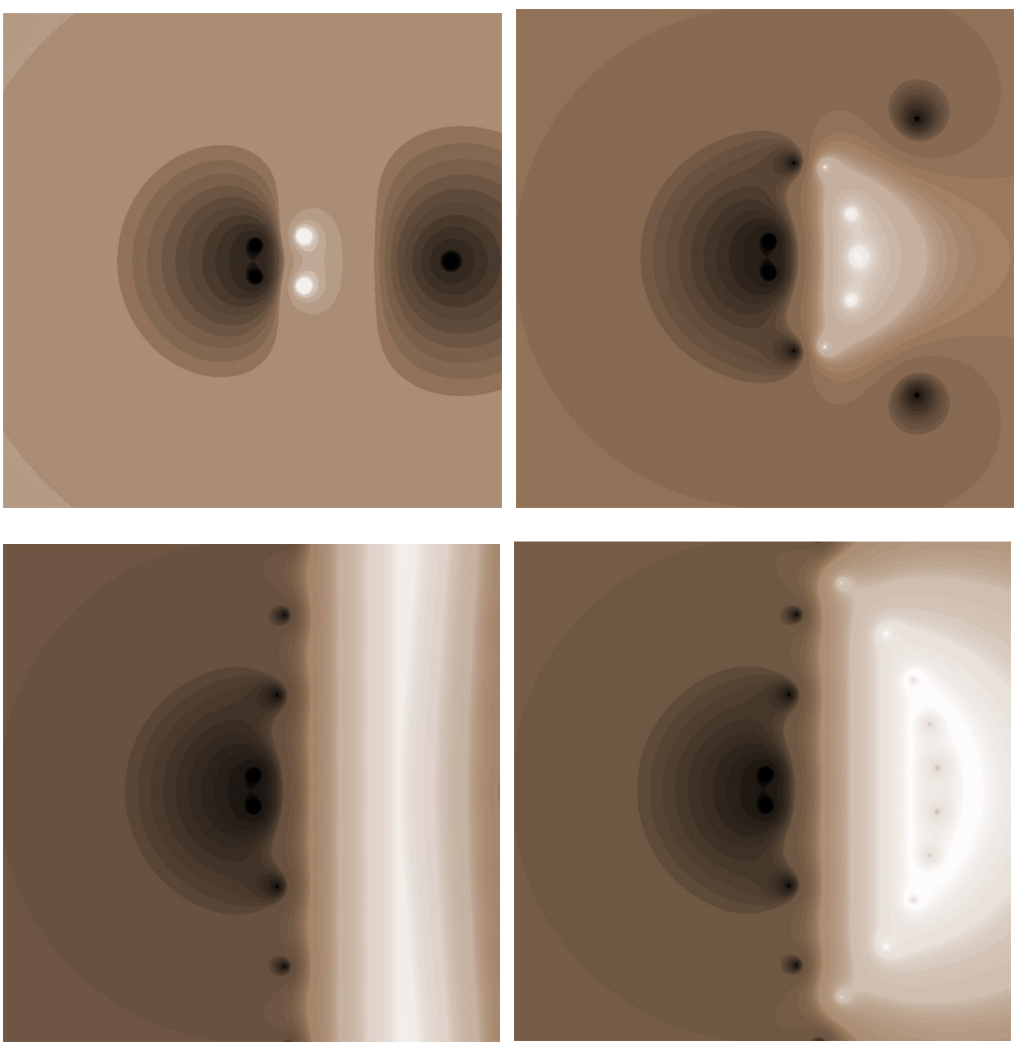
Images of Continued Fraction Expansions I

John Gill September 2012

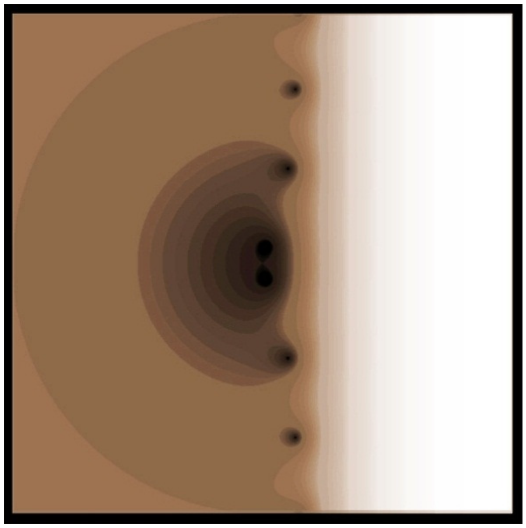
Some analytic functions have continued fraction expansions whose approximants are rational functions. Thus the approximants of the continued fraction have both fixed points and singularities (poles). What becomes of these mathematical artifacts as as $n \to \infty$? For instance, the exponential function should not have poles, although its converging rational approximants do. The following images are simple flux graphs in which dark hues indicate very little change from z to $f(z)$, and light hues show considerable change, including poles in white.

