A Further Note on Expanding Functions into Infinite Compositions: Imagery

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Abstract: A heuristic note that further explores expansions of continuous complex functions into infinite compositions. A continuation of A Note on Expanding Functions into Infinite Compositions 1 [1].

Infinite compositions of complex functions may occur in two forms:

I Inner or right compositions: \[ T(z) = \lim_{n \to \infty} R_{k=1}^{n} t_k(z), \]

II Outer or left compositions: \[ T(z) = \lim_{n \to \infty} L_{k=1}^{n} t_k(z). \]

Convergence theory of each of these is discussed in [1]. Here, the emphasis will be on converting certain functions into infinite expansions. Figures are simple topographic images.

Consider functions

(1) \[ F(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < R \quad (R \text{ could be infinite}) \]

A subclass of these functions can be described as functional equations of the following form:

(2) \[ F(nz) = nF(z) + \rho F^m(z), \quad n \geq 2, \quad m \geq 2 \]

Using the procedure described in [1]

(3) \[ F_p(z) = R_{k=1}^{p} \left[ z + \rho \frac{z^m}{n^{k(m-1)+1}} \right] \quad \text{and} \quad z_p = n^p F \left( \frac{z}{n^p} \right) \to z \Rightarrow F(z) = F_p(z_p) \]

As one possibility, the following can be used to show uniform convergence on compact subsets of \( \mathbb{C} \): Write

(4) \[ f_k(z) = z + \rho \frac{z^m}{n^{k(m-1)+1}}, \quad \text{and} \quad F_j(z) = f_1 \circ f_2 \circ \cdots \circ f_j(z) \]
Theorem 1 (Gill, 2011) Suppose \( f_k(z) = z(1 + \eta_k(z)) \), with \( \eta_k \) analytic for \( |z| \leq R_1 \) and 
\[
|\eta_k(z)| < \varepsilon_k, \quad \sum \varepsilon_k < \infty.
\]
Choose \( 0 < r < R_1 \), and define \( R = R(r) = \frac{R_1 - r}{\prod_{k=1}^{\infty}(1 + \varepsilon_k)} \). Then 
\[
F_j(z) = f_1 \circ f_2 \circ \cdots \circ f_j(z) \to G(z) \text{ uniformly for } |z| \leq R \text{ and }
\]
\[
|G'(z)| \leq \prod_{k=1}^{\infty} (1 + \beta_k) < \infty, \quad \beta_k = \frac{R}{r} \varepsilon_k.
\]

Does \( G(z) = F(z) \)?

Uniform convergence and continuity arguments applied to 
\[
|F(z) - G(z)| \leq |F_p(z_p) - G(z_p)| + |G(z_p) - G(z)|
\]
show that, indeed, \( F(z) = G(z) \) for a region containing the origin (we leave this vague).

Example 1: \( F(3z) = 3F(z) + 2F^2(z) \) \( \Rightarrow \) \( F(z) = \mathcal{R} \left[ z + 2 \frac{Z^2}{3^{k+1}} \right] \), \( -20 < x, y < 20 \) \( n=20 \)
Example 2: \[ F(2z) = 2F(z) + F^3(z) \implies F(z) = \sum_{k=1}^{\infty} \left[ z + \frac{z^3}{2^{2k+1}} \right], \quad -40 < x, y < 40 \quad n=20 \]

Restricting

(5) \[ F(nz) = nF(z) + \rho F^2(z) \]

We find that

(6) \[ f_k(z) = z + \rho_{n,k}z^2 \quad \text{where} \quad \rho_{n,k} = \frac{\rho}{n^{k+1}} \]

And

(7) \[ g_k(z) = f_k^{-1}(z) = \frac{1}{2\rho_{n,k}} \left[ \sqrt{1 + 4\rho_{n,k}z} - 1 \right], \quad \text{(principle root) leading to} \]

(8) \[ F^{-1}(z) = G(z) = \lim_{n \to \infty} G_n(z), \quad G_n(z) = \zeta_n \circ g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \quad \text{and} \quad \zeta_n \to z \]

