Images of Complex Singularities

John Gill September, 2012

The images that follow are flux graphs of various complex functions having isolated singularities including poles, essential singularities and branch points. Usually, dark areas denote either fixed points or regions where the functional values $f(z)$ differ only slightly from z. Very light points denote singularities of one form or another.

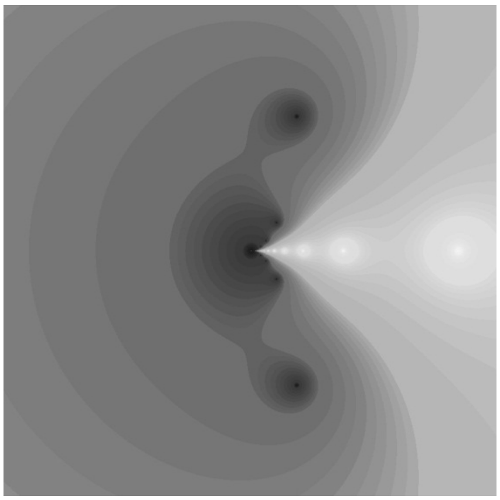

Figure 1: $f(z)=\prod_{n=1}^{\infty}\left(\frac{zn^2}{zn^2-1}\right)$, having poles $z=1,\frac{1}{4},\frac{1}{9},\frac{1}{16},\cdots$; an example of a convergent sequence of poles whose limit is a zero of the function. $-.3\le x,y\le .3$ The approximant is n=30. Note the fixed points arrayed symmetrically about the real axis.

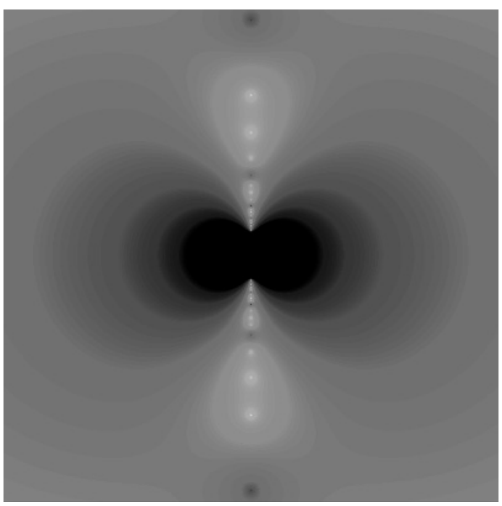

Figure 2: $f(z)=Arc\tan(z)=\frac{z}{1+}\,\frac{1^2z^2}{3+}\,\frac{2^2z^2}{5+}\,\frac{3^2z^2}{7+\cdots}$, a continued fraction expansion whose approximants are rational functions exhibiting poles. Branch points (singularities) occur at i and -i and branch lines extend in both directions, showing sequences of poles, terminating at these branch points. $-10\le x,y\le 10$. Here, the 15th approximant is compared with the 30th approximant with high speed of convergence represented by the darker areas.

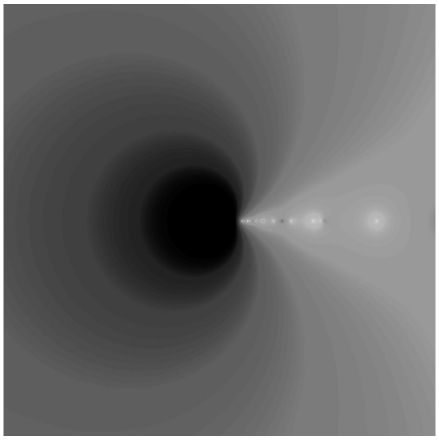

Figure 3: $f(z) = -Ln(1-z) = \frac{z}{1-}\frac{1^2 z}{2-}\frac{1^2 z}{3-}\frac{2^2 z}{4-}\frac{2^2 z}{5-\cdots}$. The approximants are rational functions; branch cut along x>1. Singularity at branch point z = 1. Poles may be seen along the cut. Here the 10th approximant is compared with the 30th approximant. $-10 \le x, y \le 10$. Again, high speed of convergence occurs in the darker regions.