Figures 1: $e^z = 1 + \frac{z}{1-}\frac{z}{2+}\frac{z}{3-}\frac{z}{2+}\frac{z}{5-}\frac{z}{2+\cdots}$ Clockwise from top left: n=5, n=10, n=20, n=40 .

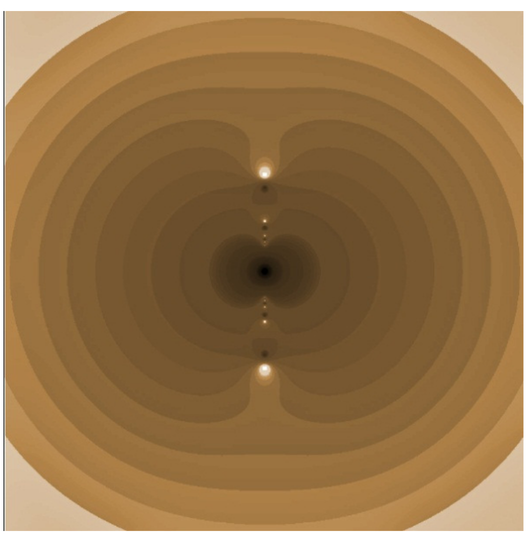
$-20 \le x, y \le 20$.

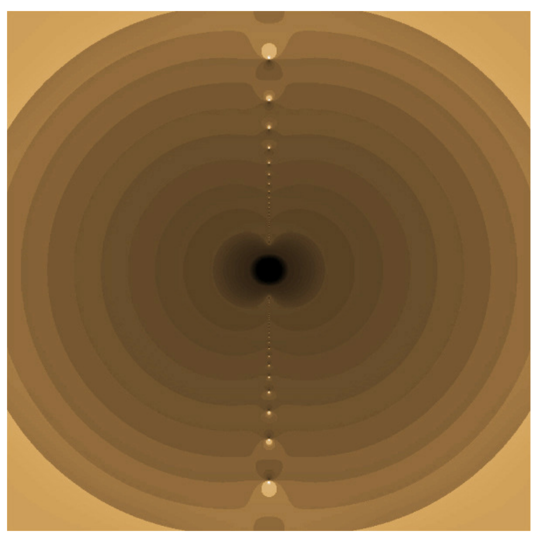


Here is the actual image of $f(z)=e^z$ over $-20 \le x, y \le 20$.



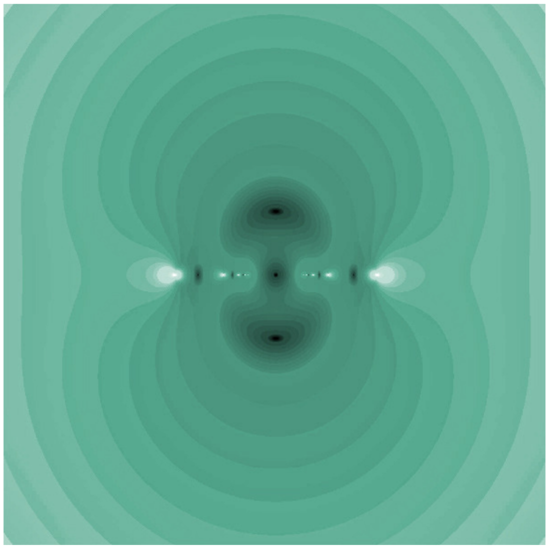
Figures 2: $Arc\tan(z)=\frac{z}{1+}\frac{1^2z^2}{3+}\frac{2^2z^2}{5+}\frac{3^2z^2}{7+\cdots}$ valid for all values of z not on the branch lines extending up from the branch point z=i and down from the branch point z=-i . Image on the left is for n=10, on the right for n=100. Both on $-10 \le x, y \le 10$. Poles are on branch lines.

