A Further Note on Expanding Functions into Infinite Compositions: Imagery

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Abstract: A heuristic note that further explores expansions of continuous complex functions into infinite compositions. A continuation of *A Note on Expanding Functions into Infinite Compositions 1 [1].*

Infinite compositions of complex functions may occur in two forms:

I Inner or right compositions: $\underset{k=1}{\overset{n}{\mathcal{R}}} t_k(z) = t_1 \circ t_2 \circ \cdots \circ t_n(z)$, $T(z) = \lim_{n\to\infty} \underset{k=1}{\overset{n}{\mathcal{R}}} t_k(z)$.

II Outer or left compositions: $\underset{k=1}{\overset{n}{\mathcal{L}}} t_k(z) = t_n \circ t_{n-1} \circ \cdots \circ t_1(z)$, $T(z) = \lim_{n\to\infty} \underset{k=1}{\overset{n}{\mathcal{L}}} t_k(z)$.

Convergence theory of each of these is discussed in [1]. Here, the emphasis will be on converting certain functions into infinite expansions. Figures are simple topographic images.

Consider functions

(1) $F(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, $|z| < R$ (R could be infinite)

A subclass of these functions can be described as functional equations of the following form:

(2) $F(nz) = nF(z) + \rho F^m(z), \; n \geq 2, \; m \geq 2$

Using the procedure described in [1]

(3) $F_p(z) = \underset{k=1}{\overset{p}{\mathcal{R}}} \left[z + \rho \frac{z^m}{n^{k(m-1)+1}} \right]$ and $z_p = n^p F\left(\frac{z}{n^p}\right) \rightarrow z \quad \Rightarrow F(z) \equiv F_p(z_p)$

As one possibility, the following can be used to show uniform convergence on compact subsets of $\mathbb{C}$: Write

(4) $f_k(z) = z + \rho \frac{z^m}{n^{k(m-1)+1}}$, and $F_j(z) = f_1 \circ f_2 \circ \cdots \circ f_j(z)$

Theorem 1 (Gill, 2011) Suppose $f_k(z)=z\left(1+\eta_k(z)\right)$, with η_k analytic for $|z|\le R_1$ and

$|\eta_k(z)|<\varepsilon_k$, $\sum\varepsilon_k<\infty$. Choose $0<r<R_1$, and define $R=R(r)=\dfrac{R_1-r}{\prod_{k=1}^{\infty}(1+\varepsilon_k)}$. Then

$F_j(\mathrm{z})=f_1\circ f_2\circ\cdots\circ f_j(\mathrm{z}) \;\to\; \mathrm{G(z)}$ uniformly for $|z|\le R$ and

$$|G'(z)|\le\prod_{k=1}^{\infty}(1+\beta_k)<\infty\ ,\quad \beta_k=\frac{R_1}{r}\varepsilon_k \qquad \blacksquare$$

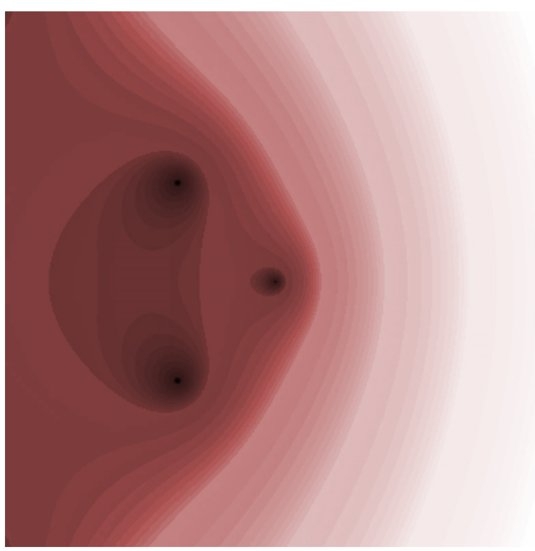
Does $G(z)=F(z)$?

