A Note on Integrals & Hybrid Contours in the Complex Plane

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Abstract: Contour integrals can be expressed graphically as simple vectors arising from a secondary contour.

We start with a well-behaved (continuous, analytic, etc.) function in the complex plane:

(1) \( \phi(z,t) \), with \( z \in S \) a convex set in \( \mathbb{C} \) and \( t \in [0,1] \). \( \phi(z,t) \in S \) \( \forall z \in S \) and \( \forall t \in [0,1] \).

Let

(2) \( g_{k,n}(z) = z + \eta_{k,n} \phi(z, \frac{k}{n}) \), with \( g_{k,n}(z) \in S \), \( 0 < k \leq n \)

Require \( 0 < \eta_{1,n} < \eta_{2,n} < \cdots < \eta_{n,n} = 1 \) and \( \lim_{n \to \infty} \eta_{k,n} = 0 \), where \( k = 1,2,\ldots,n \).

Set \( G_{1,n}(z) = g_{1,n}(z) \), \( G_{k,n}(z) = g_{k,n}(G_{k-1,n}(z)) \) and \( G_n(z) = G_{n,n}(z) \)

with \( G(z) = \lim_{n \to \infty} G_n(z) \), when that limit exists.

Written in an alternate iterative form

(3) \( z_{k,n} = z_{k-1,n} + \eta_{k,n} \phi(z_{k-1,n}, \frac{k}{n}), \quad z_{0,n} \equiv \alpha \),

the distribution of points forms a Zeno contour

(4) \( \gamma_n(\alpha) = \{z_{k,n}\}_{k=1}^{n} \Rightarrow \gamma(\alpha) = \lim_{n \to \infty} \gamma_n(\alpha) \).
The word Zeno denotes the infinite number of actions required in a finite time period if $\eta_{k,n}$ describes a partition of the time interval $[0,1]$.

$\varphi(z,t)$ is associated with a unique time-dependent vector field (TDVF):

\[(5) \quad F: F(z,t) \text{, where } F(z,t) = \varphi(z,t) + z.\]

Under benign conditions (3) admits an equivalent closed form:

\[(6) \quad \gamma(\alpha): z = z(t), \quad \frac{dz}{dt} = \varphi(z,t), \quad z(0) = \alpha.\]

Now suppose another well-behaved function $f(z,t) \in S$ is introduced. Set

\[(7) \quad \varphi^*(z,t) = f(z,t) \cdot \varphi(z,t) \text{ and create a new, hybrid contour in the following way:} \]

\[(8) \quad z_{k,n}^* = z_{k-1,n}^* + \eta_{k,n} \cdot \varphi^*(z_{k-1,n}, \frac{k-1}{n}), \quad z_{0,n}^* = \alpha,\]

with $\gamma^*_n(\alpha) = \{z_{k,n}^*\}$ and $\gamma^*_n(\alpha) \to \gamma'(\alpha)$ as $n \to \infty$.

Observe that $\varphi^*$ is a function of points on the original Zeno contour. So that we now have two contours: $\gamma(\alpha)$ (in green) and $\gamma^*(\alpha)$ (in red) that are siamese, i.e., originating at the same point. The underlying TDVF is illustrated by vector clusters (black for $t=0$, green for $t=1$):
Example 1:

Rewrite (3) as

\[ \sum_{k=1}^{n} f(z_{k-1,n}, \frac{k-1}{n}) \cdot (z_{k,n} - z_{k-1,n}) = \frac{1}{n} \sum_{k=1}^{n} f(z_{k-1,n}, \frac{k-1}{n}) \cdot \varphi(z_{k-1,n}, \frac{k-1}{n}) \Rightarrow \]

(9) \[ \int_{\gamma(\alpha)} fdz = \left[ \alpha + \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(z_{k-1,n}, \frac{k-1}{n}) \right] - \alpha = G'(\alpha) - \alpha \]

Thus the value of the integral is essentially the vector connecting \( \alpha \) to \( G'(\alpha) \).
In Example 1 above \( \varphi = \frac{2x}{1+2t} + \frac{2(y-2t^2)}{1+2t}i \), \( f = \varphi \) and \( \alpha = 1.5i \). Then

\[
\int_{\gamma(\alpha)} \varphi dz = 1.6627 + 4.0006i
\]

If \( f = \varphi \), a very well-behaved function,

\[
\int_{\gamma(z_0)} \varphi(z) \, dz = \int_0^1 \varphi(z) \, dz = \int_0^1 \varphi^2(z(t)) \, dt \quad \text{or} \quad \int_{z_0}^{z(1)} \varphi(z) \, dz \quad \text{for analytic} \ \varphi.
\]