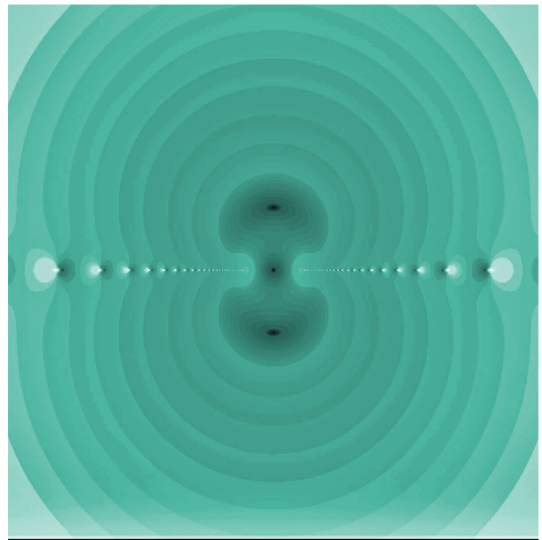




Figures 3: $Ln\left(\frac{1+z}{1-z}\right)=\frac{2z}{1-}\frac{1^2z^2}{3-}\frac{2^2z^2}{5-}\frac{3^2z^2}{7-\cdots}$ on $-10\le x,y\le 10$. On the left n=10, and on the right n=100. Branch points at z=1 and z=-1, with branch lines (with poles) extending outwards.

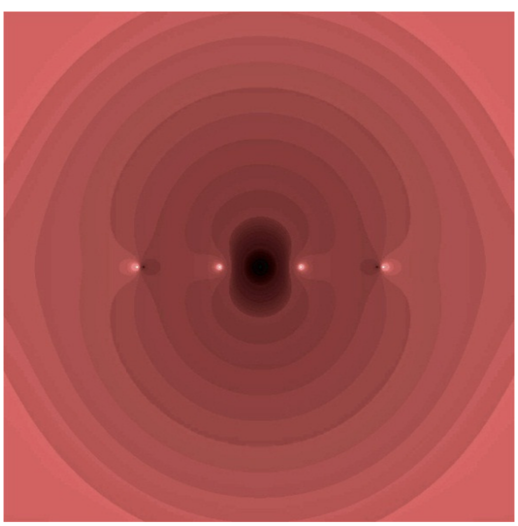
Convergence of the CF everywhere except on branch lines.

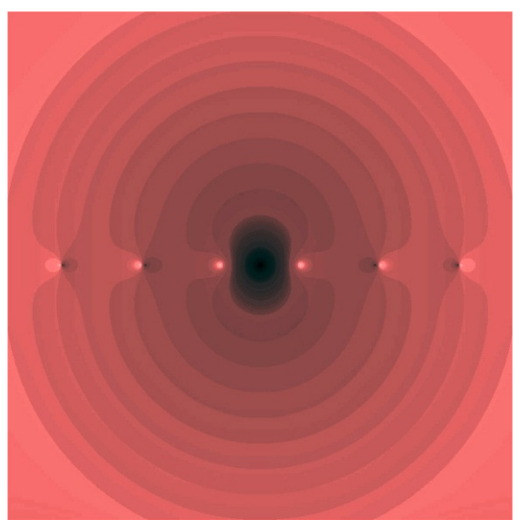




Figures 4: $Tan(z)=\frac{z}{1-}\frac{z^2}{3-}\frac{z^2}{5-}\frac{z^2}{7-\cdots}$ valid for $z\in\mathbb{C}$ (includes point at infinity)

On the left n=5 and on the right n=100. $-10\le x,y\le 10$.





Here the poles are not mere artifacts, but represent points where $Tan(z)=\infty$.

Figures 5: $F(z)=\frac{z}{1-}\frac{\alpha_1 z(1-\alpha_1 z)}{1-}\frac{\alpha_2 z(1-\alpha_2 z)}{1-}\frac{\alpha_3 z(1-\alpha_3 z)}{1-\cdots}$ where $\alpha_n=\frac{1}{n}$. This is a (modified) *fixed-point* CF with the two fixed points of each linear fractional transformation $T_n(w)=\frac{\alpha_n z(1-\alpha_n z)}{1-w}$ displayed in the partial numerators. In the left image n=5, in the right n=100 with $-12\le x,y\le 12$. Convergence (in the extended plane) for any $|z|\le R$ may be ascertained by applying Worpitzky's theorem to the tail end of the CF.

