Classroom Notes: Images of Fixed-Point Continued Fractions

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Abstract: Computer generated images in the complex plane of iterations of certain analytic continued fractions.

Fixed-point continued fractions have the form

\[
\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots \text{ where } \alpha_k, \beta_k \text{ are the fixed points of the function}
\]

\[
f_k(\zeta) = \frac{\alpha_k \beta_k}{\alpha_k + \beta_k - \zeta}.
\]

The nth approximant of the continued fraction then can be written

\[
F_n(\zeta) = f_1 \circ f_2 \circ \cdots \circ f_n(\zeta).
\]

If \( \alpha_k = \alpha_k(z) \) and \( \beta_k = \beta_k(z) \), we have

\[
F_n(z,\zeta) = f_1 \circ f_2 \circ \cdots \circ f_n(z,\zeta).
\]

The following images show magnitudes of simple complex flux (SCF), \(|F_n(z,0)|\), or fixed-point flux (FPF), \(|F_n(z,0) - z|\). Dark=small, light=large. Several variations on FPCFs are shown, as well.  [Liberty Basic V 4.04 (2013)]

Example 1:  \( f_k(z,\zeta) = \frac{kz}{k+z-\zeta}, -6 \leq x, y \leq 6, n=10 \)  SCF & FPF
Example 2: 

\[ f_1(z, \zeta) = \frac{(1 + ik)^2 z}{1 + ik + z - \zeta} \quad \text{and} \quad f_2(z, \zeta) = \frac{(0.5 + ik)^2 z}{0.5 + ik + z - \zeta}, \quad -6 \leq x, y \leq 6, \quad n = 10 \quad \text{SCF} \]

A Variation on fixed-point CFs. 

Note: a programming error produced these two images!

Example 3: 

\[ f_k(z, \zeta) = \frac{kz(1 - kz)}{1 - \zeta}, \quad -6 \leq x, y \leq 6, \quad n = 10 \quad \text{SCF} \]
Example 4: \[ f_k(z, \zeta) = \frac{kz(1-kz)}{\alpha_k z} \left( 1 - \frac{\alpha_k + z - \zeta}{\alpha_k z} \right), \quad \alpha_k = 1+ik, \quad -6 \leq x, y \leq 6, \quad n = 10 \quad SCF \]

![Example 4 Diagram](image)

Example 5: \[ f_k(z, \zeta) = \frac{(1-ik)z}{1-ik + z - \zeta}, \quad -7 \leq x, y \leq 7, \quad n = 10 \quad SCF \]

![Example 5 Diagram](image)
Example 6: \( f(z, \zeta) = \frac{z}{k + z - \zeta}, \ -6 \leq x, y \leq 6, \ n = 10 \) SCF. Another variation on FPCFs.

Example 7: \( f_k(z, \zeta) = \frac{kz^2}{k + z - \zeta}, \ -10 \leq x, y \leq 10, \ n = 10 \) SCF. Another variation on FPCFs.
Example 8: \( f(z, \zeta) = \frac{(1 + 2ki)^2 z}{1 + 2ki + z - \zeta} \), \(-10 \leq x, y \leq 10\), \( n = 10 \) SCF. Another variation on FPCFs.

Example 9: \( f(z, \zeta) = \frac{i(1 + ik)z}{1 + ik + z - \zeta} \), \(-8 \leq x, y \leq 8\), \( n = 10 \) SCF. Variation on FPCFs.