A Note on Integrals & Hybrid Contours in the Complex Plane

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Abstract: Contour integrals can be expressed graphically as simple vectors arising from a secondary contour.

We start with a well-behaved (continuous, analytic, etc.) function in the complex plane:

(1) $\varphi(z,t)$, with $z \in S$ a convex set in $\mathbb{C}$ and $t \in [0,1]$. $\varphi(z,t) \in S \;\; \forall z \in S$ and $\forall t \in [0,1]$

Let

(2) $g_{k,n}(z) = z + \eta_{k,n}\varphi(z, \frac{k-1}{n})$, with $g_{k,n}(z) \in S$, $0 < k \le n$

Require $0 < \eta_{1,n} < \eta_{2,n} < \cdots < \eta_{n,n} = 1$ and $\lim_{n\to\infty} \eta_{k,n} = 0$, where $k = 1,2,...,n$.

Set $G_{1,n}(z) = g_{1,n}(z)$, $G_{k,n}(z) = g_{k,n}\left(G_{k-1,n}(z)\right)$ and $G_n(z) = G_{n,n}(z)$

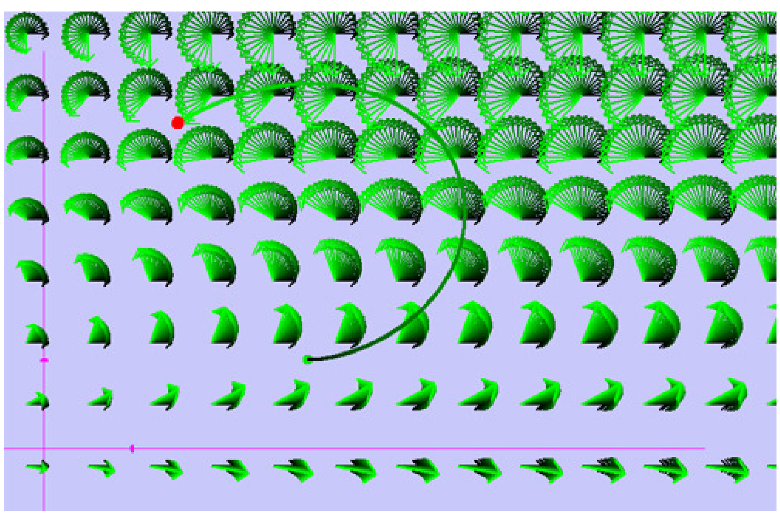
with $G(z) = \lim_{n\to\infty} G_n(z)$, when that limit exists.

Written in an alternate iterative form

(3) $z_{k,n} = z_{k-1,n} + \eta_{k,n}\varphi(z_{k-1,n}, \frac{k-1}{n}),\; z_{0,n} \equiv \alpha$,

the distribution of points forms a *Zeno contour*

(4) $\gamma_n(\alpha) = \left\{ z_{k,n} \right\}_{k=1}^{n} \;\; \Rightarrow \;\; \gamma(\alpha) = \lim_{n\to\infty} \gamma_n(\alpha)$.



The word *Zeno* denotes the infinite number of actions required in a finite time period if $\eta_{k,n}$ describes a partition of the time interval [0,1].

$\varphi(z,t)$ is associated with a unique *time-dependent vector field* (TDVF):

(5) $\mathbb{F} : F(z,t)$, where $F(z,t) = \varphi(z,t) + z$.

Under benign conditions (3) admits an equivalent closed form:

(6) $\gamma(\alpha) : z = z(t), \quad \dfrac{dz}{dt} = \varphi(z,t), \quad z(0) = \alpha$.

Now suppose another well-behaved function $f(z,t) \in S$ is introduced. Set

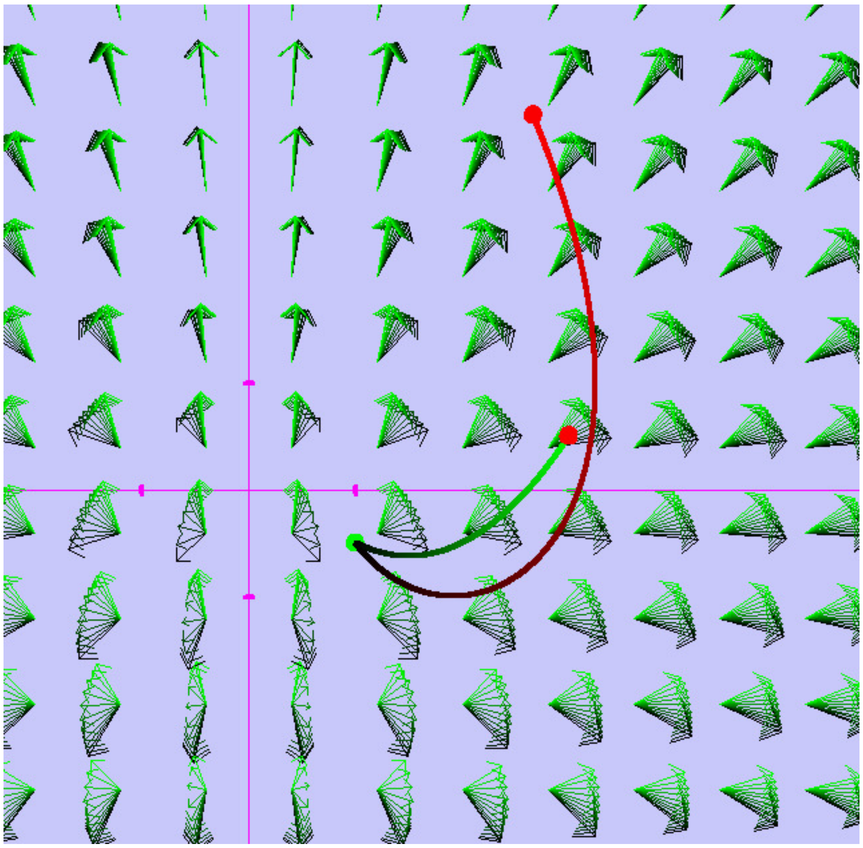
(7) $\varphi^*(z,t) = f(z,t) \cdot \varphi(z,t)$ and create a new, *hybrid contour* in the following way:

(8) $z^*_{k,n} = z^*_{k-1,n} + \eta_{k,n} \cdot \varphi^*(z_{k-1,n}, \tfrac{k-1}{n}), \quad z_{0,n} \equiv \alpha$,

with $\gamma^*_n(\alpha) = \{z^*_{k,n}\}$ and $\gamma^*_n(\alpha) \to \gamma^*(\alpha)$ as $n \to \infty$

Observe that φ^* is a function of <u>points on the original Zeno contour</u> . So that we now have two contours: $\gamma(\alpha)$ (in green) and $\gamma^*(\alpha)$ (in red) that are *siamese*, i.e., originating at the same point. The underlying TDVF is illustrated by vector clusters (black for t=0, green for t=1):

Example 1:



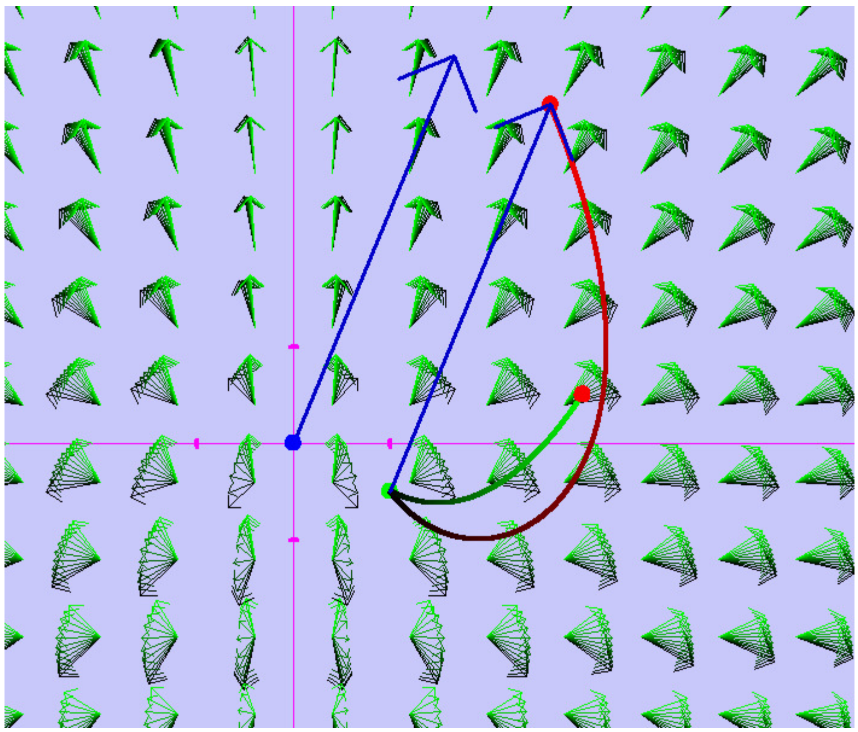
Rewrite (3) as $\sum_{k=1}^{n} f(z_{k-1,n}, \tfrac{k-1}{n}) \cdot (z_{k,n} - z_{k-1,n}) = \frac{1}{n} \sum_{k=1}^{n} f(z_{k-1,n}, \tfrac{k-1}{n}) \cdot \varphi(z_{k-1,n}, \tfrac{k-1}{n}) \quad \Rightarrow$

$$(9) \qquad \int_{\gamma(\alpha)} f dz \; = \; \left[\alpha + \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*}(z_{k-1,n}, \tfrac{k-1}{n}) \right] - \alpha \; = \; G^{*}(\alpha) - \alpha$$

Thus the value of the integral is essentially the vector connecting α to $G^{*}(\alpha)$.