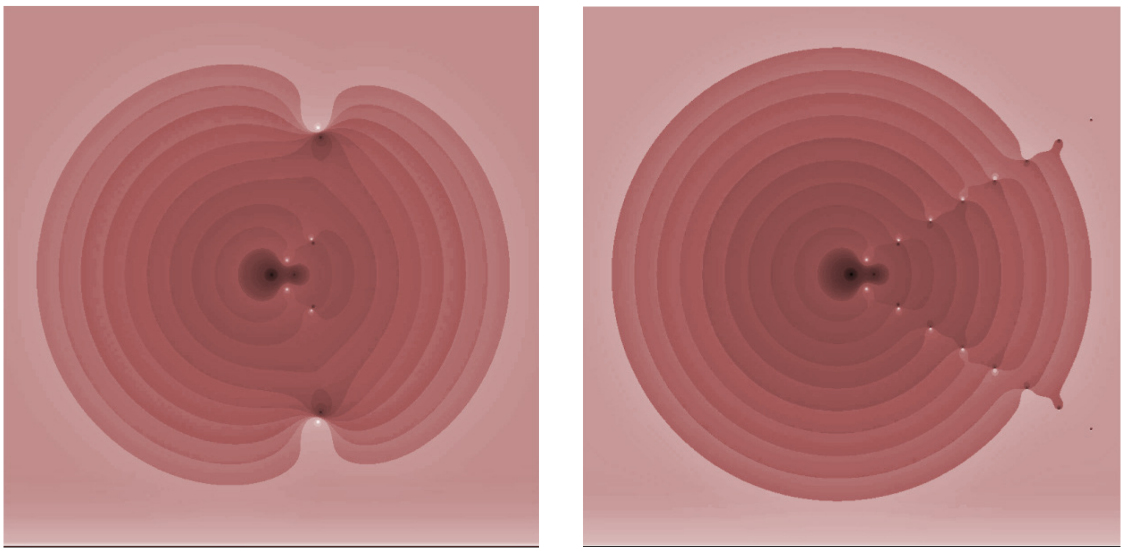
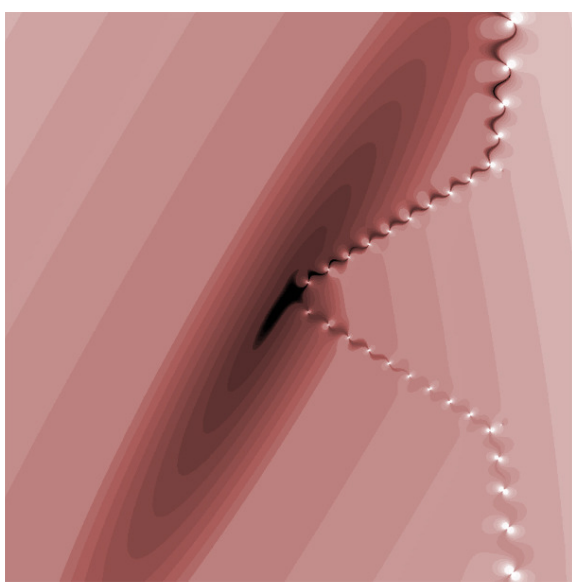


