdots \circ f_1(\zeta)$

$z = 2+3i$

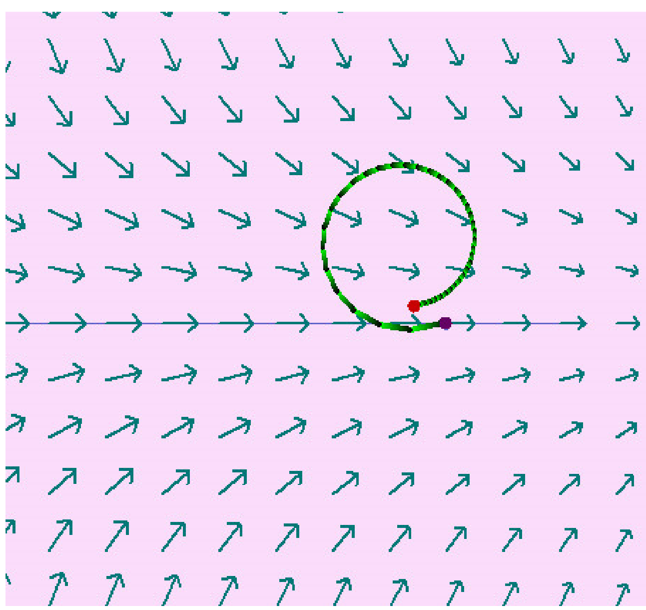


Example 13 : $\zeta(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots$, $f_n^{(t)}(\zeta)=\left(1+\frac{1}{\zeta}\cdot\frac{1}{n^s}\right)^t\cdot\zeta$, $s=1.5+.5i$, $n=50$



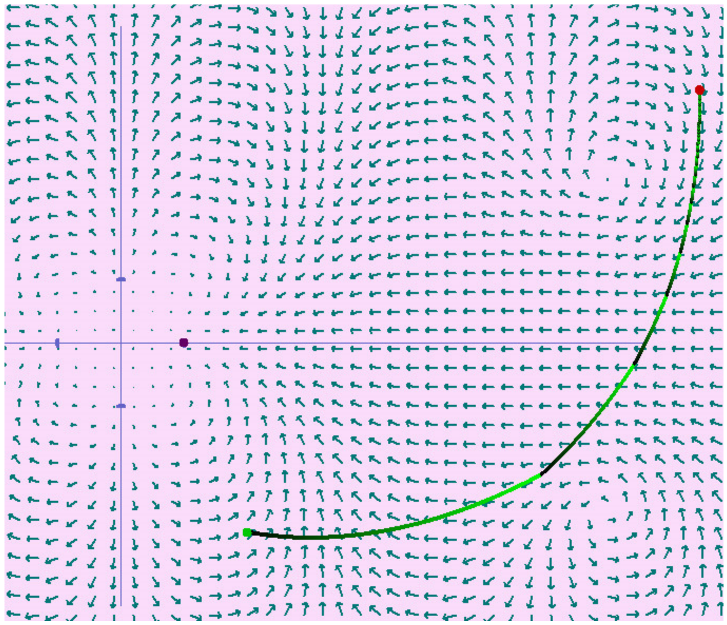
$\zeta(s)$ has an attracting fixed point $\zeta(\alpha)=\alpha\approx 1.82$.

Example 14 : $\zeta_{10}(s)=\frac{1}{10^s}+\frac{1}{11^s}+\frac{1}{12^s}+\cdots$, $s=1.2+4i$, $n=40$



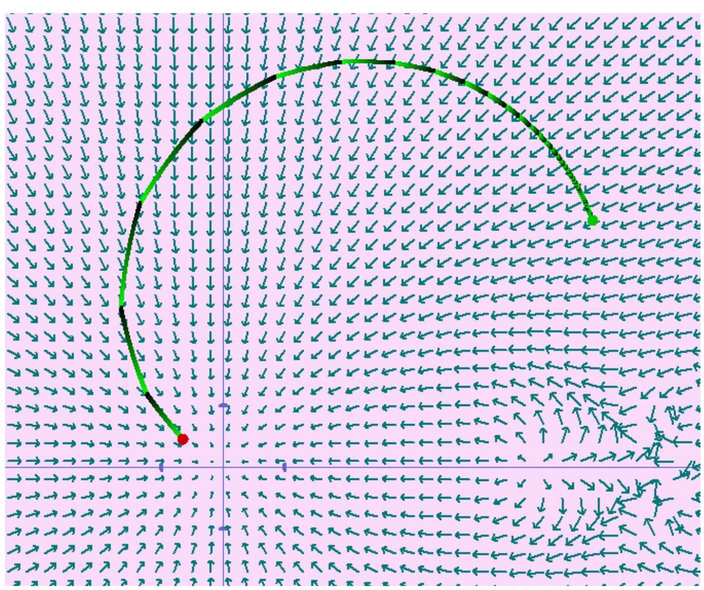
Example 15: $Sin(z)=z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\pi^2 n^2}\right)$, $f_n^{(t)}(\zeta)=\left(1-\frac{z^2}{\pi^2 n^2}\right)^t\cdot\zeta$, $z=2-3i$, $n=80$,