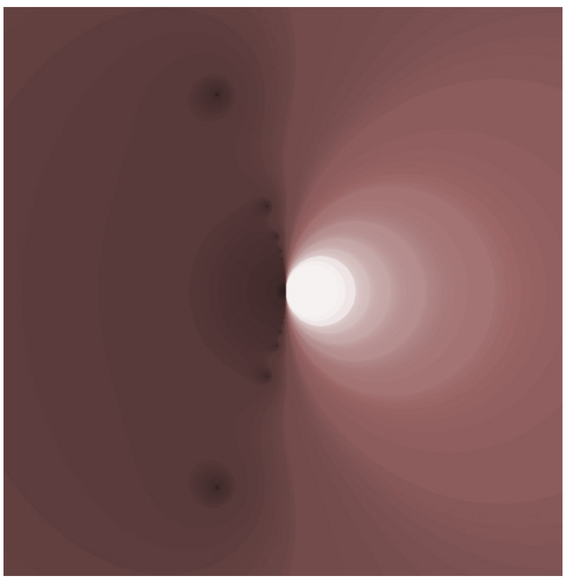

Figure 4: $f(z) = e^{\frac{1}{z}}$. Note the fixed points to the left of the imaginary axis. Here, $-.3 \le x, y \le .3$. An essential singularity occurs at z = 0. Although the function is relatively well-behaved to the left of the origin , which actually is an attractor in this region, the same cannot be said of the region to the right of the origin. This kind of mixed behavior is not uncommon in essential singularities. Simple flux graph with $|f(z)-z|$ small = dark.

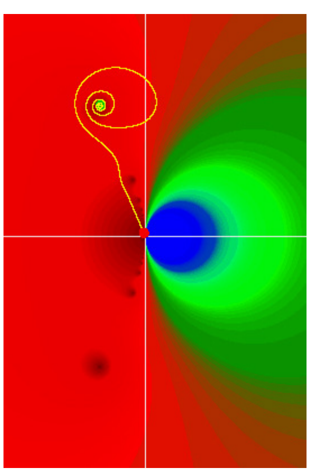

Figure 4b: In this image of $f(z) = e^{\frac{1}{z}}$, a Zeno contour [1] winds its way around and away from a repeller (-.072+2i) and toward the attractor $z = 0^-$.

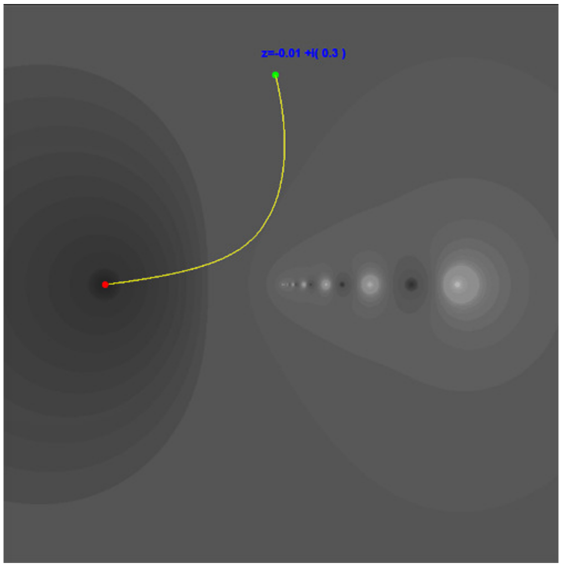

Figure 5: $f(z)=1+\sum_{n=1}^{\infty}\frac{1}{2^n z-1}$, rational approximation for n=50, $-.4 \le x, y \le .4$. Poles at $z=\frac{1}{2},\frac{1}{4},\frac{1}{8},\cdots$. The sequence of poles, with fixed points interspersed, converges to the origin.

The Zeno contour identifies an attractor, $\alpha \approx -.2534+0i$.

