Some analytic functions have continued fraction expansions whose approximants are rational functions. Thus the approximants of the continued fraction have both fixed points and singularities (poles). What becomes of these mathematical artifacts as as $n \to \infty$? For instance, the exponential function should not have poles, although its converging rational approximants do. The following images are simple flux graphs in which dark hues indicate very little change from $z$ to $f(z)$, and light hues show considerable change, including poles in white.

**Figures 1**: $e^z = 1 + \frac{z}{1 - \frac{z}{2 + \frac{z}{3 - \frac{z}{5 - \frac{z}{2 + \cdots}}}}}$ Clockwise from top left: n=5, n=10, n=20, n=40.

$-20 \leq x, y \leq 20$. 
Here is the actual image of \( f(z) = e^z \) over \(-20 \leq x, y \leq 20\).

Figures 2: \( \text{Arc tan}(z) = \frac{z + 1^2 z^2 + 2^2 z^4 + 3^2 z^6}{1 + 3 + 5 + 7 + \cdots} \) valid for all values of \( z \) not on the branch lines extending up from the branch point \( z = i \) and down from the branch point \( z = -i \). Image on the left is for \( n = 10 \), on the right for \( n = 100 \). Both on \(-10 \leq x, y \leq 10\). Poles are on branch lines.
Figures 3: \[ \ln \left( \frac{1+z}{1-z} \right) = \frac{2z}{1} + \frac{2z^2}{3} + \frac{2z^3}{5} + \frac{2z^4}{7} + \cdots \] on \(-10 \leq x, y \leq 10\). On the left \(n=10\), and on the right \(n=100\). Branch points at \(z=1\) and \(z=-1\), with branch lines (with poles) extending outwards.

Convergence of the CF everywhere except on branch lines.

Figures 4: \[ \tan(z) = \frac{z}{1 - \frac{z^2}{1} - \frac{z^2}{3} - \frac{z^2}{5} - \frac{z^2}{7} - \cdots} \] valid for \(z \in \mathbb{C}\) (includes point at infinity)

On the left \(n=5\) and on the right \(n=100\). \(-10 \leq x, y \leq 10\).

Here the poles are not mere artifacts, but represent points where \(\tan(z) = \infty\).
Figures 5: \[ F(z) = \frac{z - \alpha_1 z (1 - \alpha_2 z)}{1 - 1 \alpha_1 z (1 - \alpha_2 z)} \frac{\alpha_2 z (1 - \alpha_3 z)}{1 - \alpha_2 z (1 - \alpha_3 z)} \frac{\alpha_3 z (1 - \alpha_4 z)}{1 - \alpha_3 z (1 - \alpha_4 z)} \ldots \] where \( \alpha_n = \frac{1}{n} \). This is a (modified) fixed-point CF with the two fixed points of each linear fractional transformation \( T_n(w) = \frac{\alpha_n z (1 - \alpha_n z)}{1 - w} \) displayed in the partial numerators. In the left image \( n=5 \), in the right \( n=100 \) with \(-12 \leq x, y \leq 12\). Convergence (in the extended plane) for any \( |z| \leq R \) may be ascertained by applying Worpitzky’s theorem to the tail end of the CF.

Figure 6: \[ F(z) = \frac{\alpha_1 z (1 - \alpha_2 z)}{1 - 1 \alpha_1 z (1 - \alpha_2 z)} \frac{\alpha_2 z (1 - \alpha_3 z)}{1 - \alpha_2 z (1 - \alpha_3 z)} \frac{\alpha_3 z (1 - \alpha_4 z)}{1 - \alpha_3 z (1 - \alpha_4 z)} \ldots , \ n=50, \ -20 \leq x, y \leq 20. \ \alpha_n = \frac{1}{n} . \]
Figure 7: $F(z) = \frac{1 \cdot z}{z+1} - \frac{2 \cdot z}{z+2} + \frac{3 \cdot z}{z+3} - \ldots$, another fixed-point CF. $n=50$. $-70 \leq x, y \leq 70$

On the right: $n=50$. $-25 \leq x, y \leq 25$.

Figures 8: $F(z) = \frac{1 - a}{z + 1} - \frac{1 - a}{z + 1} + \frac{2 - a}{z + 1} + \frac{3 - a}{z + 1} - \ldots$, found in an asymptotic expansion of the incomplete Gamma function. $a=0$. On the left, $n=5$ and on the right, $n=50$. $-1.5 \leq x, y \leq 1.5$. 
Figure 9: \( F(z) = \frac{-z(z+1)}{1+} + \frac{2z(2z-1)}{1+} + \frac{3z(3z+1)}{1+} + \frac{4z(4z-1)}{1+} \cdots, \, n = 50, \, -3 \leq x, y \leq 3. \)

Fixed-point CF

Figure 9: \( F(z) = \frac{1}{1+} \frac{(-1+i)z}{1+} \frac{(2+i)z}{1+} \frac{(-3+i)z}{1+} \frac{(4+i)z}{1+} \cdots, \, n=100, \, -3 \leq x, y \leq 3. \) A Zeno contour locates an attractor: \( \alpha = 1.1491(1+i). \)
Figure 10: \( F(z) = z + \frac{z}{\zeta + z} + \frac{z}{\zeta + z} + \frac{z}{\zeta + z} + \cdots, \quad \zeta = 1 + i. \quad n = 50, \quad -4 \leq x, y \leq 4. \)

A Zeno contour identifies an attractor: \( \alpha = -1.0769(1 + i). \)

Figure 11: \( \frac{1}{z} \ln(1 + z) = \frac{1}{1} - \frac{1^2 z}{2} + \frac{1^2 z}{2} - \frac{2^2 z}{3} + \frac{2^2 z}{3} - \frac{3^2 z}{4} + \frac{3^2 z}{4} - \frac{3^2 z}{5} + \cdots, \quad -6 \leq x, y \leq 6 \quad \text{and} \quad n = 100. \)

Branch point at \( z = -1, \) branch line \( x < -1. \) A Zeno contour identifies a transitory and minor attractor on the branch line (coupled closely with a pole): \( \alpha = -3.2331. \)
References:
