Images of Virtual Integrals

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Definition: Zeno contour. Let \( g_{k,n}(z) = z + \eta_{k,n} \varphi(z) \) where \( z \in S \) and \( g_{k,n}(z) \in S \) for a convex set \( S \) in the complex plane. Require \( \lim_{n \to \infty} \eta_{k,n} = 0 \), where (usually) \( k = 1,2,\ldots,n \). Set \( G_{1,n}(z) = g_{1,n}(z) \), \( G_{k,n}(z) = g_{k,n}(G_{k-1,n}(z)) \) and \( G_n(z) = G_{n,n}(z) \) with \( G(z) = \lim_{n \to \infty} G_n(z) \), when that limit exists. The Zeno contour is a graph of this iteration. The word Zeno denotes the infinite number of actions required in a finite time period if \( \eta_{k,n} \) describes a partition of the time interval \([0,1]\). Normally, \( \varphi(z) = f(z) - z \) for a vector field, \( \mathbb{F} = f \). The alternative notation \( G_n(z) = \sum_{k=1}^n g_{k,n}(z) \) is also available. Euler’s method is a finite example of a ZC.

I. Begin with \( \eta_{k,n} = \frac{1}{n} \) and \( g_{k,n}(z) \equiv z + \frac{1}{n} \varphi(z) \) with \( \varphi(z) \) continuous on a domain \( S \), and \( z \in S \Rightarrow g_{k,n}(z) \in S \). (If the underlying vector field is time-dependent, \( g_{k,n}(z) \equiv z + \frac{1}{n} \varphi(z, \frac{k}{n}) \))

Thus \( G_{n,n}(z) = z + \frac{1}{n} \varphi(z) + \frac{1}{n} \varphi(G_{1,n}(z)) + \frac{1}{n} \varphi(G_{2,n}(z)) + \cdots + \frac{1}{n} \varphi(G_{n-1,n}(z)) \).

Now, imagine a function

\[
\psi(z,t), t \in [0,1] \quad \text{and} \quad \psi\left(z, \frac{k}{n}\right) = \lim_{m \to \infty} \varphi\left(G_{mk-1,mn}(z)\right), \text{ with } \int_0^1 \psi(z,t) \, dt \text{ defined:}
\]

\[
G_n(z) - z = \frac{1}{n} \psi\left(z, \frac{1}{n}\right) + \frac{1}{n} \psi\left(z, \frac{2}{n}\right) + \frac{1}{n} \psi\left(z, \frac{3}{n}\right) + \cdots + \frac{1}{n} \psi\left(z, \frac{n}{n}\right) \approx \int_0^1 \psi(z,t) \, dt
\]

And for \( t \) irrational, \( \psi(z,t) = \lim_{\tau \to t} \psi(z, \tau) \) for rational \( t_r \).

The existence of this function (and the integral) is equivalent to the convergence of the Zeno contour. \( \int_0^1 \psi(z,t) \, dt \) is more a virtual integral since its analytical form can be murky at times.

Then:

\[
\lambda(z) = \int_0^1 \psi(z,t) \, dt = G(z) - z
\]

which is valid for both normal VFs and TDVFs.
Write the recurrence sequence (for TDVF) as

\[ Z(z_0, \frac{k}{n}) = Z(z_0, \frac{(k-1)}{n}) + \frac{1}{n} \varphi(Z(z_0, \frac{(k-1)}{n}), \frac{k}{n}) \]

Assuming \( Z = Z(z_0, t) \), one concludes

\[ \frac{\Delta_{k,n}Z}{\Delta_{n}t} = \varphi\left(Z(z_0, \frac{(k-1)}{n}), \frac{k}{n}\right) \Rightarrow \frac{dZ}{dt} = \varphi(Z(z_0, t), t), \quad t \in [0,1] \]. So that

\[ \psi(z,t) = \varphi(z(t), t) \Rightarrow \lambda(z_0) = \int_{0}^{1} \varphi(z(t), t) \, dt = z(1) - z(0) \]

where \( Z(t) = F(z,t) \) when \( Z(t) \) can be determined.

**Example:** \( \varphi(z) = z^2 \) produces \( \frac{dz}{dt} = z^2 \) which gives \( Z = Z(z_0, t) = \frac{1}{1/z_0 - t} \), and that in turn gives \( \int_{0}^{1} \psi(z,t) \, dt = \lambda(z) = z^2 / (1 - z) \). Here \( \psi(z,t) = Z^2 = z^2 / (1 - zt)^2 \). Image of \( \lambda(z) \):

![Image of \( \lambda(z) \)](image)

The images here represent simple complex topography: dark=small absolute values of \( \lambda(z) \), light=large absolute values of \( \lambda(z) \).
**Example:** \( \varphi(z) = e^z \). Here \( Z(z,t) = z - \ln(1 - e^t) \) and \( \psi(z,t) = e^z/(1 - e^t) \). Thus \( \lambda(z) = \ln\left(\frac{1}{1 - e^z}\right) \). Image of \( \lambda(z) \) over \([-10<x,y<10] \) and \([-1<x,y<1]\):
Example: \( \varphi(x + iy) = x\cos(4y) + iy\sin(4x) \). \( Z(t) \) not available. \(-6 < x, y < 6\), \( n = 50 \).

Image of \( \lambda(z) = \int_0^1 \psi(z,t) dt \):

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Example: (a) \( \varphi(x + iy) = x\cos(4y + x) + iy\sin(4x + y) \). \( Z(t) \) not available. \(-4 < x, y < 4\), \( n = 80 \).
And (b) \( \varphi^*(x + iy, t) = x\cos(4ty + x) + iy\sin(4tx + y) \) for the time-dependent vector field

\( \varphi^*(z,t) + z, \ t: 0 \rightarrow 1 \). \( \lambda(z) = \int_0^1 \psi(z,t) dt \):

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(a) (b)
Example: \( \varphi(x + iy, t) = 4x \cos(y + 4\sin(xyt)) + iy \sin(x + 4\cos(xyt)), -4 < x, y < 4, n=80 \)

\[ \lambda(z) = \int_0^1 \psi(z, t) dt : \]

Example: \( \frac{dz}{dt} = \varphi(z) = \frac{z}{1 + \frac{z}{2 + \frac{z}{3 + \ldots}}} \), \(-20 < x, y < 20, n=50\). \( \lambda(z) = \int_0^1 \psi(z, t) dt : \)