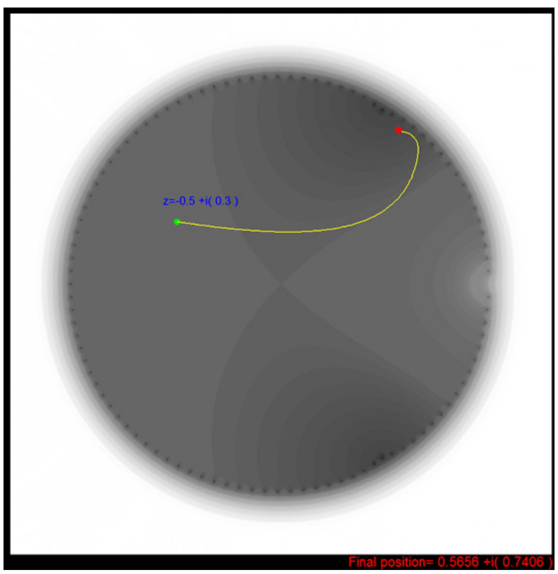

Figure 6: $f(z)=1+z+z^2+z^3+\cdots$, n=100 approximant. $-1.2 \le x, y \le 1.2$. The singularity at z = 1 is barely noticeable for this value of n compared to those points |z|>1. The dark dots around the rim are fixed points for $f(z)=1+z+\cdots+z^{100}$. The Zeno contour has located an attractor: $\alpha \approx .5656+.7406i$.

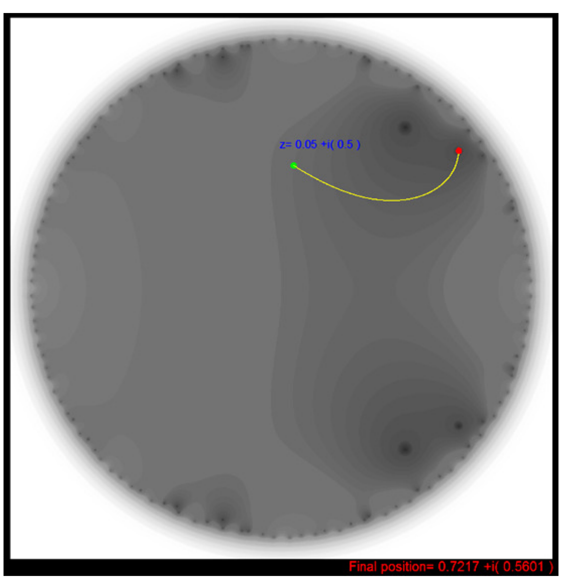

Figure 7: The initial expansion (n = 7) of the lacunary function $f(z)=1+\sum_{n=1}^{\infty} z^{2^n}$,

$-1.1 \le x, y \le 1.1$. Fixed points crowd the circumference of the unit circle. A Zeno contour seeks out an attractor: $\alpha \approx .7217+.5601i$. The white band around the circumference illustrates what will become a dense set of singularities that prevent analytic continuation.

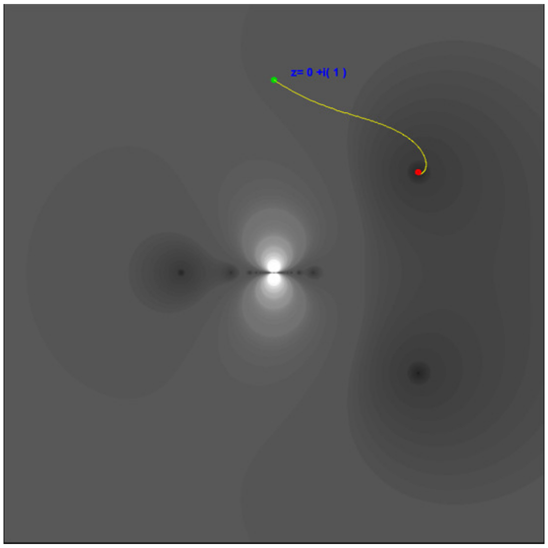

Figure 8: $f(z) = Cos\left(\frac{1}{z}\right)$ with essential singularity $z = 0$, $-1.4 \le x, y \le 1.4$. A Zeno contour has identified an attractor $\alpha \approx .7492 + .5223i$. Note the cluster of fixed points along the real axis near the origin.

