Images of Infinite Compositions 1

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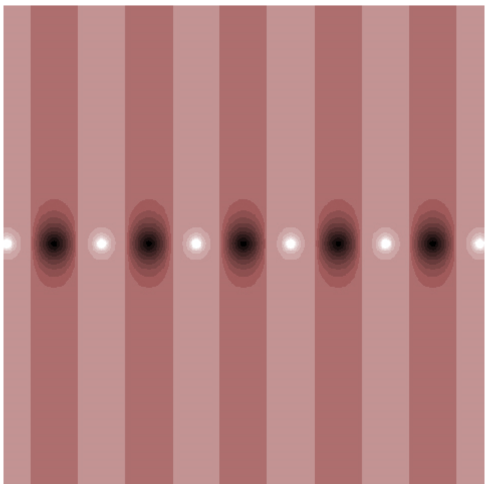
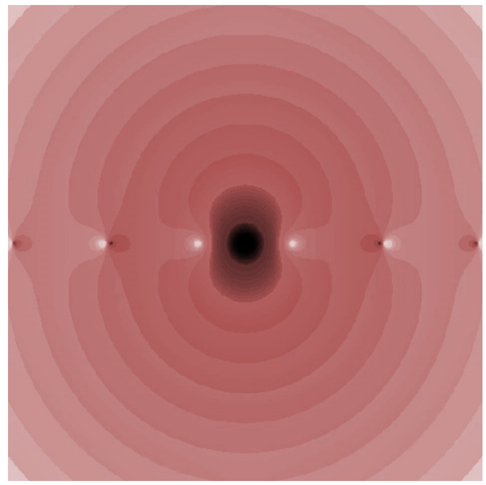
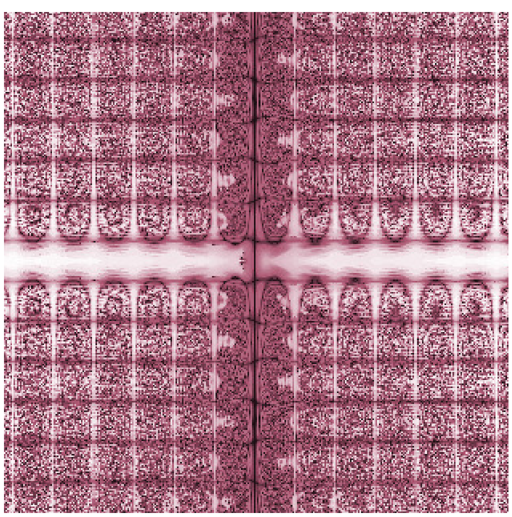
Abstract: Images of infinite compositions – see *Technical Notes* at end.

1. $Tan(z)=\underset{k=1}{\overset{\infty}{\mathcal{R}}}\left[\frac{z}{1-\frac{1}{4^k}z^2}\right]$, n=5 and $Tan(z)-z$, n=20, -8<x,y<8



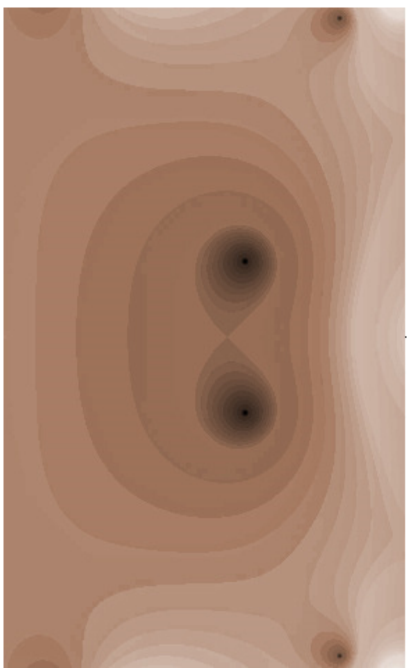


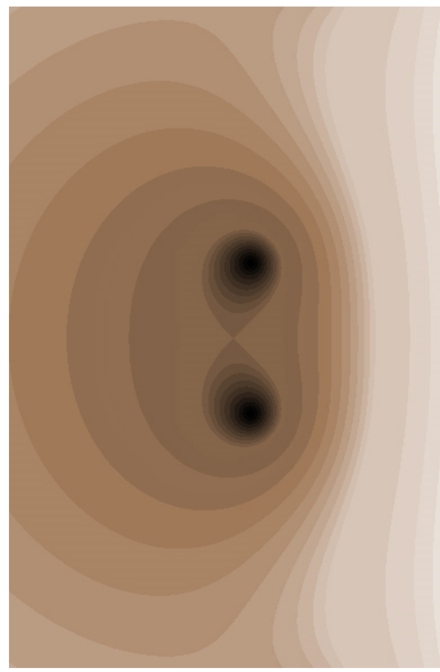
2. $F_{30}(z)=\underset{k=1}{\overset{30}{\mathcal{L}}}\left[z+\frac{1}{2^k}\frac{xCos(y)+i\ ySin(x)}{1+z}\right]$, $[-20<x,y<20]$



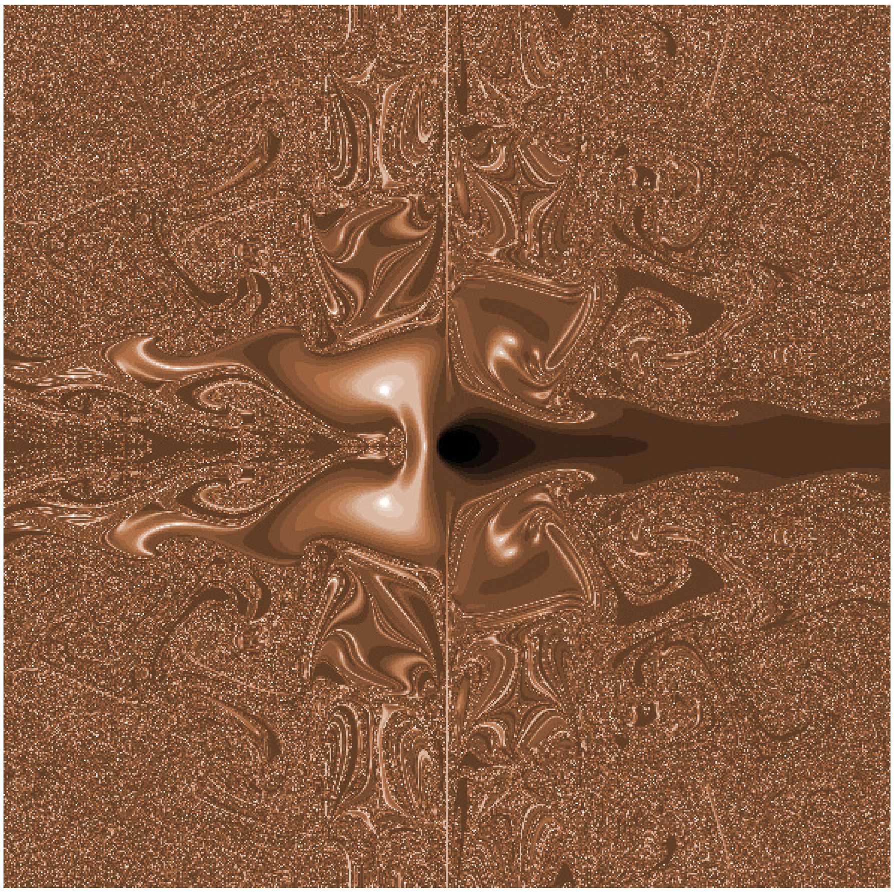
3. $F(z) = e^z = 1 + \mathcal{R}_{k=1}^{\infty}\left(\frac{z^2}{2^{k+1}} + z\right)$, $F(z) - z$, $-4 < x < 3.2, -6 < y < 6$, $n = 50$

and $G(z) = 1 + \mathcal{L}_{k=1}^{\infty}\left[z + \frac{z^2}{2^{k+1}}\right]$, $-4 < x < 4, -6 < y < 6$, $G(z) - z$