In *Example 1* above $\varphi = \frac{2x}{1+2t} + \frac{2(y-2t^2)}{1+2t}i$, $f = \varphi$ and $\alpha = 1 + .5i$. Then

$$\int_{\gamma(\alpha)} \varphi dz \approx 1.6627 + 4.0006i$$



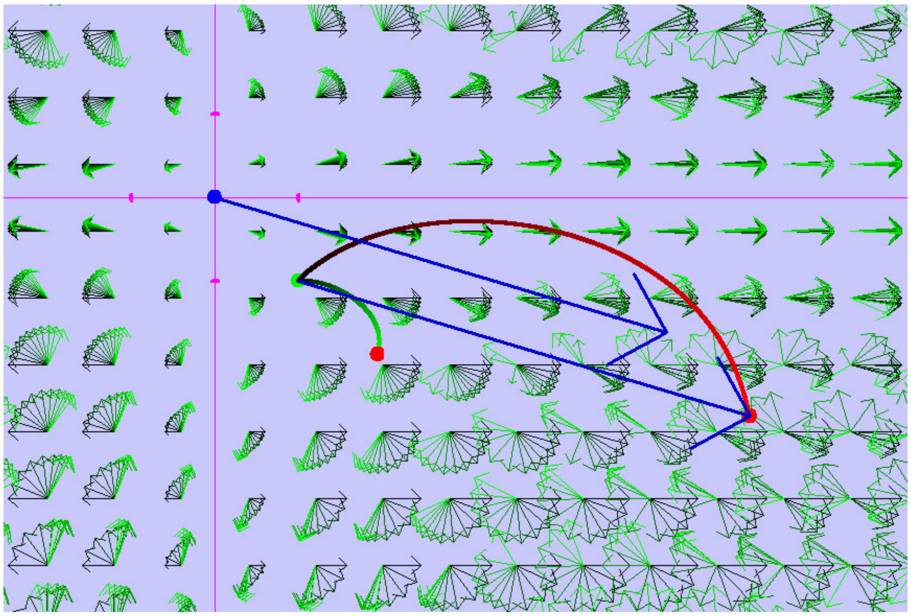
If $f = \varphi$, a *very* well-behaved function,

(10) $\int_{\gamma(z_0)} \varphi(z)\, dz = \int_0^1 \varphi(z)\frac{dz}{dt}dt = \int_0^1 \varphi^2(z(t))\, dt$ <u>or</u> $\int_{z_0}^{z(1)} \varphi(z)\, dz$ for *analytic* φ .

The notation $\int_{\gamma(\alpha)} fdz$ represents $\int_{\gamma(\alpha)} f(z)\, dz$ or $\int_0^1 f(z(t),t)\varphi(z(t),t)dt$

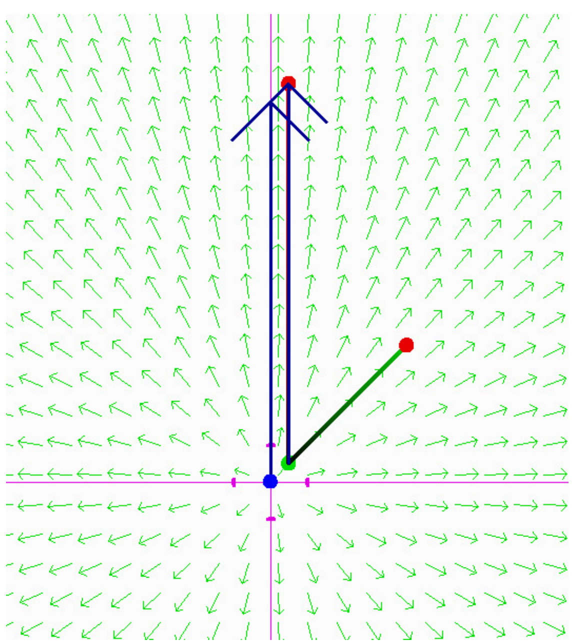
Example 2: $\varphi = xCos(yt) + iySin(xt)$, $f = (2x+t) + i(x-y-t)$, $\alpha = 1-i \quad \Rightarrow$