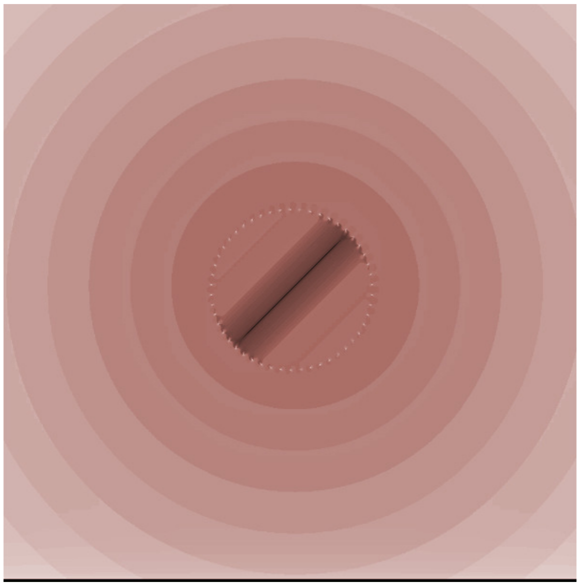
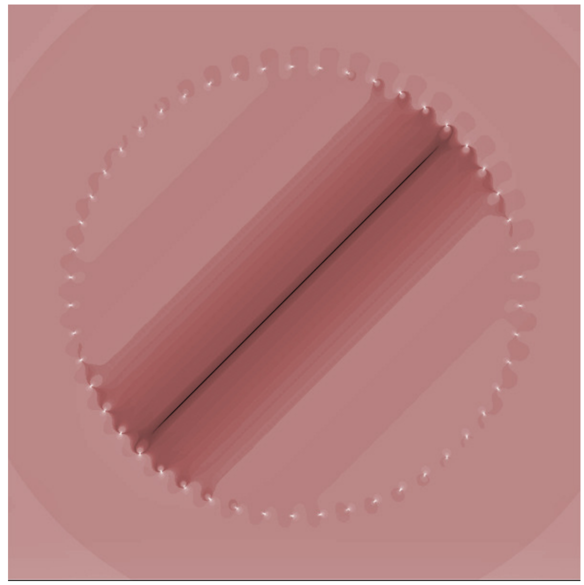
Figure 6: $F(z)=\frac{\alpha_1 z(1-\alpha_1 z)}{1-}\frac{\alpha_2 z(1-\alpha_2 z)}{1-}\frac{\alpha_3 z(1-\alpha_3 z)}{1-\cdots}$, n=50, $-20\le x,y\le 20$. $\alpha_n=\frac{1}{n}$.



Fixed-point CF

Figure 7: $F(z)=\frac{1\cdot z}{z+1-}\frac{2\cdot z}{z+2-}\frac{3\cdot z}{z+3-\cdots}$, another *fixed-point CF* . n=50. $-70\le x,y\le 70$

On the right: n=50. $-25\le x,y\le 25$.

Figures 8: $F(z)=\frac{1}{z+}\frac{1-a}{1+}\frac{1}{z+}\frac{2-a}{1+}\frac{2}{z+}\frac{3-a}{1+}\frac{3}{z+\cdots}$, found in an asymptotic expansion of the incomplete Gamma function. a=0. On the left, n=5 and on the right, n=50. $-1.5\le x,y\le 1.5$.