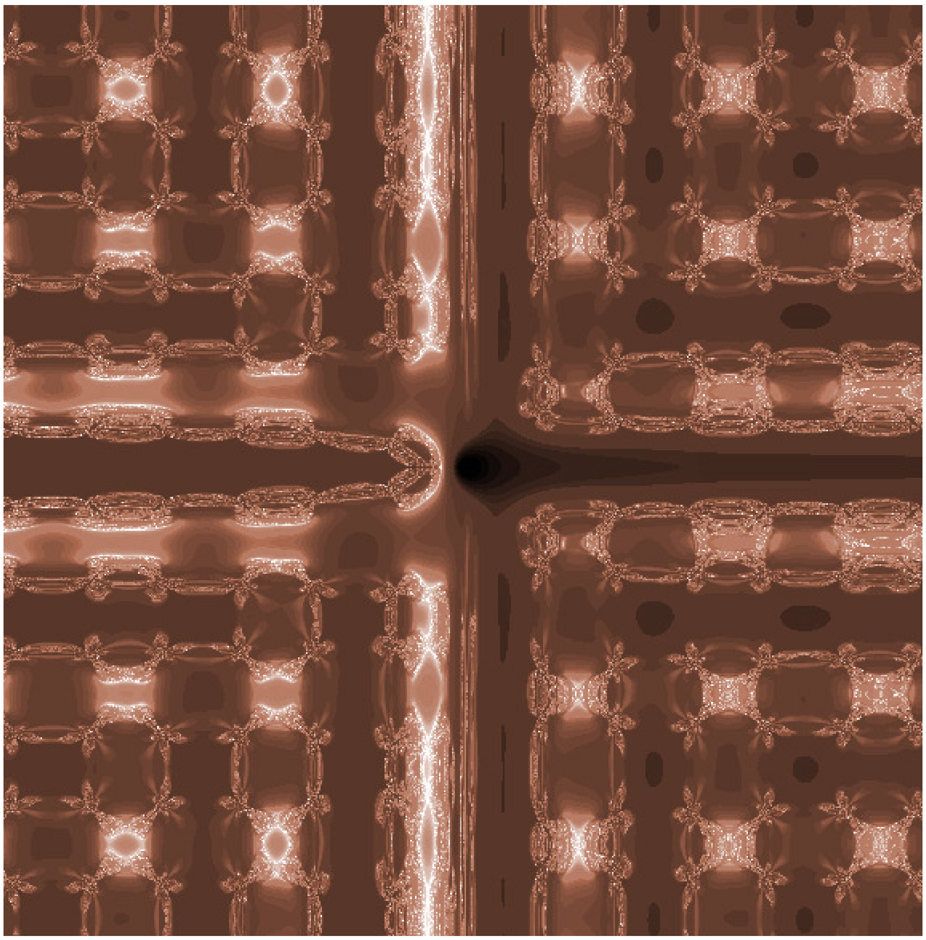




4. $F_{40}(x+iy)=\underset{k=1}{\overset{40}{\mathcal{R}}}\left(\frac{x+iy}{1+\frac{1}{2^k}\left(xCos(y)+iySin(x)\right)}\right),\ -20<x,y<20$