$$\int_{\gamma(\alpha)} f dz \approx 5.4003 - 1.6223i$$



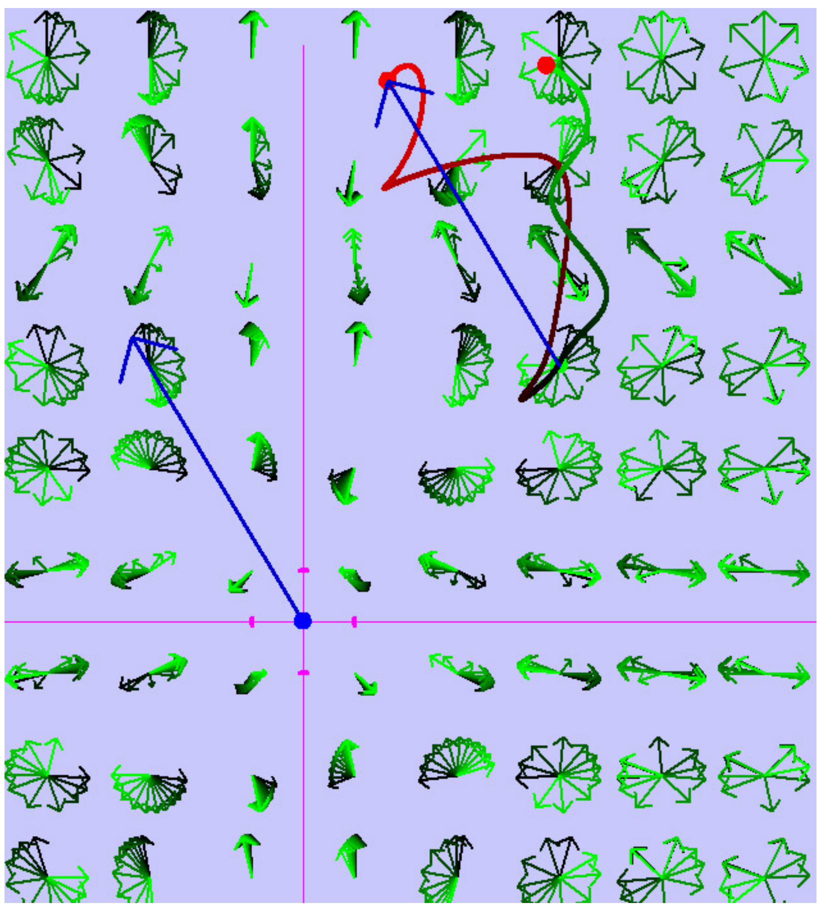
Example 3: A simple case where direct evaluation is easily done:

$$\varphi(z) = 2z \quad \Rightarrow \quad z(t) = \alpha e^{2t} \quad , \; f(z,t) = zt \quad , \quad \alpha = .5(1+i) \quad \Rightarrow \quad \int_{\gamma(\alpha)} f dz \approx 10.2995i$$



Example 4: $\varphi = xCos(xt + y) + ySin(xt - y)i$, $f = 2Sin(x + yt) - 2Cos(x - yt)i$, $\alpha = 5(1+i)$

$$\Rightarrow \quad \int_{\gamma(\alpha)} fdz \approx -3.3396 + 5.5439i$$



***Virtual Integral* vs secondary contour**: Assume $\varphi = \varphi(z)$. Previously, a *virtual integral* was established in the following way:

$$\gamma_n(\alpha): \quad z_{n,n} = \alpha + \frac{1}{n}\varphi(\alpha) + \frac{1}{n}\varphi(z_{1,n}) + \frac{1}{n}\varphi(z_{2,n}) + \cdots + \frac{1}{n}\varphi(z_{n-1,n})$$

Now, by slight of hand, define

$$\psi(\alpha,t) \ , t \in [0,1] \quad \underline{\text{and}} \quad \psi\left(\alpha, \frac{k}{n}\right) \equiv \lim_{m\to\infty} \varphi\left(z_{mk-1,mn}\right), \text{ with } \int_0^1 \psi(\alpha,t)\,dt \text{ defined :}$$

$$G_n(\alpha) - \alpha = \frac{1}{n}\psi\left(\alpha, \frac{1}{n}\right) + \frac{1}{n}\psi\left(\alpha, \frac{2}{n}\right) + \frac{1}{n}\psi\left(\alpha, \frac{3}{n}\right) + \cdots + \frac{1}{n}\psi\left(\alpha, \frac{n}{n}\right) \approx \int_0^1 \psi(\alpha,t)dt$$

So that $$\int_0^1 \psi(\alpha,t)dt = G(\alpha) - \alpha .$$

Under <u>perfect</u> circumstances $\gamma(\alpha): z = z(t)$ has a pleasant closed form and $\psi(\alpha,t) = \varphi(z(t))$, $\alpha = z(0)$. For example, $\varphi(z,t) = 2zt \Rightarrow \varphi(z(t)) = 2\alpha t e^{t^2} = \psi(\alpha,t)$

Therefore $\lambda(\alpha) = \int_0^1 \varphi(z(t))dt = G(\alpha) - \alpha$, whereas, in this note, $\int_0^1 \varphi^2(z(t))dt = G^*(\alpha) - \alpha$.

