Images of Virtual Integrals

John Gill December 2013

<u>Definition</u>: *Zeno contour*. Let $g_{k,n}(z)=z+\eta_{k,n}\varphi(z)$ where $z\in S$ and $g_{k,n}(z)\in S$ for a convex set S in the complex plane. Require $\lim\limits_{n\to\infty}\eta_{k,n}=0$, where (usually) $k=1,2,...,n$. Set $G_{1,n}(z)=g_{1,n}(z)$, $G_{k,n}(z)=g_{k,n}\left(G_{k-1,n}(z)\right)$ and $G_n(z)=G_{n,n}(z)$ with $G(z)=\lim\limits_{n\to\infty}G_n(z)$, when that limit exists. The *Zeno contour* is a graph of this iteration. The word *Zeno* denotes the infinite number of actions required in a finite time period if $\eta_{k,n}$ describes a partition of the time interval [0,1]. Normally, $\varphi(z)=f(z)-z$ for a vector field, $\mathbb{F}=f$. The alternative notation $G_n(z)=\mathop{\mathcal{L}}\limits_{k=1}^{n}g_{k,n}(z)$ is also available. *Euler's method* is a finite example of a ZC.

I. Begin with $\eta_{k,n}=\dfrac{1}{n}$ and $g_{k,n}(z)\equiv z+\dfrac{1}{n}\varphi(z)$ with $\varphi(z)$ continuous on a domain S , and $z\in S\Rightarrow g_{k,n}(z)\in S$. (If the underlying vector field is *time-dependent*, $g_{k,n}(z)\equiv z+\dfrac{1}{n}\varphi(z,\tfrac{k}{n})$)

Thus $$G_{n,n}(z)=z+\frac{1}{n}\varphi(z)+\frac{1}{n}\varphi(G_{1,n}(z))+\frac{1}{n}\varphi(G_{2,n}(z))+\cdots+\frac{1}{n}\varphi(G_{n-1,n}(z)).$$

Now, imagine a function

$$\psi(z,t)\ ,\ t\in[0,1]\quad \underline{\text{and}}\quad \psi\left(z,\frac{k}{n}\right)\equiv\lim_{m\to\infty}\varphi\left(G_{mk-1,mn}(z)\right),\ \text{with}\ \int_0^1\psi(z,t)\,dt\ \text{defined:}$$

$$G_n(z)-z=\frac{1}{n}\psi\left(z,\frac{1}{n}\right)+\frac{1}{n}\psi\left(z,\frac{2}{n}\right)+\frac{1}{n}\psi\left(z,\frac{3}{n}\right)+\cdots+\frac{1}{n}\psi\left(z,\frac{n}{n}\right)\approx\int_0^1\psi(z,t)dt$$

And for t irrational, $\psi(z,t)=\lim\limits_{t_r\to t}\psi(z,t_r)$ for rational t_r .

The existence of this function (and the integral) is equivalent to the convergence of the Zeno contour. $\int_0^1\psi(z,t)\,dt$ is more a *virtual* integral since its analytical form can be murky at times.

Then: $\boxed{\lambda(z)=\int_0^1\psi(z,t)\,dt=G(z)-z}$ which is valid for both normal VFs and TDVFs .

Write the recurrence sequence (for TDVF) as

$$Z\left(z_0,\tfrac{k}{n}\right)=Z\left(z_0,\tfrac{(k-1)}{n}\right)+\frac{1}{n}\varphi\left(Z\left(z_0,\tfrac{(k-1)}{n}\right),\tfrac{k}{n}\right)$$