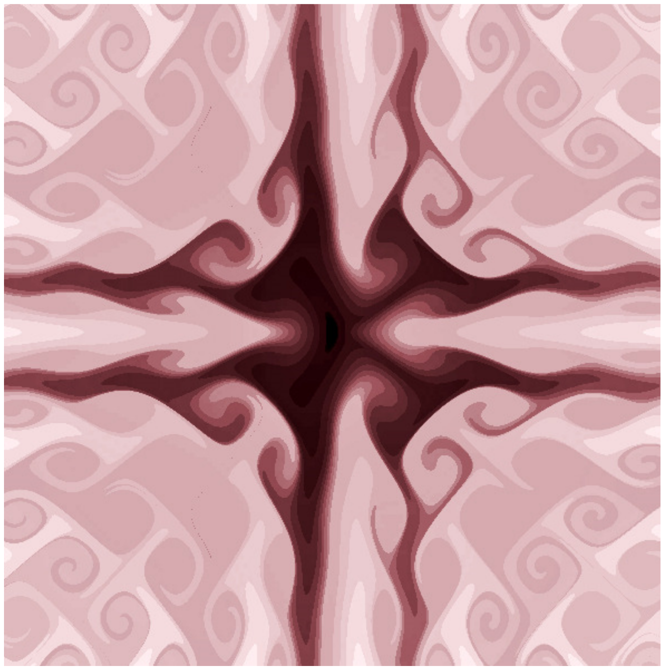
5. $G_{40}(x+iy)=\underset{k=1}{\overset{40}{\mathcal{L}}}\left(\frac{x+iy}{1+\frac{1}{2^k}\left(xCos(y)+iySin(x)\right)}\right)$, $-20<x,y<20$



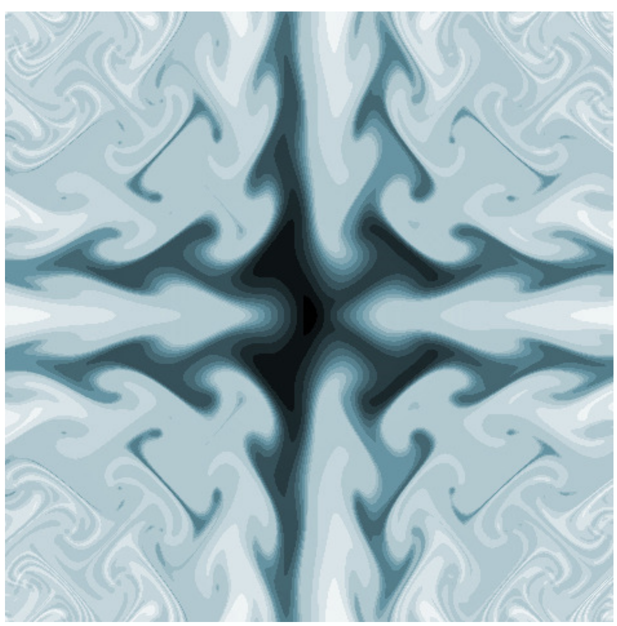
Definition: $\lambda^*(z) := z + \lambda(z) = z + \int_0^1 \psi(z,t)dt = \int_0^1 (z + \psi(z,t))dt =: \int_0^1 \psi^*(z,t)dt$,

Collapsed Virtual Integral (see Technical Notes)

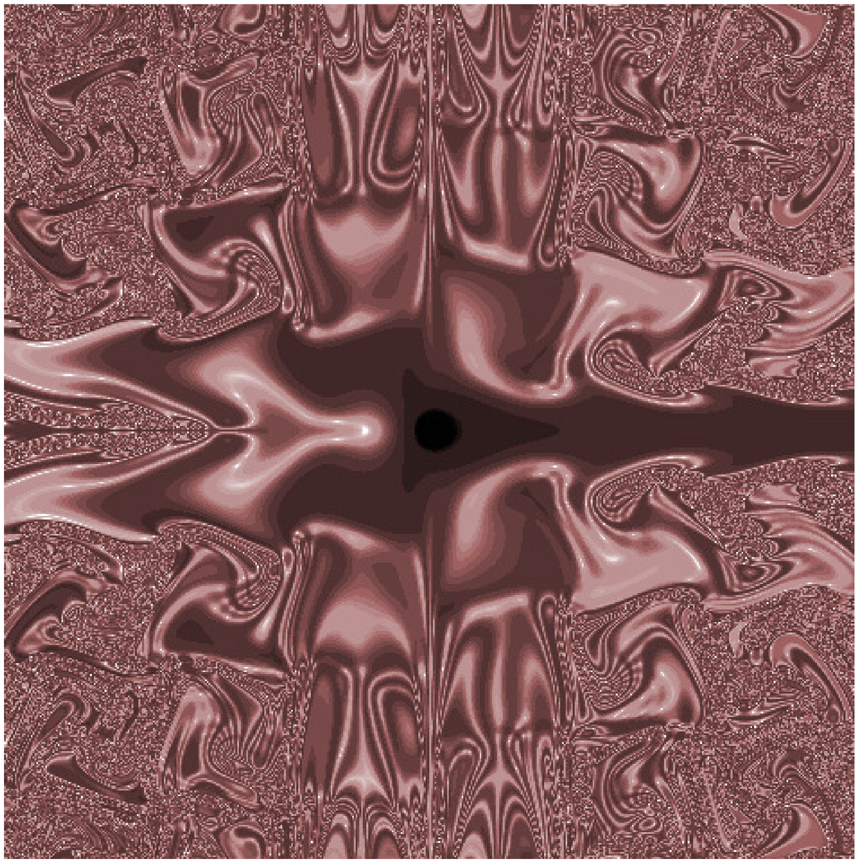
6. $g_{k,n}(z) = z + \frac{1}{n}\left(xCos(y) + iySin(x)\right)$, $\lambda^*(z) = \lim_{n\to\infty}\left(g_{n,n} \circ g_{n-1,n} \circ \cdots \circ g_{1,n}(z)\right)$, $-20 < x, y < 20$