However, it is usually the case that $\psi(\alpha,t)$ cannot be easily described as $\varphi(z(t))$ and

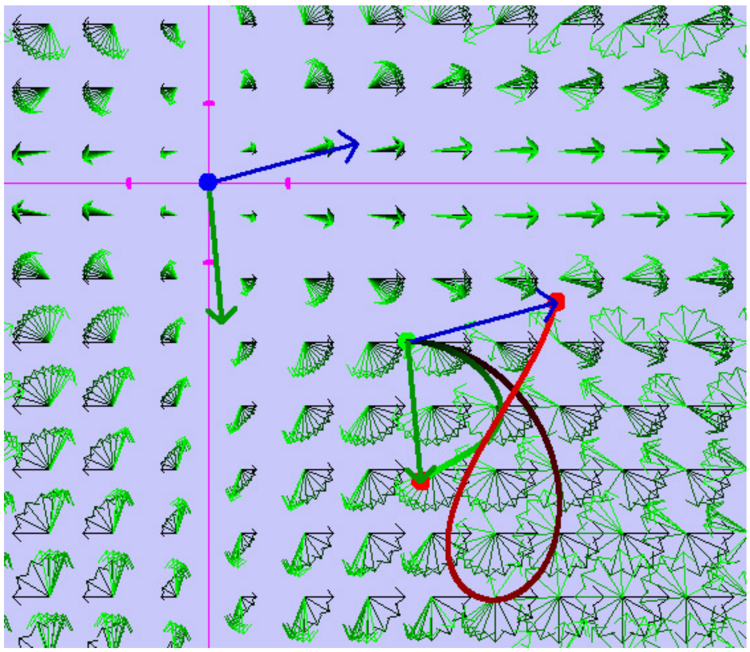
$\lambda(\alpha) = \int_0^1 \psi(\alpha,t)dt = G(\alpha) - \alpha$ has a “virtual” integrand.

The notation $\int_0^1 \varphi^2(z(t))dt = G^*(\alpha) - \alpha$ is used even when $z(t)$ is indescribable.

Example 5: $\varphi = xCos(yt) + iySin(xt)$, $\alpha = 2.5 - 2i \quad \Rightarrow$

$$\lambda(\alpha) = \int_0^1 \psi(z,t)dt \; = \; G(\alpha) - \alpha \; \approx \; .1789 - 1.7624i \, ,$$

and $$\int_0^1 \varphi^2(z(t))dt \; = \; G^*(\alpha) - \alpha \; \approx \; 1.8830 + .4950i \, .$$ (Results hold for $\varphi(z,t)$)



Simplified, heuristic description of the process:

When the functions are analytic and integrations in closed forms are possible, we have

$$\frac{dz}{dt} = \varphi(z) \;\Rightarrow\; z = z(t) \quad \text{and} \quad \frac{d\zeta}{dt} = f(z)\cdot\varphi(z)$$, leading to

$$d\zeta = f(z(t))\cdot\varphi(z(t))\cdot dt = f(z)dz \quad \Rightarrow \zeta(1) - \zeta(0) = \int_{\gamma(\alpha)} f(z)dz = \int_0^1 f(z(t))\cdot\varphi(z(t))\cdot dt$$

Results are valid for $f = f(z,t)$ and $\varphi = \varphi(z,t)$.

Equation solving . . .

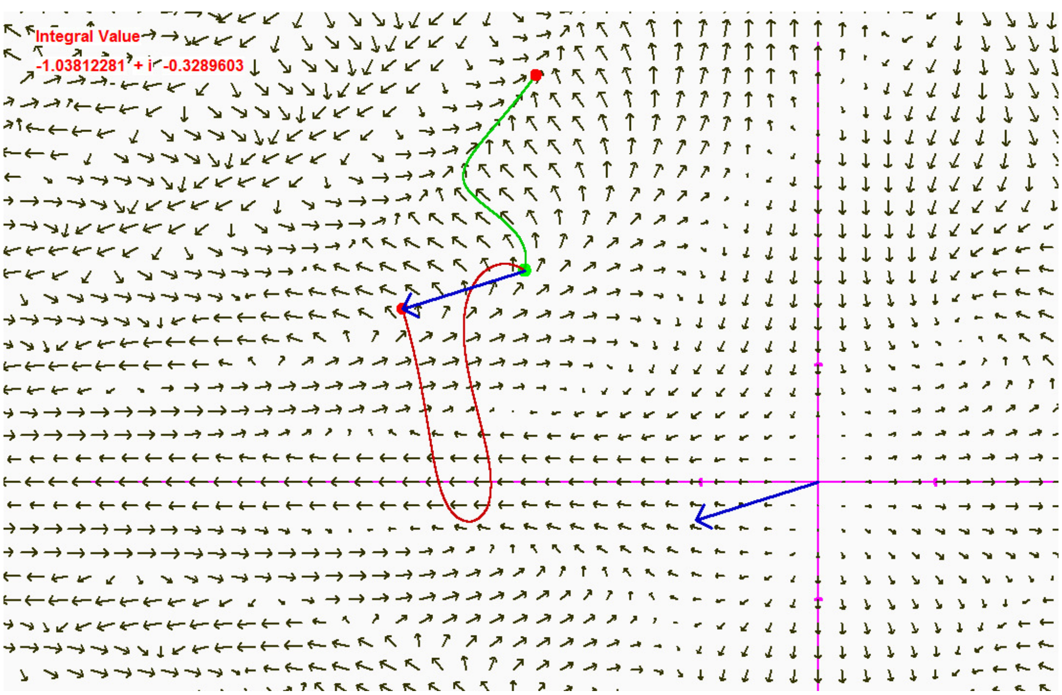
Let us restrict our discussion to $f = \varphi$. Then what has been described previously fits the format

$$T(\varphi,\alpha) = \beta \quad \Leftrightarrow \quad \int_{\gamma(\alpha)} \varphi(z)\, dz \;=\; \beta$$

That is to say $T : C(S) \times S \to C$ where $C(S)$ is the set of complex functions continuous on the set S and C is the complex plane.

Example 6: $\varphi(z) = \varphi(x+iy) = xCos(xy) + iySin(xy)$, with associated vector field $\mathbb{F} : F(z) = \varphi(z) + z$. Thus

$$\mathrm{T}(xCos(xy) + iySin(xy), -2.5 + 1.8i) \;=\; \int_{\gamma(-2.5+1.8i)} \varphi(z)\, dz \;\approx -1.038 - .3289i$$



Example 7: Determine α satisfying the equation $\mathrm{T}(2z,\alpha) = 2\mathrm{i}$. From $\dfrac{dz}{dt} = 2z$, one finds

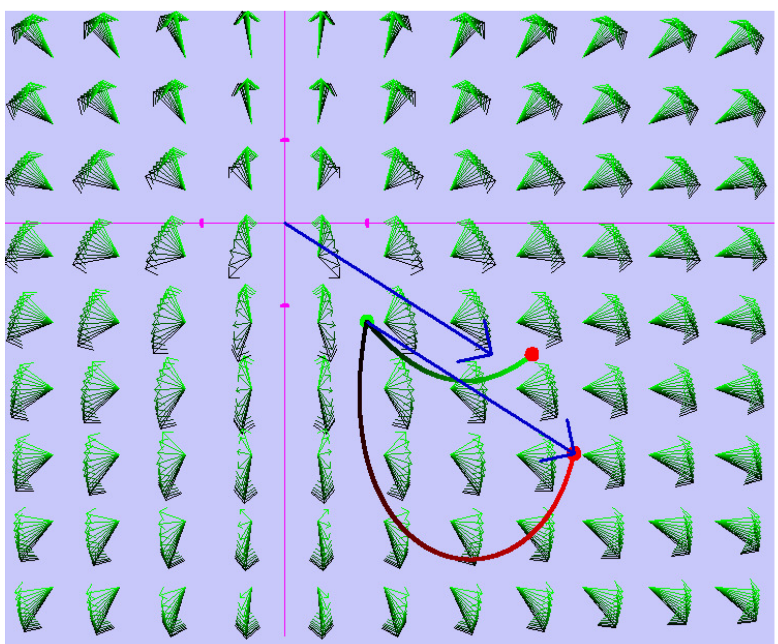
$z(t) = \alpha e^{2t} \Rightarrow z(1) = \alpha e^{2}$. Therefore $2\int_{\alpha}^{\alpha e^2} z\,dz = 2i \;\Rightarrow\; \alpha \approx \pm.1366(1+i)$

Example 8: Given $z(t) = \alpha(1+2t) + 2t^2 i,\ \alpha = 1-1.2i$, find β.

From $\varphi(z,t) = \dfrac{dz}{dt}$ we see that $\dfrac{dz}{dt} = 2\alpha + 4ti = \dfrac{2z}{1+2t} + \left(\dfrac{4t(1+t)}{1+2t}\right)i = \varphi(z,t)$.

Hence $\mathbb{F}: f(z) = \left(\dfrac{3+2t}{1+2t}\right)x + i\left(y + 2\dfrac{y+2t(1+t)}{1+2t}\right)$, a TDVF