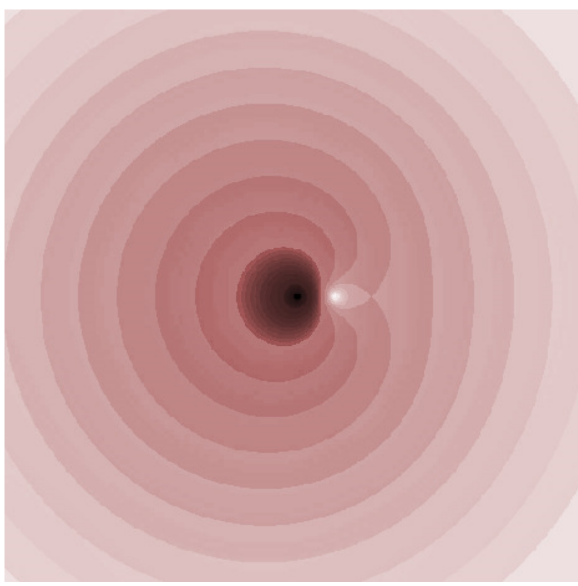
Assuming $Z=Z\left(z_0,t\right)$, one concludes

$$\frac{\Delta_{k,n}Z}{\Delta_n t}=\varphi\left(Z\left(z_0,\tfrac{(k-1)}{n}\right),\tfrac{k}{n}\right) \;\Rightarrow\; \frac{dZ}{dt}=\varphi\left(Z\left(z_0,t\right),t\right),\; t\in[0,1].$$ So that

$$\boxed{\psi(z,t)=\varphi(z(t),t) \;\Rightarrow\; \lambda(z_0)=\int_0^1\varphi(z(t),t)\,dt=z(1)-z(0)}$$

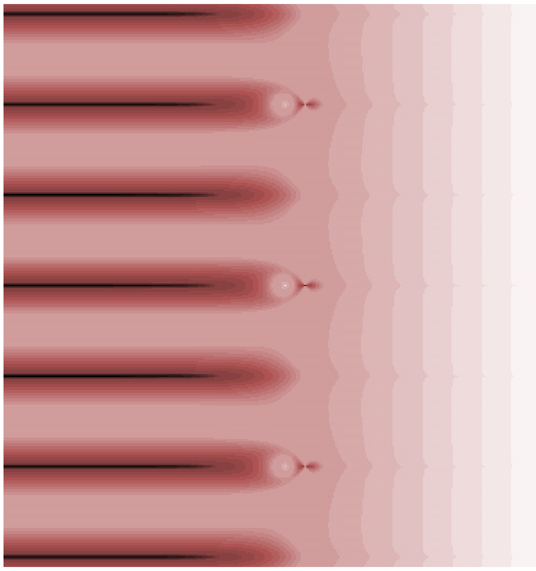
where $Z(t)=F(z,t)$ <u>when</u> $Z(t)$ can be determined.

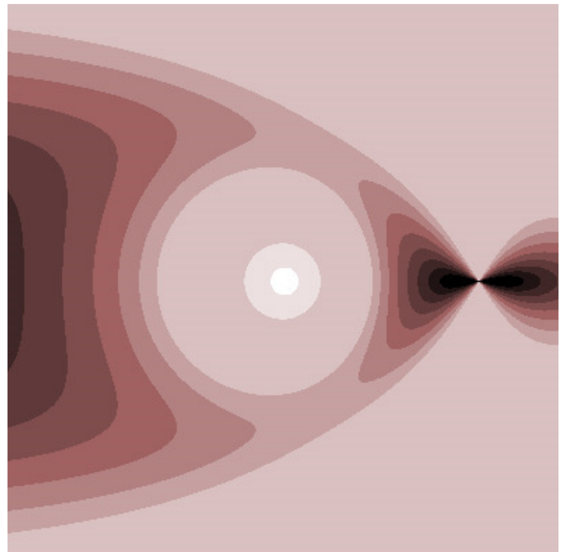
<u>*Example*</u>: $\varphi(z)=z^2$ produces $\frac{dz}{dt}=z^2$ which gives $Z=Z\left(z_0,t\right)=\frac{1}{1/z_0\;-\;t}$, and that in turn gives $\int_0^1\psi\left(z,t\right)dt=\lambda(z)=z^2/(1-z)$. Here $\psi(z,t)=Z^2=z^2/(1-zt)^2$. Image of $\lambda(z)$:



The images here represent simple complex topography: dark=small absolute values of $\lambda(z)$, light =large absolute values of $\lambda(z)$.