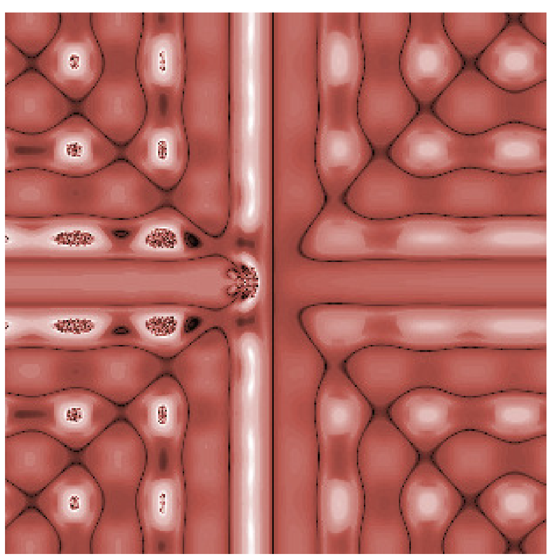
7. $g_n(z) = z + \frac{1}{2^n}\left(xCos(y) + iySin(x)\right)$, $\mathrm{F}(z) = \lim_{n\to\infty}\left(g_1 \circ g_2 \circ \cdots \circ g_n(z)\right)$, $-20 < x, y < 20$



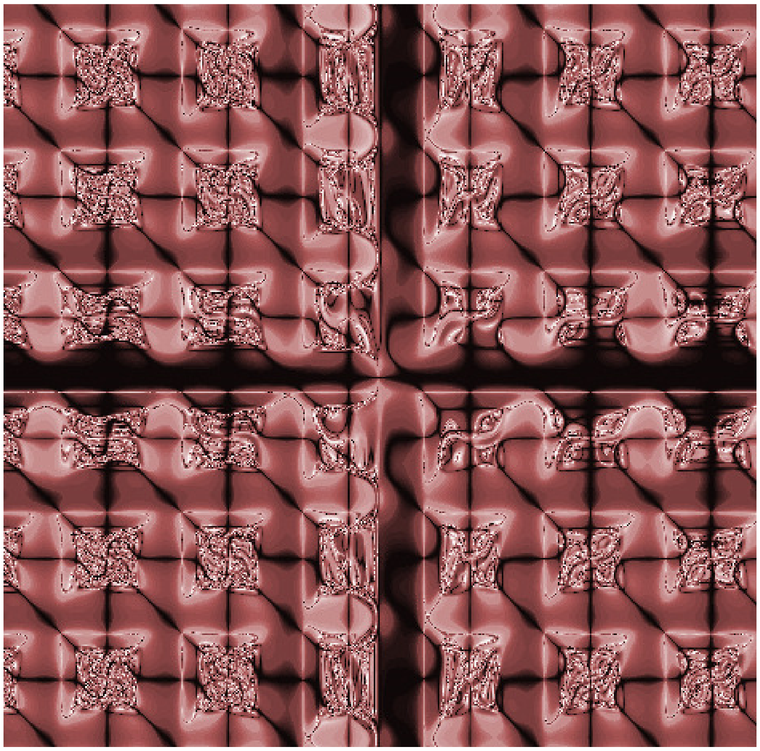
8. $F_{40}(x+iy) = \mathcal{R}_{k=1}^{40}\left(\frac{x+iy}{1+\frac{1}{4^k}\left(xCos(y)+iySin(x)\right)}\right), \quad -20<x,y<20$