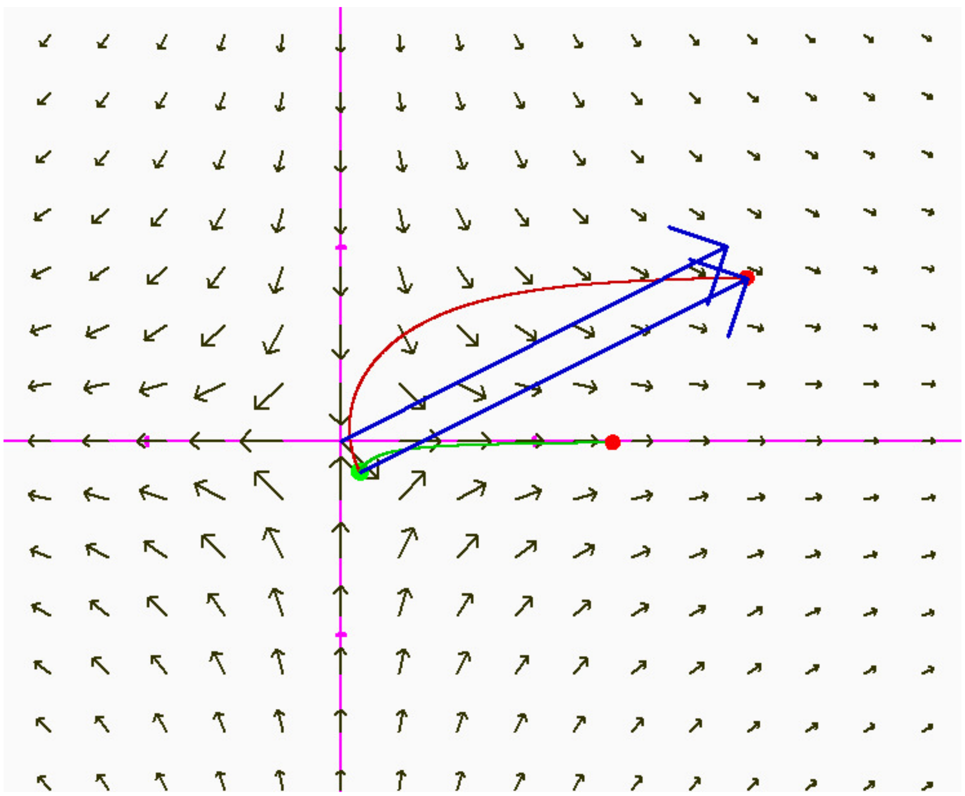
And $\int_0^1 \varphi^2(z(t))\,dt = 4\int_0^1 (\alpha + 2ti)^2\,dt \approx 2.5046 - 1.5950i$



Example 9: $\varphi = \frac{1}{z}$ and $\beta = 2 + i$. Solve for α: $\mathrm{T}(\varphi, \alpha) = \beta$. From $\frac{dz}{dt} = \frac{1}{z}$,

$z(t) = \sqrt{2t + \alpha^2}$ and the problem looks like this: $\int_{\alpha}^{\sqrt{2+\alpha^2}} \frac{1}{z}\, dz = 2 + i$. Thus $\sqrt{2+\alpha^2} = \alpha \cdot e^{2+i}$

gives a solution: $\alpha \approx .10168 - .16128i$.



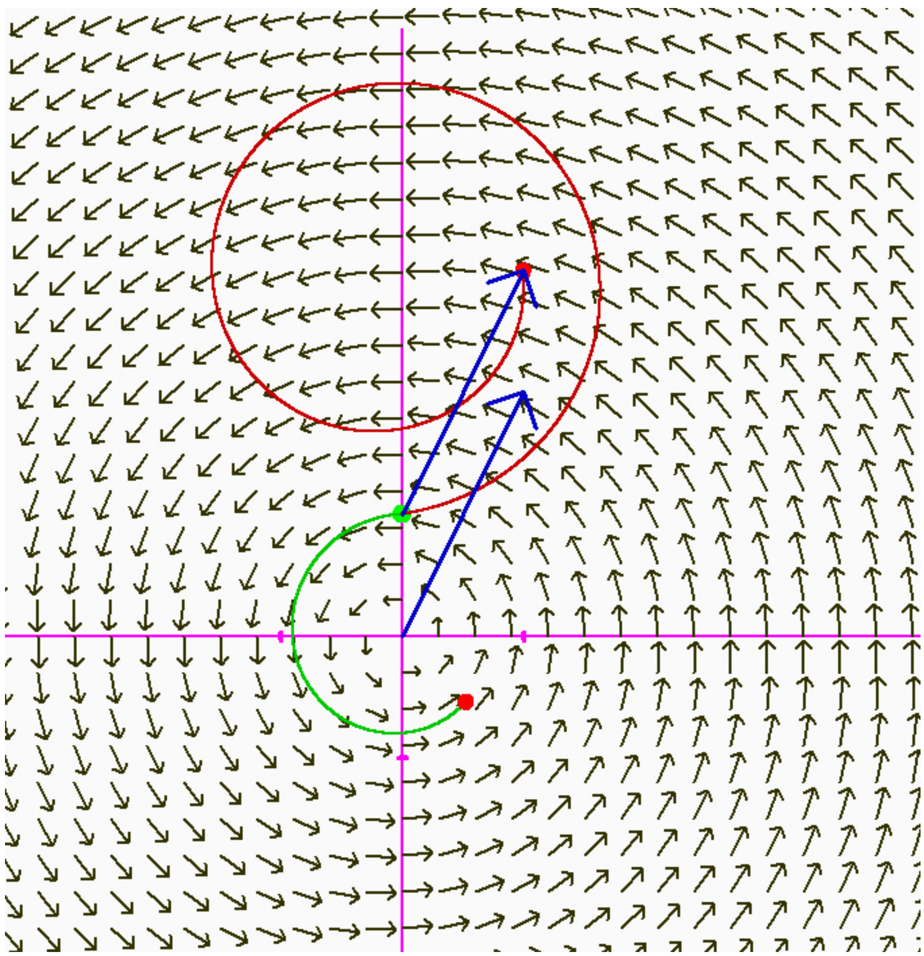
Attempting to solve $\mathrm{T}(\varphi, \alpha) = \beta$ for φ requires more effort and may entail restricting the functional form.