The notation \( \int_{\gamma(\alpha)} \varphi \, dz \) represents \( \int_{\gamma(\alpha)} f \, dz \) or \( \int_{\gamma(\alpha)} f(z(t), t) \varphi(z(t), t) \, dt \).
Example 2: \( \varphi = x\cos(yt) + iy\sin(xt) \), \( f = (2x + t) + i(x - y - t) \), \( \alpha = 1 - i \) \( \Rightarrow \)

\[
\int_{\gamma(\alpha)} f dz \approx 5.4003 - 1.6223i
\]

Example 3: A simple case where direct evaluation is easily done:

\( \varphi(z) = 2z \) \( \Rightarrow \) \( z(t) = \alpha e^{zt} \), \( f(z, t) = zt \), \( \alpha = 5(1 + i) \) \( \Rightarrow \)

\[
\int_{\gamma(\alpha)} f dz \approx 10.2995i
\]
Example 4: $\varphi = x\cos(xt + y) + y\sin(xt - y)i$, $f = 2\sin(x + yt) - 2\cos(x - yt)i$, $\alpha = 5(1 + i)$

$$\Rightarrow \int_{y(\alpha)} f\,dz = -3.3396 + 5.5439i$$
Virtual Integral vs secondary contour: Assume $\varphi = \varphi(z)$. Previously, a virtual integral was established in the following way:

$$\gamma_n(\alpha) : \quad z_{n,n} = \alpha + \frac{1}{n} \varphi(\alpha) + \frac{1}{n} \varphi(z_{1,n}) + \frac{1}{n} \varphi(z_{2,n}) + \cdots + \frac{1}{n} \varphi(z_{n-1,n})$$

Now, by slight of hand, define

$$\psi(\alpha, t), \ t \in [0,1] \quad \text{and} \quad \psi\left(\alpha, \frac{k}{n}\right) \equiv \lim_{m \to \infty} \varphi(z_{mk-n}) \quad \text{with} \quad \int_0^1 \psi(\alpha, t) \, dt \ \text{defined :}$$

$$G_n(\alpha) - \alpha = \frac{1}{n} \psi\left(\alpha, \frac{1}{n}\right) + \frac{1}{n} \psi\left(\alpha, \frac{2}{n}\right) + \frac{1}{n} \psi\left(\alpha, \frac{3}{n}\right) + \cdots + \frac{1}{n} \psi\left(\alpha, \frac{n}{n}\right) \approx \int_0^1 \psi(\alpha, t) \, dt$$

So that

$$\int_0^1 \psi(\alpha, t) \, dt = G(\alpha) - \alpha.$$

Under perfect circumstances, $\gamma(\alpha) : z = z(t)$ has a pleasant closed form and

$$\psi(\alpha, t) = \varphi(z(t)) , \ \alpha = z(0) . \ \text{For example,} \ \varphi(z, t) = z \Rightarrow \varphi(z(t)) = 2 \alpha t e^{\alpha t} = \psi(\alpha, t)$$

Therefore

$$\lambda(\alpha) = \int_0^1 \varphi(z(t)) \, dt = G(\alpha) - \alpha , \ \text{whereas, in this note,} \ \int_0^1 \varphi^2(z(t)) \, dt = G^*(\alpha) - \alpha .$$

However, it is usually the case that $\psi(\alpha, t)$ cannot be easily described as $\varphi(z(t))$ and

$$\lambda(\alpha) = \int_0^1 \psi(\alpha, t) \, dt = G(\alpha) - \alpha \ \text{has a “virtual” integrand.}$$

The notation $\int_0^1 \varphi^2(z(t)) \, dt = G^*(\alpha) - \alpha$ is used even when $z(t)$ is indescribable.
Example 5: \( \varphi = x\cos(yt) + iy\sin(xt) \), \( \alpha = 2.5 - 2i \) \( \Rightarrow \)

\[
\lambda(\alpha) = \int_0^1 \psi(z,t) dt = G(\alpha) - \alpha = .1789 - 1.7624i ,
\]

and

\[
\int_0^1 \varphi^*(z(t)) dt = G^*(\alpha) - \alpha = 1.8830 + .4950i . \text{ (Results hold for } \varphi(z,t) \text{)}
\]

**Simplified, heuristic description of the process:**

When the functions are analytic and integrations in closed forms are possible, we have

\[
\frac{dz}{dt} = \varphi(z) \Rightarrow z = z(t) \quad \text{and} \quad \frac{d\zeta}{dt} = f(z) \cdot \varphi(z) , \text{ leading to}
\]

\[
d\zeta = f(z(t)) \cdot \varphi(z(t)) \cdot dt = f(z)dz \quad \Rightarrow \quad \zeta(1) - \zeta(0) = \int_{z(0)}^1 f(z)dz = \int_0^1 f(z(t)) \cdot \varphi(z(t)) \cdot dt
\]

Results are valid for \( f = f(z,t) \) and \( \varphi = \varphi(z,t) \).
Equation solving . . .