9. $G_{20}(x+iy) = \mathcal{L}_{k=1}^{20}\left(\frac{xCos(y)+iySin(x)}{1+\frac{1}{2^k}\left(x+iy\right)}\right), \quad -20<x,y<20$



10. $G_{20}(x+iy)=\underset{k=1}{\overset{20}{\mathcal{L}}}\left[\frac{xSin(y)+iyCos(x)}{1+\frac{1}{2^k}\left(xe^{4Cos(y)}+iye^{4Sin(x)}\right)}\right], \quad -20<x,y<20$



Technical Notes

Infinite compositions of analytic functions occur in two forms:

I Inner or right compositions: $\underset{k=1}{\overset{n}{\mathcal{R}}}t_k(z)=t_1\circ t_2\circ\cdots\circ t_n(z)$, $T(z)=\lim_{n\to\infty}\underset{k=1}{\overset{n}{\mathcal{R}}}t_k(z)$.

II Outer or left compositions: $\underset{k=1}{\overset{n}{\mathcal{L}}}t_k(z)=t_n\circ t_{n-1}\circ\cdots\circ t_1(z)$, $T(z)=\lim_{n\to\infty}\underset{k=1}{\overset{n}{\mathcal{L}}}t_k(z)$.

Theory of convergence includes the following:

Theorem 1 (Gill) Let $\{g_n\}$ be a sequence of complex functions defined on $S=(|z|<M)$. Suppose there exists a sequence $\{\rho_n\}$ such that $\sum_{k=1}^{\infty}\rho_n<\infty$ and $|g_n(z)-z|<C\rho_n$ if $|z|<M$. Set $\sigma=C\sum_{1}^{\infty}\rho_k$ and $R_0=M-\sigma>0$. Then, for every $z\in S_0=\left(|z|<R_0\right)$,
$G_n(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z)\ \to\ G(z)$, uniformly on compact subsets of S_0.

Theorem 2 (Gill) Consider functions $\{f_n(z)\}$ analytic for $|z|\le R_0$. If $|f_n(z)-z|\le C\rho^n$ for $0\le\rho<1$ and $|z|\le R_0=R+\sigma$, where $\sigma=C\dfrac{\rho}{1-\rho}$, then
$\lim_{n\to\infty}\left(f_1\circ f_2\circ\cdots\circ f_n(z)\right)=F(z)$ for $|z|\le R$. Convergence is uniform on compact subsets.

Theorem 3 (Lorentzen) Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D. Suppose there exists a compact set $\Omega\subset D$ such that for each n, $f_n(D)\subset\Omega$. Then $F_n(z)=f_1\circ f_2\circ\cdots\circ f_n(z)$ converges uniformly in D to a constant function F(z) = λ.

Theorem 4 (Gill) Let $\{g_n\}$ be a sequence of functions analytic on a simply-connected domain D and continuous on the closure of D. Suppose there exists a compact set $\Omega\subset D$ such that $g_n(D)\subset\Omega$ for all n. Define $G_n(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z)$. Then $G_n(z)\to\alpha$ uniformly on the closure of D *if and only if* the sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in Ω converge to the number α.