$G_n(\zeta)=f_n\circ f_{n-1}\circ\cdots\circ f_1(\zeta)$



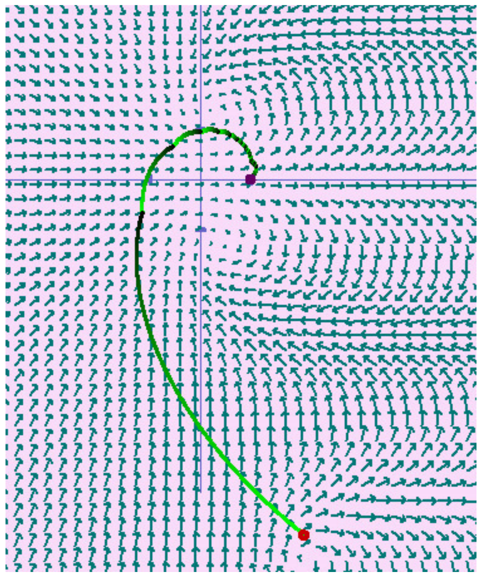
Example 16 : $f_n(z)=e^{\frac{z}{n^2+10}}(z-5)+5$, $K_n^t(z)=e^{\frac{ntz}{n^2+10}}$, $z=6+3i$, $n=50$.

$F_n(z)=f_1\circ f_2\circ\cdots\circ f_n(z)$.

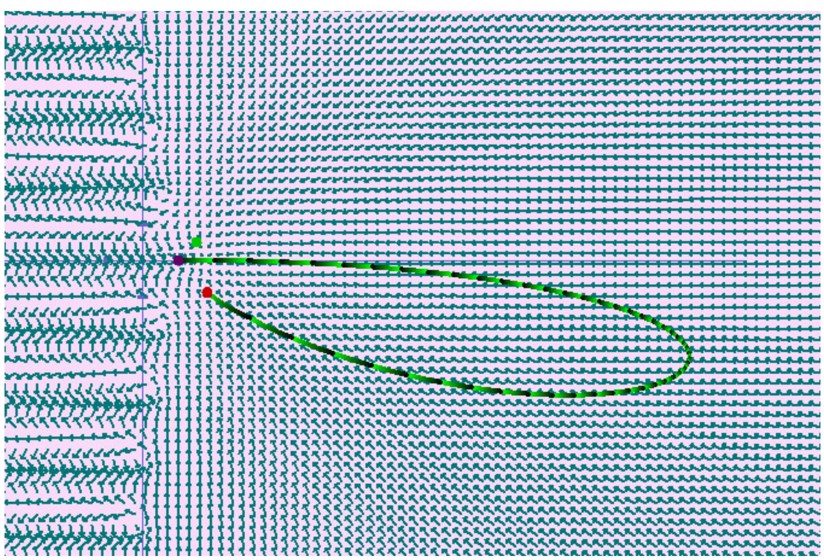


Example 17 : $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots, \quad f_n^{(t)}(\zeta) = \left(\frac{z}{n}\right)^t \cdot (\zeta - \alpha_n) + \alpha_n, \quad \alpha_n = \frac{1}{1 - \frac{z}{n}},$

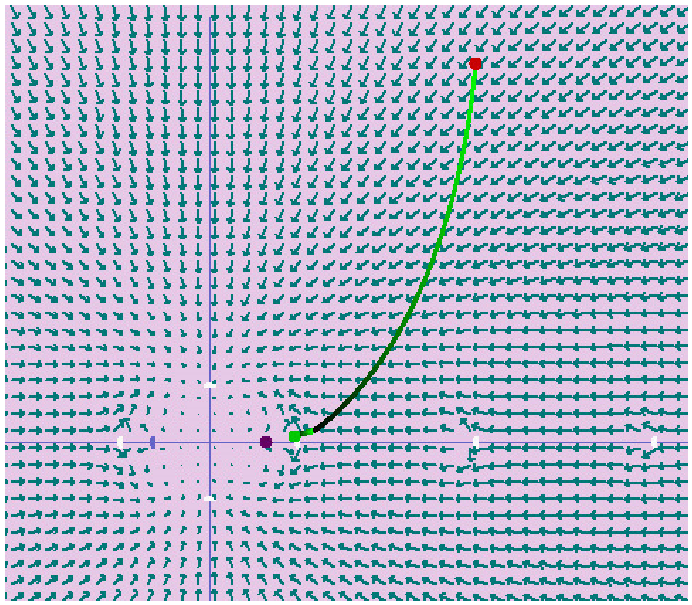
$F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(1) \approx e^z$, $z = 2 + 5i$, $n = 20$



Example 18 : $\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \quad f_n^{(t)}(z) = 1 + \left(\left(1 - \frac{1}{n}\right)^s\right)^t \cdot z$. s = 1.5+.5i n=50



Example 19 : $Tan(z) = \mathcal{R}_{k=1}^{\infty}\left[\frac{z}{1-\frac{1}{4^k}z^2}\right]$, $f_n^{(t)}(z) = \left(\frac{1}{1-\frac{1}{4^n}z^2}\right)^t \cdot z$, $Tan(1.5+.1i) \approx 4.7+6.7i$



[1] J. Gill, *A Natural Continuous Interpolating Structure for Continued Fractions*, J. Comp & Appl Mathematics, 105 (1999).