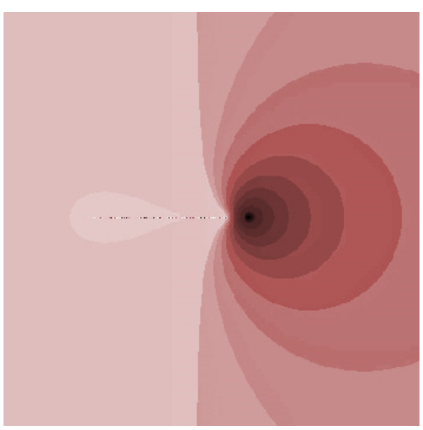
Example: $\varphi(z)=e^{z}$. Here $Z(z,t)=z-Ln(1-e^{z}t)$ and $\psi(z,t)=e^{z}/(1-e^{z}t)$. Thus $\lambda(z)=Ln\left(\frac{1}{1-e^{z}}\right)$. Image of $\lambda(z)$ over [-10<x,y<10] and [-1<x,y<1]:

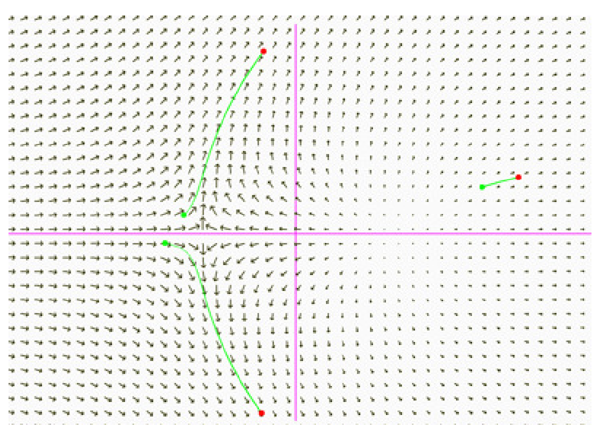




Example: $\varphi(z)=\frac{z-\alpha}{z-\beta}$. However, $Z-z_0=t+(\alpha-\beta)\cdot Ln\left(\frac{Z+\alpha}{z_0+\alpha}\right)$ allows no closed formulation of $\psi(z_0,t)$, although $\lambda(z)=\int_0^1\psi(z,t)dt=1+(\alpha-\beta)\cdot Ln\left(\frac{Z(z,1)+\alpha}{z+\alpha}\right)$.

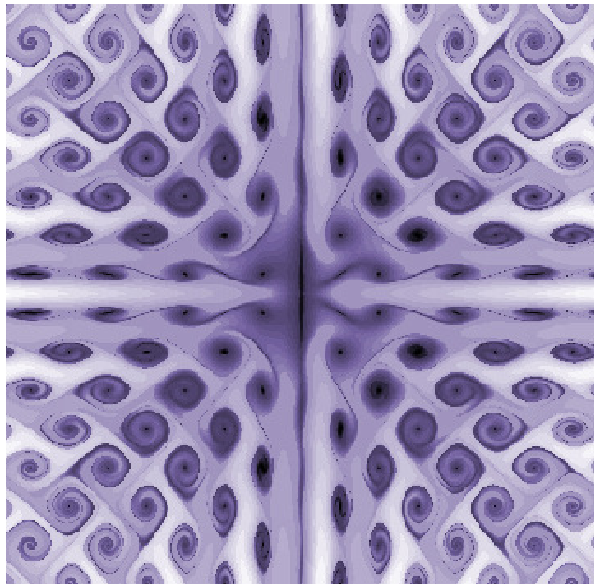
Image of $\lambda(z)$, ($\alpha=1$, $\beta=-1$, n=50, -6<x,y<6) and the associated vector field $f(z)=z+\frac{z-1}{z+1}$ with three Zeno contours:





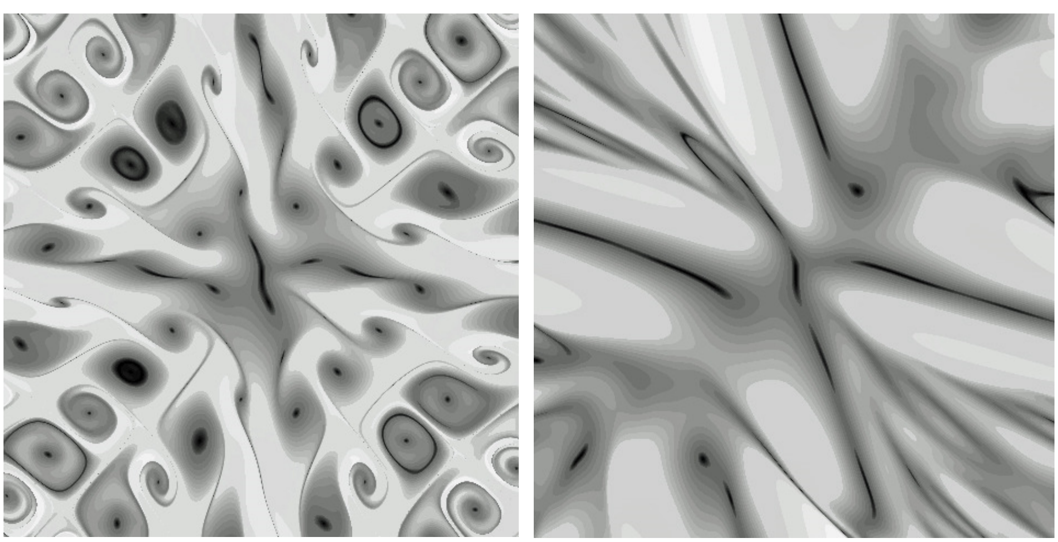
Example: $\varphi(x+iy) = xCos(4y) + iySin(4x)$. $Z(t)$ not available. -6<x,y<6 , n=50.

Image of $\lambda(z) = \int_0^1 \psi(z,t)dt$:



Example: (a) $\varphi(x+iy) = xCos(4y+x) + iySin(4x+y)$. ($Z(t)$ not available). -4<x,y<4. n=80.
And (b) $\varphi^*(x+iy,t) = xCos(4ty+x) + iySin(4tx+y)$ for the *time-dependent vector field*

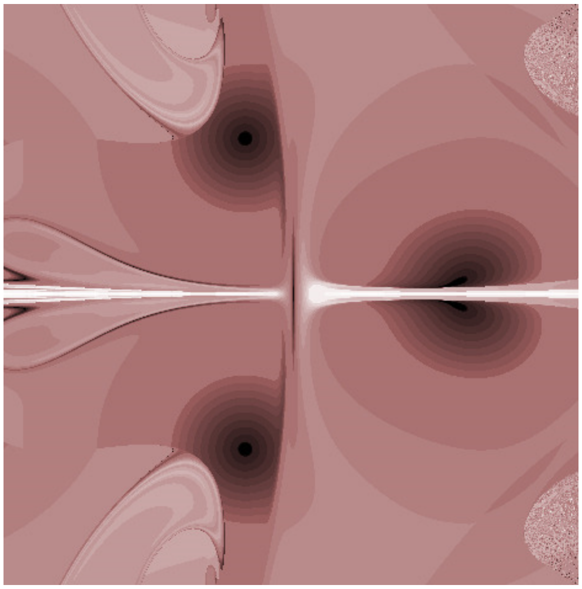
$\varphi^*(z,t) + z$, $t: 0 \to 1$. $\lambda(z) = \int_0^1 \psi(z,t)dt$:



(a) (b)

Example: $\varphi(x+iy,t) = 4xCos(y+4Sin(xyt)) + i4ySin(x+4Cos(xyt))$, $-4 < x, y < 4$, n=80

$$\lambda(z) = \int_0^1 \psi(z,t)dt :$$



Example: $\frac{dz}{dt} = \varphi(z) = \cfrac{z}{1+\cfrac{z}{2+\cfrac{z}{3+\cdots}}}$, $-20 < x, y < 20$, $n = 50$. $\lambda(z) = \int_0^1 \psi(z,t)dt :$