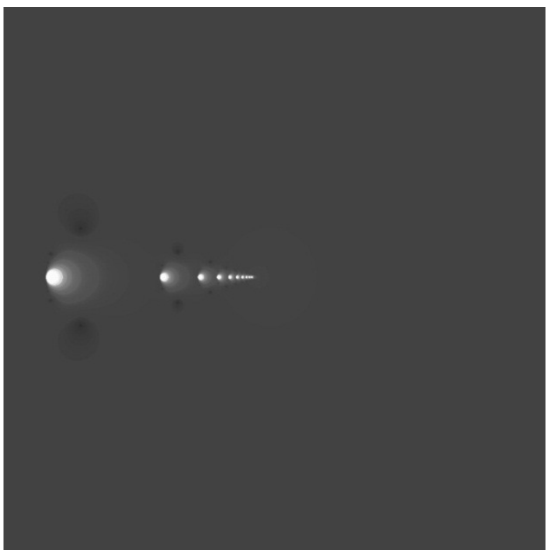

Figure 9: $f(z) = \prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2} e^{\frac{1}{nz+1}}\right)$, having a chain of essential singularities at $z = -1, \frac{-1}{2}, \frac{-1}{3}, \cdots$ converging to z = 0, where $f(0) \approx 17.145$. Here $-1.2 \le x, y \le 1.2$ and $n = 10$. Note the shadowy fixed points above and below the singularities.

Figure 10: $f(z) = \lim_{n \to \infty} \prod_{k=1}^{n}\left[1 + \frac{8}{k^2} e^{\frac{k}{nz+k}}\right]$, defined by Tannery's Theorem [2] in a region excluding [-1,0), for on that interval the set of essential singularities of $f(z)$ is dense (thus becoming a set of non-isolated singularities). Here $-1 \le x, y \le 1$, n = 100. In the region $(|z| > 2)$, for example, the limit function is constant: $f(z) \approx 406.7$.

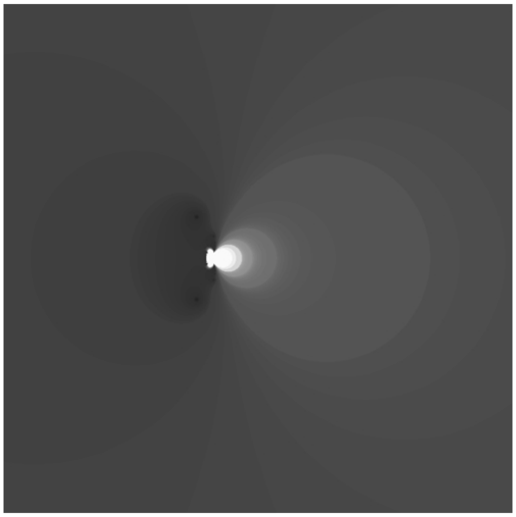

Figure 11: $f(z)=\prod_{n=1}^{\infty}\left[1+\frac{8}{n^2}e^{\frac{1}{z}}\right]$, somewhat similar to the functions in figures 9 and 10, but simpler. One essential singularity at z = 0, but occurring an infinite number of times. $-1\le x,y\le 1$.

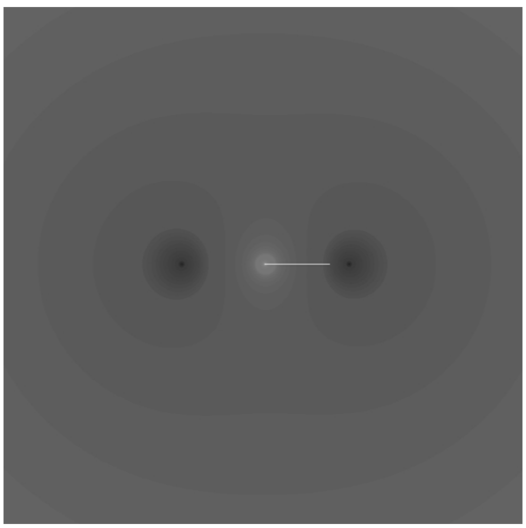

Figure 12: $f(z)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2\left(z-k/n\right)}$, defined by Tannery's Theorem, converges everywhere except on the interval [0,1] to $f(z)=\frac{\pi^2}{6z}$ (note the two fixed points). There exists a set of poles dense in [0,1].

Here $-4\le x,y\le 4$, n = 100.