Definition: *Zeno contour*. Let $g_{k,n}(z)=z+\eta_{k,n}\varphi(z)$ where $z\in S$ and $g_{k,n}(z)\in S$ for a convex set S in the complex plane. Require $\lim_{n\to\infty}\eta_{k,n}=0$, where (usually) $k=1,2,...,n$. Set $G_{1,n}(z)=g_{1,n}(z)$, $G_{k,n}(z)=g_{k,n}\left(G_{k-1,n}(z)\right)$ and $G_n(z)=G_{n,n}(z)$ with $G(z)=\lim_{n\to\infty}G_n(z)$, when that limit exists. The *Zeno contour* is a graph of this iteration. The word *Zeno* denotes the infinite number of actions required in a finite time period if $\eta_{k,n}$ describes a partition of the time interval [0,1]. Normally, $\varphi(z)=f(z)-z$ for a vector field, $\mathbb{F}=f$. The alternative notation $G_n(z)=\mathop{\mathcal{L}}_{k=1}^{n}g_{k,n}(z)$ is also available. *Euler's method* is a finite example of a ZC.

Begin with $\eta_{k,n} = \frac{1}{n}$ and $g_{k,n}(z) \equiv z + \frac{1}{n}\varphi(z)$ with $\varphi(z)$ continuous on a domain S, and

$z \in S \Rightarrow g_{k,n}(z) \in S$. (If the underlying vector field is *time-dependent*, $g_{k,n}(z) \equiv z + \frac{1}{n}\varphi(z, \frac{k}{n})$)

Thus $\quad G_{n,n}(z) = z + \frac{1}{n}\varphi(z) + \frac{1}{n}\varphi(G_{1,n}(z)) + \frac{1}{n}\varphi(G_{2,n}(z)) + \cdots + \frac{1}{n}\varphi(G_{n-1,n}(z))$.

Now, *imagine* a function

$$\psi(z,t)\ ,\ t \in [0,1] \quad \underline{\text{and}} \quad \psi\left(z, \frac{k}{n}\right) \equiv \lim_{m \to \infty} \varphi\left(G_{mk-1,mn}(z)\right), \text{ with } \int_0^1 \psi(z,t)\,dt \text{ defined :}$$

$$G_n(z) - z = \frac{1}{n}\psi\left(z, \frac{1}{n}\right) + \frac{1}{n}\psi\left(z, \frac{2}{n}\right) + \frac{1}{n}\psi\left(z, \frac{3}{n}\right) + \cdots + \frac{1}{n}\psi\left(z, \frac{n}{n}\right) \approx \int_0^1 \psi(z,t)dt$$

And for t irrational, $\psi(z,t) = \lim_{t_r \to t} \psi(z, t_r)$ for rational t_r .

The existence of this function (and the integral) is equivalent to the convergence of the Zeno contour. $\int_0^1 \psi(z,t)\,dt$ is more a *virtual* integral since its analytical form can be murky at times.

Then: $$\boxed{\lambda(z) = \int_0^1 \psi(z,t)\,dt = G(z) - z}$$ which is valid for both normal VFs and TDVFs .

Write the recurrence sequence (for TDVF) as $\quad Z\left(z_0, \frac{k}{n}\right) = Z\left(z_0, \frac{(k-1)}{n}\right) + \frac{1}{n}\varphi\left(Z\left(z_0, \frac{(k-1)}{n}\right), \frac{k}{n}\right)$

Assuming $Z = Z(z_0, t)$, one concludes $\frac{\Delta_{k,n} Z}{\Delta_n t} = \varphi\left(Z\left(z_0, \frac{(k-1)}{n}\right), \frac{k}{n}\right) \Rightarrow \frac{dZ}{dt} = \varphi\left(Z(z_0,t), t\right)$,

$t \in [0,1]$.

Then: $$\boxed{\psi(z,t) = \varphi(z(t),t) \Rightarrow \lambda(z_0) = \int_0^1 \varphi(z(t),t)\,dt = z(1) - z(0)}$$

All images derived from mathematics/graphics programs the author has written in Visual Basic/Liberty Basic.