Let us restrict our discussion to $f = \varphi$. Then what has been described previously fits the format

$$T(\varphi, \alpha) = \beta \iff \int_{\gamma(\alpha)} \varphi(z) \, dz = \beta$$

That is to say $T: C(S) \times S \to C$ where $C(S)$ is the set of complex functions continuous on the set $S$ and $C$ is the complex plane.

Example 6: $\varphi(z) = \varphi(x + iy) = x\cos(xy) + iy\sin(xy)$, with associated vector field $\mathbb{F}: F(z) = \varphi(z) + z$. Thus

$$T(x\cos(xy) + iy\sin(xy), -2.5 + 1.8i) = \int_{\gamma(-2.5+1.8i)} \varphi(z) \, dz \approx -1.038 - 3289i$$
Example 7: Determine $\alpha$ satisfying the equation $T(2z, \alpha) = 2i$. From $\frac{dz}{dt} = 2z$, one finds

$z(t) = e^{2t} \Rightarrow z(1) = e^{2}$. Therefore $\int_{\alpha}^{\alpha} 2i \, dz = 2i \Rightarrow \alpha \approx \pm 1.366(1+i)$

Example 8: Given $z(t) = \alpha(1+2t)+2t^2 i$, $\alpha = 1-1.2i$, find $\beta$.

From $\varphi(z,t) = \frac{dz}{dt}$ we see that

$$\frac{dz}{dt} = 2\alpha + 4ti = \frac{2z}{1+2t} + \left( \frac{4t(1+t)}{1+2t} \right)i = \varphi(z,t).$$

Hence $F: f(z) = \left( \frac{3+2t}{1+2t} \right)x + i \left( y + 2 \frac{y+2t(1+t)}{1+2t} \right)$, a TDVF.

And $\int_{0}^{1} \varphi^2(z(t)) \, dt = 4 \int_{0}^{1} (\alpha+2ti)^2 \, dt = 2.5046-1.5950i$
Example 9: \( \varphi = \frac{1}{z} \) and \( \beta = 2 + i \). Solve for \( \alpha \): \( \mathcal{T}(\varphi, \alpha) = \beta \). From \( \frac{dz}{dt} = \frac{1}{z} \),

\[ z(t) = \sqrt{2t + \alpha^2} \]

and the problem looks like this:

\[ \int_{\alpha}^{\sqrt{2 + \alpha^2}} \frac{1}{z} \, dz = 2 + i \]

Thus \( \sqrt{2 + \alpha^2} = \alpha \cdot e^{2i} \)
gives a solution: \( \alpha \approx 0.10168 - 0.16128i \).

Attempting to solve \( \mathcal{T}(\varphi, \alpha) = \beta \) for \( \varphi \) requires more effort and may entail restricting the functional form.
Example 10: Solve for $C$:

$$T(Cz, i) = 1 + 2i$$

From $\frac{dz}{dt} = Cz$ one obtains $z(t) = i e^{Ct}$. Therefore $\int_{i}^{i e^{C}} z \, dz = 1 + 2i$. Solving for $C$ requires numerical techniques for intrinsic functions.

$$C \approx -0.2724 + 3.9117.$$
Example 11: Solve for $C$: \[ T(iCz,1) = 2 + 3i \quad \Rightarrow \quad C = -6 + 4i \]

Example 12: Given \[ z(t) = C\alpha(t^2 + 1) - 2ti \] and $\alpha = 1$, solve $T(\varphi,1) = 4 - 2i$ for $\varphi$.

$\varphi = 2(C\alpha t - i)$ and numerical computations give $C = -1.9734 + 1.8800i$
Final Comments . . .

Setting $\lambda(\alpha) = \int_0^1 \psi(\alpha, t) dt$ and $\rho(\alpha) = \frac{1}{f(G(\alpha))}$, it is possible to write

$$\lambda(\alpha) = \rho(\alpha) \int_{\gamma(\alpha)} (f(z) - zf'(z)) \, dz = \alpha(1 - f(\alpha)\rho(\alpha))$$

Given a primary and a secondary contour,

$$\gamma(\alpha): z_{k,n} = z_{k-1,n} + \eta_{k,n} \varphi(z_{k-1,n}, \frac{k-1}{n}) \quad \text{and} \quad \gamma^*(\alpha): z_{k,n}^* = z_{k-1,n}^* + \eta_{k,n} T(z_{k-1,n}, \frac{k-1}{n})$$

$$\int_{\gamma(\alpha)} \frac{T(z)}{\varphi(z)} \, dz = G^*(\alpha) - \alpha$$

Enough of this trivia. I'm getting bored!