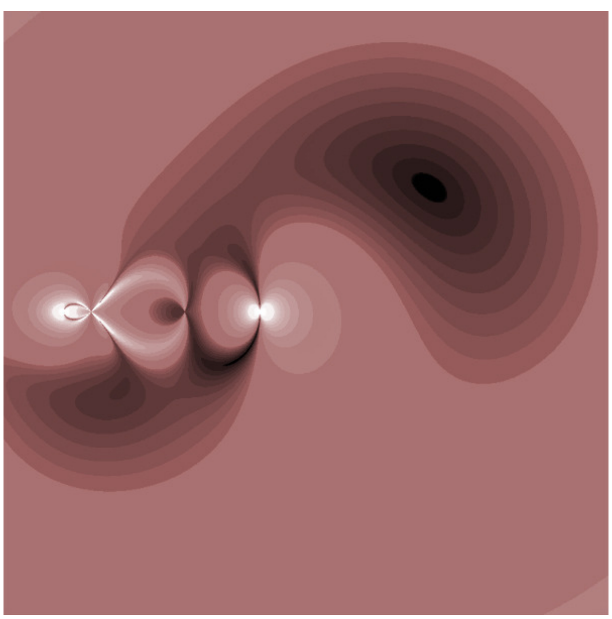
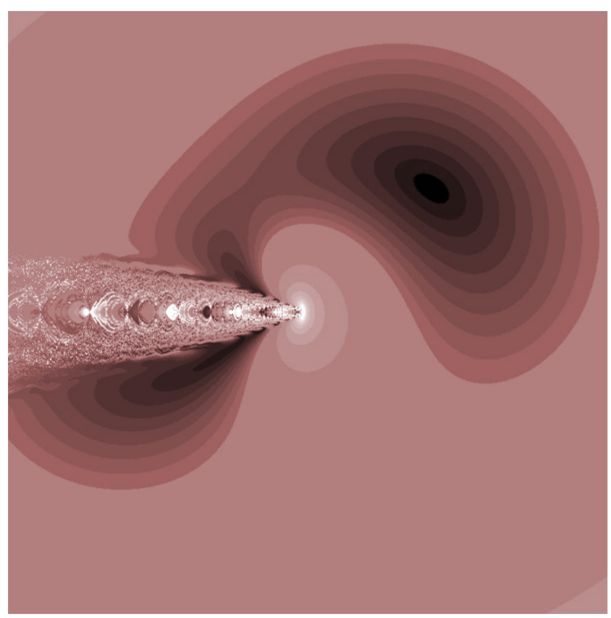
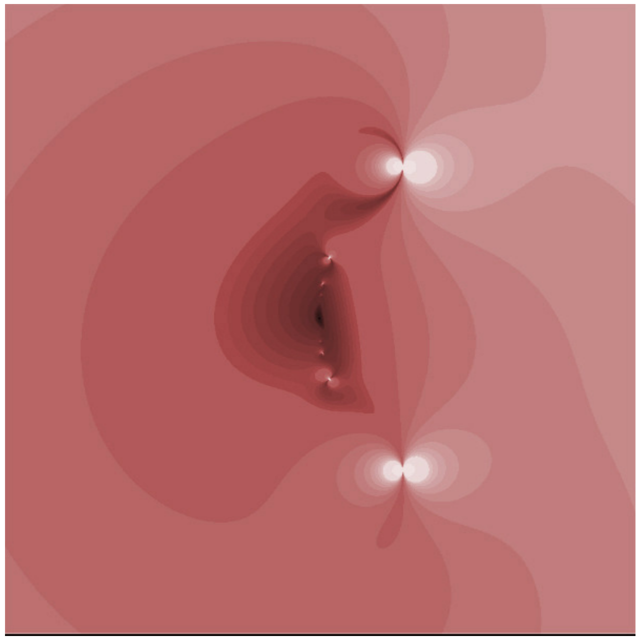
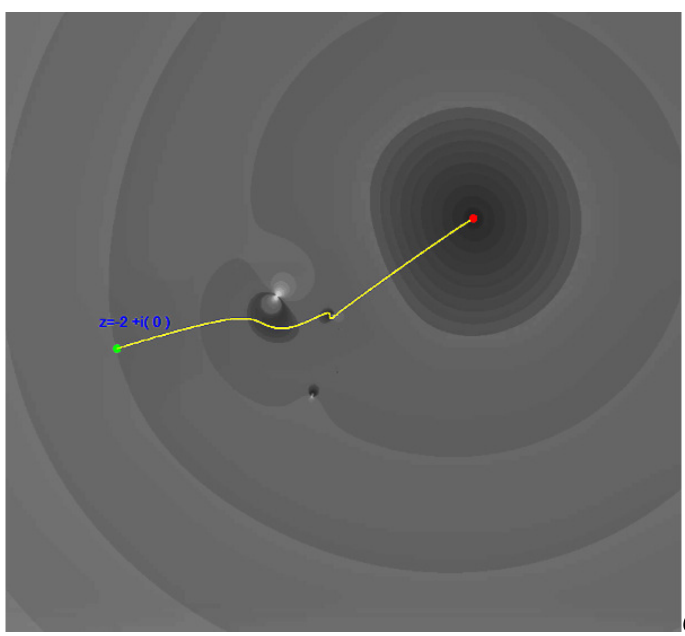



Figure 9: $F(z)=\frac{-z(z+1))}{1+}\frac{2z(2z-1)}{1+}\frac{3z(3z+1))}{1+}\frac{4z(4z-1)}{1+\cdots}$, $n=50$, $-3\le x,y\le 3$.