Example 10: Solve for C :

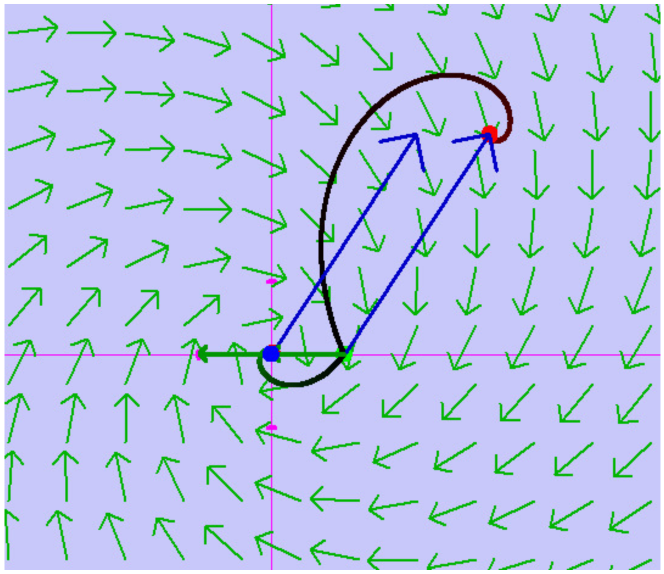
$$\mathrm{T}(Cz,i)=1+2i$$

From $\dfrac{dz}{dt}=Cz$ one obtains $z(t)=ie^{Ct}$. Therefore $C\int_{i}^{ie^{C}} z\ dz\ =\ 1+2i$. Solving for C requires numerical techniques for intrinsic functions.

$$C\approx -.2724+3.9117\,.$$

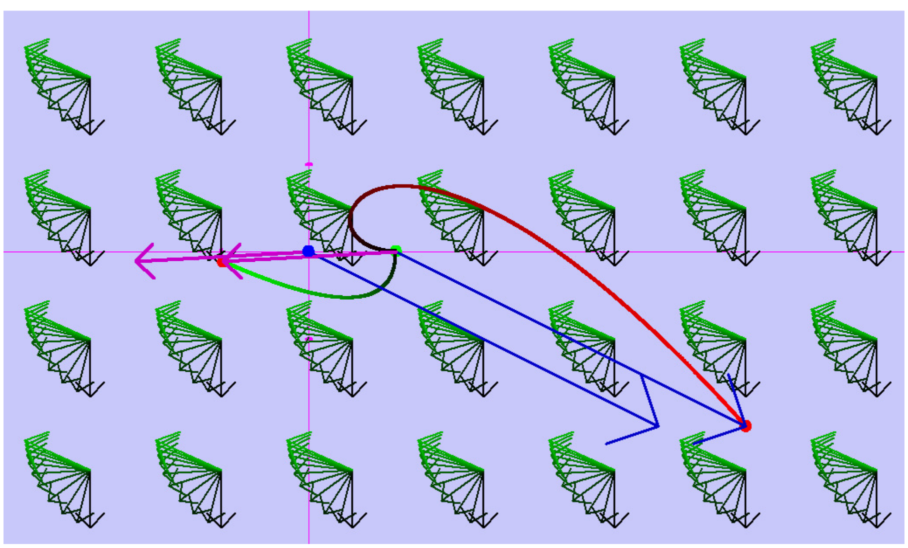


Example 11: Solve for C : $\mathrm{T}(iCz,1)=2+3i \quad \Rightarrow \quad C=-6+4i$



Example 12: Given $z(t)=C\alpha(t^2+1)-2ti$ and $\alpha=1$, solve $T(\varphi,1)=4-2i$ for φ.

$\varphi=2(C\alpha t-i)$ and numerical computations give $C\approx-1.9734+1.8800i$



Final Comments . . .

Setting $\lambda(\alpha)=\int_0^1 \psi(\alpha,t)dt$ and $\rho(\alpha)=\dfrac{1}{f(G(\alpha))}$, it is possible to write

$$\lambda(\alpha)=\rho(\alpha)\int_{\gamma(\alpha)} \left(f(z)-zf'(z)\right) dz \;-\; \alpha\left(1-f(\alpha)\rho(\alpha)\right)$$

Given a primary and a secondary contour,

$$\gamma(\alpha):\ z_{k,n}=z_{k-1,n}+\eta_{k,n}\varphi(z_{k-1,n},\tfrac{k-1}{n}) \quad \text{and} \quad \gamma^{*}(\alpha): z^{*}_{k,n}=z^{*}_{k-1,n}+\eta_{k,n}T(z_{k-1,n},\tfrac{k-1}{n})$$

$$\int_{\gamma(\alpha)} \frac{T(z)}{\varphi(z)}\, dz \;=\; G^{*}(\alpha)-\alpha$$

Enough of this trivia. I'm getting bored!