Uniform convergence and continuity arguments applied to

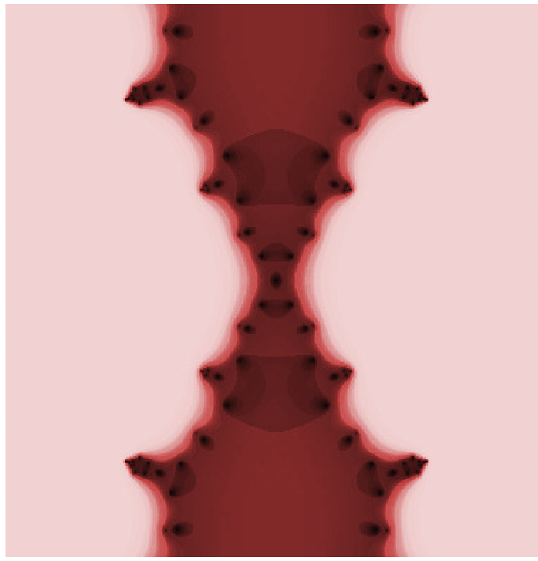
$$|F(z)-G(z)|\le|F_p(z_p)-G(z_p)|+|G(z_p)-G(z)|$$

show that, indeed, $F(z)=G(z)$ for a region containing the origin (we leave this vague).

Example 1: $F(3z)=3F(z)+2F^2(z) \quad\Rightarrow\quad F(z)=\underset{k=1}{\overset{\infty}{\mathcal{R}}}\left[z+2\dfrac{z^2}{3^{k+1}}\right]$, -20<x,y<20 n=20



Example 2: $F(2z) = 2F(z) + F^3(z) \quad \Rightarrow \quad F(z) = \underset{k=1}{\overset{\infty}{\mathcal{R}}} \left[z + \frac{z^3}{2^{2k+1}} \right]$, -40<x,y<40 n=20



Restricting

(5) $$F(nz) = nF(z) + \rho F^2(z)$$

We find that

(6) $f_k(z) = z + \rho_{n,k} z^2$ where $\rho_{n,k} = \dfrac{\rho}{n^{k+1}}$

And

(7) $g_k(z) = f_k^{-1}(z) = \dfrac{1}{2\rho_{n,k}} \left[\sqrt{1 + 4\rho_{n,k} z} \; - \; 1 \right]$, (principle root) leading to

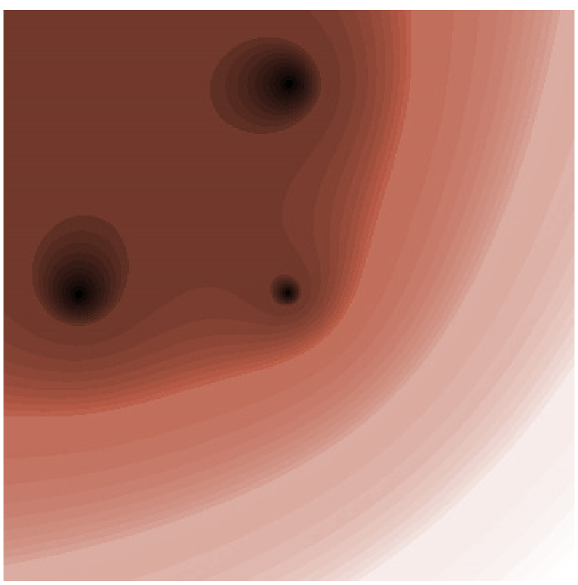
(8) $F^{-1}(z) = G(z) = \lim_{n\to\infty} G_n(z)$, $G_n(z) = \zeta_n \circ g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$ and $\zeta_n \to z$

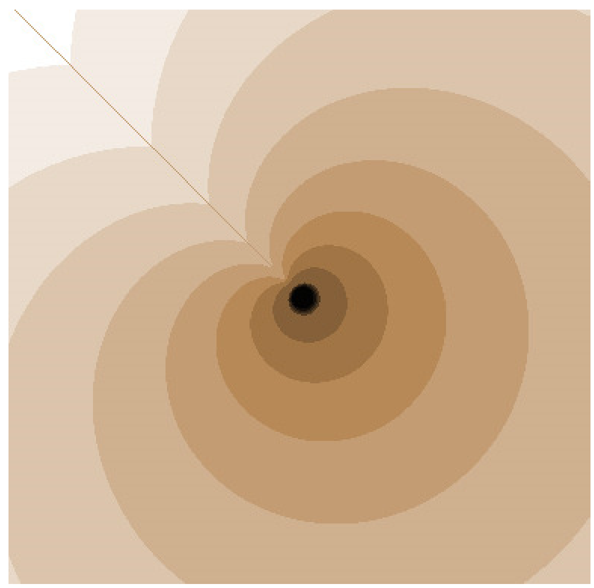
Convergence of the inverse composition can be determined, for example, by the following theorem:

Theorem 2 (Gill, 2011) Let $\{g_n\}$ be a sequence of complex functions defined on $S=(|z|<M)$. Suppose there exists a sequence $\{\rho_n\}$ such that $\sum_{k=1}^{\infty}\rho_n < \infty$ and $|g_n(z)-z| < C\rho_n$ if $|z|<M$. Set $\sigma = C\sum_1^{\infty}\rho_k$ and $R_0 = M-\sigma>0$. Then, for every $z\in S_0 = \left(|z|<R_0\right)$,

$G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \;\; \to G(z)$, uniformly on compact subsets of S_0 .