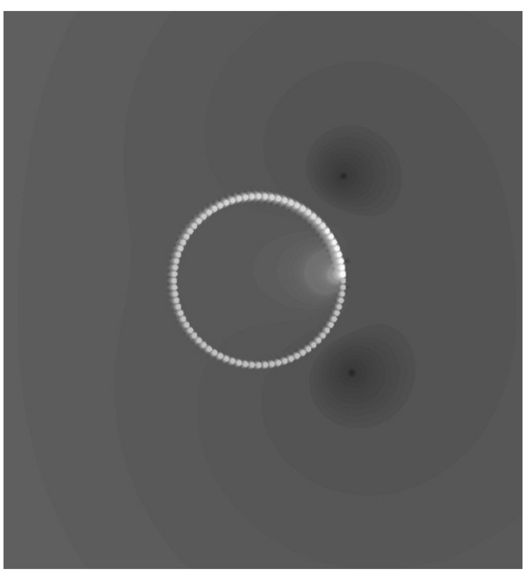

Figure 13: "*Dense ring of essential singularities*" :

$f(z)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}e^{\frac{1}{\exp(i2\pi k/n)-z}}$, converging away from $(|z|=1)$ to $f(z)=\frac{\pi^2}{6}e^{\frac{1}{1-z}}$ by Tannery's theorem. Here n = 100 and $-3\le x,y\le 3$.

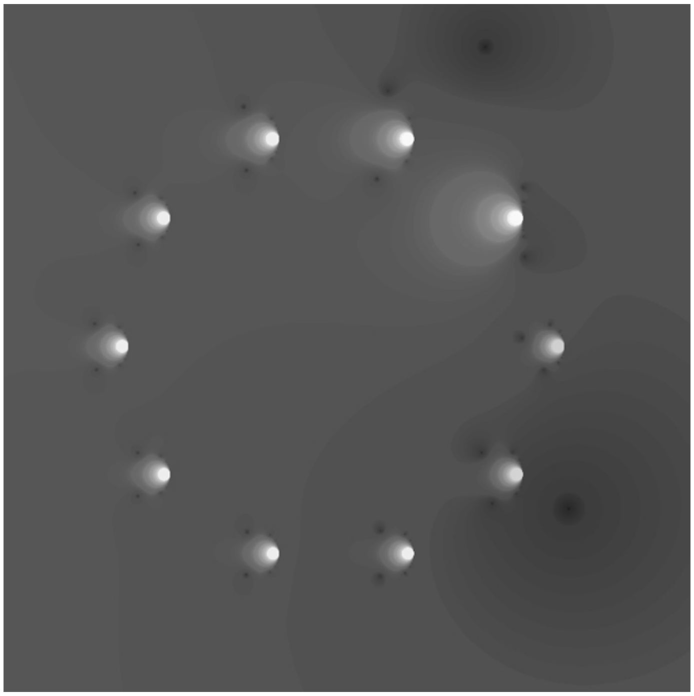

Figure 13b: For an early approximant, n = 10, the essential singularities glow resolutely, shadowed by associated minimal displacement or fixed points. $-1.4 \le x, y \le 1.4$.

Figure 14: "*Dense square of essential singularities*":

$$f(z) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} \sum_{m=1}^{n} \left(\frac{1}{m^2} \exp\left(\frac{1}{z - \left(\frac{k}{n} + i\frac{m}{n}\right)} \right) \right)$$

Outside the unit square, $f(z) = \dfrac{\pi^4}{36} \cdot e^{\frac{1}{z}}$ (check out the fixed point to the right: $\alpha \approx 3.578$), but inside the square there is a dense set of essential singularities. The expression converges due to Tannery's theorem. Here $-4 \le x, y \le 4$ and $n = 20$.