Convergence of the inverse composition can be determined, for example, by the following theorem:
Theorem 2 (Gill, 2011) Let \( \{g_n\} \) be a sequence of complex functions defined on \( S=(|z|<M) \). Suppose there exists a sequence \( \{\rho_n\} \) such that \( \sum_{k=1}^{\infty} \rho_k < \infty \) and \( |g_n(z) - z| < C\rho_n \) if \( |z| < M \). Set \( \sigma = C\sum_{k=1}^{\infty} \rho_k \) and \( R_0 = M - \sigma > 0 \). Then, for every \( z \in S_0 = (|z|<R_0) \),
\[
G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \to G(z), \text{ uniformly on compact subsets of } S_0.
\]

Example 3: \( F(z): F(3z) = 3F(z) + (1 + i)F^2(z), \quad F^{-1}(z) = G(z) \quad -20<x,y<20 \quad n=20 \)
Now consider

\[ F(nz) = nF(z) + T(z) \cdot F^m(z), \quad n \geq 2, \quad m \geq 2 \quad \text{and} \quad T(z) \cdot F^m(z) = \beta z^2 + \beta z^3 + \cdots, \]

which gives

\[ F_p(z_0) = \left( z + T \left( \frac{Z_0}{z^n} \right) \frac{Z^m}{n^{(m-1)}} \right) \circ Z_0 \]

**Example 4:** \( F(2z) = 2F(z) + z \cdot F^3(z) \quad \Rightarrow \quad F_p(z_0) = \left( z + \frac{Z_0}{2^{3k+1}} \right) \circ Z_0 \quad -40 < x, y < 40 \quad n=20 \)

**Example 5:** \( F(2z) = 2F(z) + e^z \cdot F^3(z) \quad \Rightarrow \quad F_p(z_0) = \left( z + \frac{Z_0}{2^{3k+1}} \right) \circ Z_0 \quad -40 < x, y < 40 \quad n=20 \)
Example 6: \( F(3z) = 3F(z) + z^2F^2(z) \implies F_p(z_0) = \bigcup_{k=1}^{p} \left[ z + \left( \frac{z_0}{3^k} \right)^2, \frac{z_0}{3^k+1} \right] \circ z_0 \), \(-45<x,y<45\) \( n=30 \)

Example 7: \( F(2z) = 2F(z) + \frac{z^3}{F(z)} \implies F_p(z_0) = \bigcup_{k=1}^{p} \left[ z + \frac{z_0^3}{4^kZ} \right] \circ z_0 \), \(-50<x,y<50\) \( n=20 \)
Example 8: \[ F(2iz) = 2iF(z) + F^2(z) \Rightarrow F(z) = \sum_{k=1}^{\infty} \mathcal{R} \left[ z + \frac{z^2}{(2i)^{k+1}} \right] \]  
\(-40 < x, y < 40\)  
n=30

\[ |F(z)| \]

\[ |F(z) - z| \]

\(-20 < x < 70, -40 < y < 40\)
Example 9: \( F(2iz) = 2iF(z) + F^3(z) \implies F(z) = \sum_{k=1}^{\infty} \left[ z + \frac{z^3}{(2i)^{2k+1}} \right] \quad -40<x,y<40 \quad n=30 \)

Example 10: \( F(3z) = 3F(z) + z^2 e^{F(z)} \implies F(z_0) = \sum_{k=1}^{\infty} \left[ z + \frac{z^2}{3^{k+1}} \cdot e^{\frac{z}{3^k}} \right] \circ z_0 \quad -20<x,y<20 \quad n=20 \)
Example 11: \( F(2z) = \frac{2z}{1-F(z)} \Rightarrow F(z_0) = \frac{\mathcal{R}}{k=1} \left[ \frac{z_0}{1 - z / 2^k} \right] \circ z_0 \quad \text{for} \quad -50 < x < 80, -50 < y < 50 \quad n=30 \).

Another way to look at this expansion is as a continued fraction: \[ F(z) = \frac{z}{1 - \frac{z}{2 - \frac{z}{4 - \frac{z}{8 - \ldots}}} \right] \]

Example 12: \( F(2z) = \frac{z + F(z)}{1 + F(z)} \Rightarrow F(z_0) = \frac{\mathcal{R}}{k=1} \left[ \frac{z_0 + z}{2 + z / 2^{k-1}} \right] \circ z_0 \quad \text{for} \quad -70 < x < 40, -40 < y < 40 \quad n=30 \)