Fixed-point CF

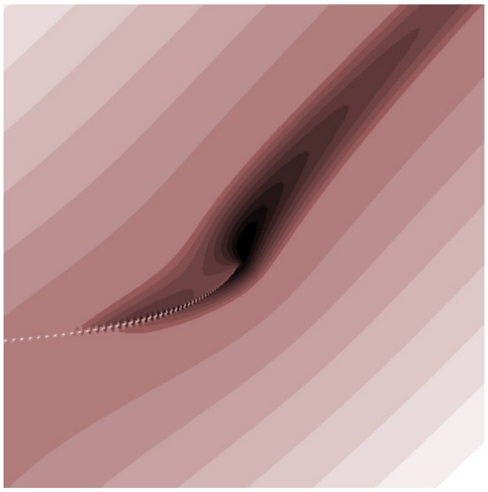
Figure 9: $F(z)=\frac{1}{1+}\frac{(-1+i)z}{1+}\frac{(2+i)z}{1+}\frac{(-3+i)z}{1+}\frac{(4+i)z}{1+\cdots}$, n=100, $-3\le x,y\le 3$. A Zeno contour locates an attractor: $\alpha=1.1491(1+i)$.



C-Fraction

Figure 10: $F(z)=z+\frac{z}{\zeta+z+}\frac{z}{\zeta+z+}\frac{z}{\zeta+z+\cdots}$, $\zeta=1+i$. $n=50$, $-4\le x,y\le 4$.

A Zeno contour identifies an attractor: $\alpha=-1.0769(1+i)$.



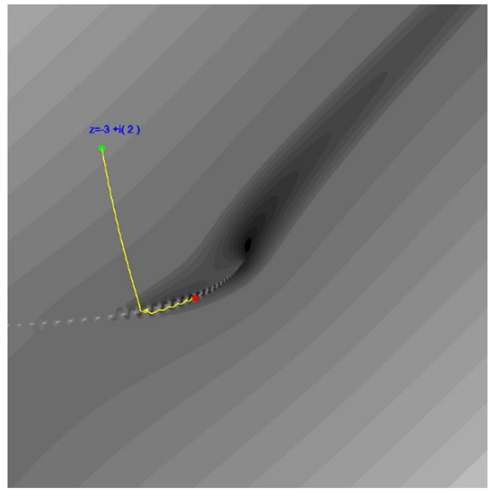
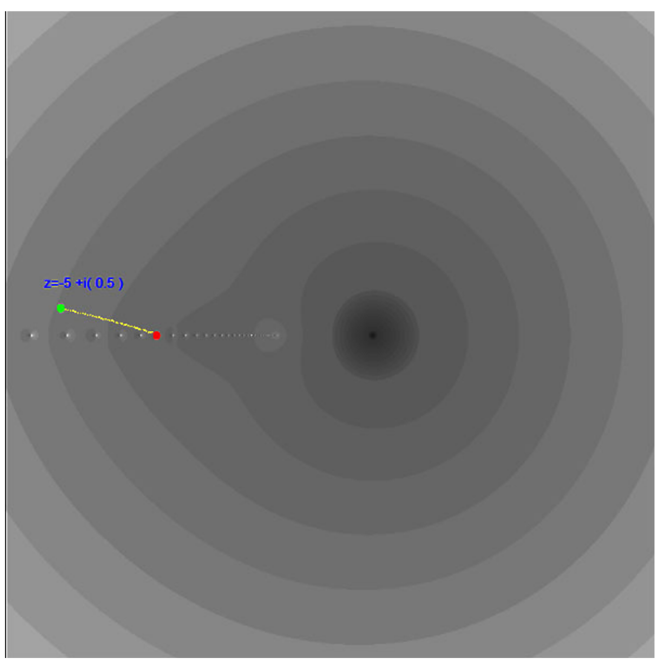


Figure 11: $\frac{1}{z}Ln(1+z)=\frac{1}{1+}\frac{1^2z}{2+}\frac{1^2z}{3+}\frac{2^2z}{4+}\frac{2^2z}{5+}\frac{3^2z}{6+}\frac{3^2z}{7+\cdots}$, $-6\le x,y\le 6$ and $n=100$.

Branch point at z=-1, branch line x<-1. A Zeno contour identifies a transitory and minor attractor on the branch line (coupled closely with a pole): $\alpha\approx-3.2331$.



References:

W. Jones, W. Thron, *Continued Fractions: Analytic Theory & Applications*, Addison-Wesley (1980)

J. Gill, *Zeno Contours in the Complex Plane*, Comm. Anal. Th. Cont. Frac. XIX (2012)