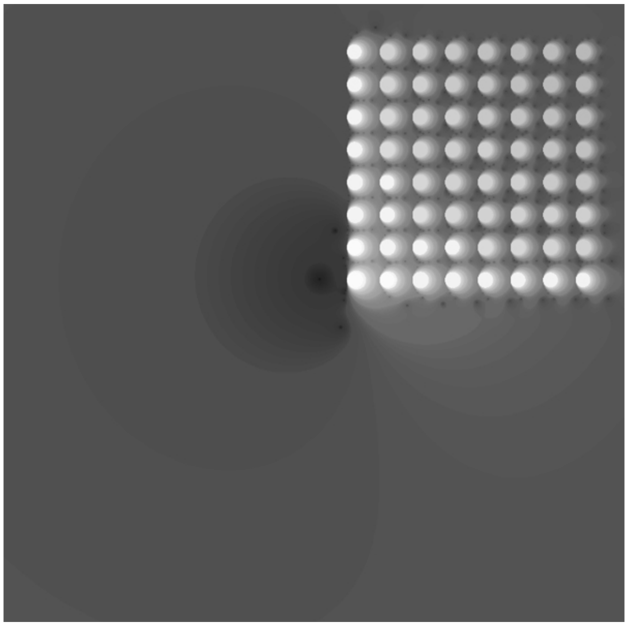

Figure 14b: Here $-1.2 \le x, y \le 1.2$ and $n = 8$.

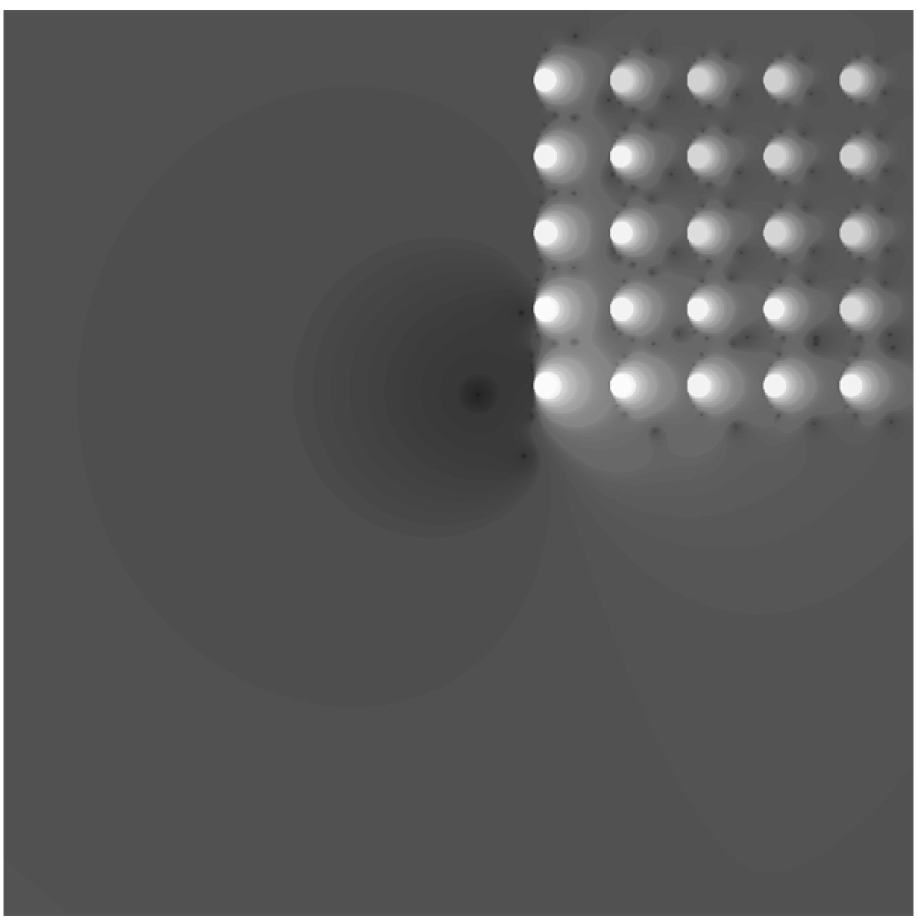

Figure 14c: Here $-1.2 \le x, y \le 1.2$ and $n = 5$. Observe the scattered minimal displacement or fixed points about the singularities.

[1] J. Gill, *Zeno Contours in the Complex Plane*, Comm. Anal. Th. Cont. Frac. Vol. XIX (2012)

[2] J. Gill, *Generalizations of the Classical Tannery's Theorem*, Comm. Anal. Th. Cont. Frac. Vol. XIX (2012)