Example 3: $F(z): F(3z) = 3F(z) + (1+i)F^2(z)$, $F^{-1}(z) = G(z)$ -20<x,y<20 n=20





F(z) G(z)

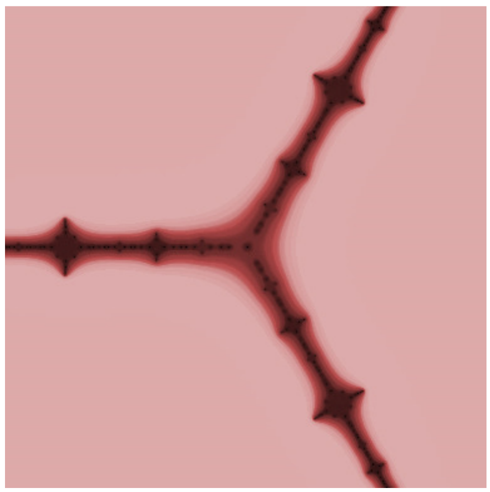
Now consider

(9) $F(nz)=nF(z)+T(z)\cdot F^m(z),\ n\geq 2,\ m\geq 2$ and $T(z)\cdot F^m(z)=\beta_2 z^2+\beta_3 z^3+\cdots$,

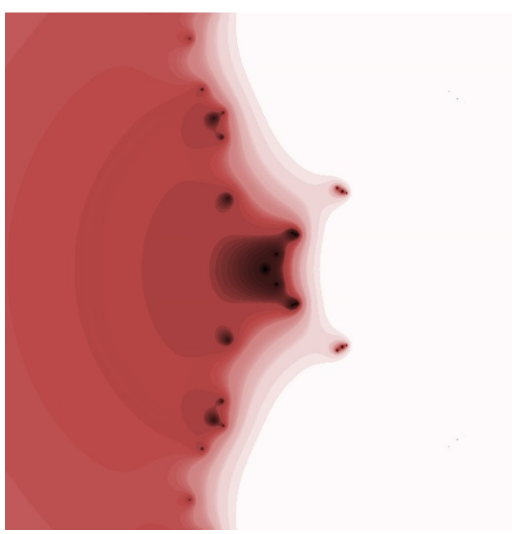
which gives

(10) $F_p(z_0)=\mathop{\mathcal{R}}_{k=1}^{p}\left[z+T\left(\frac{z_0}{n^k}\right)\frac{z^m}{n^{k(m-1)+1}}\right]\circ z_0$

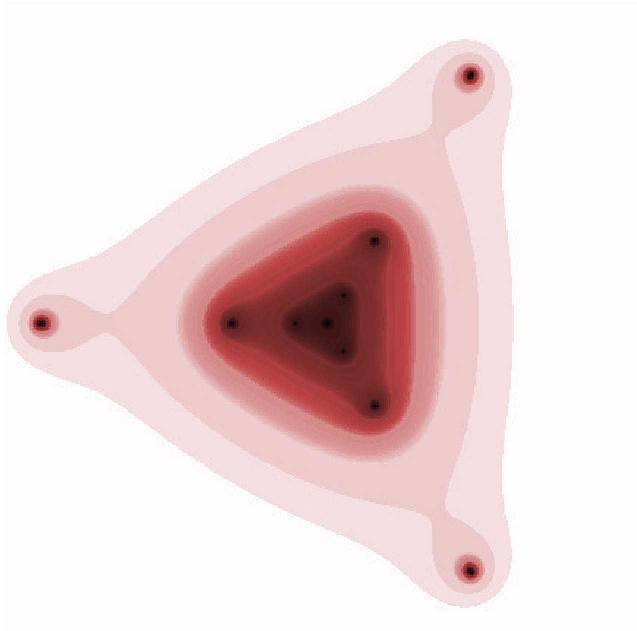
Example 4: $F(2z)=2F(z)+z\cdot F^3(z) \Rightarrow F_p(z_0)=\mathop{\mathcal{R}}_{k=1}^{p}\left[z+z_0\cdot\frac{z^3}{2^{3k+1}}\right]\circ z_0$ -40<x,y<40 n=20



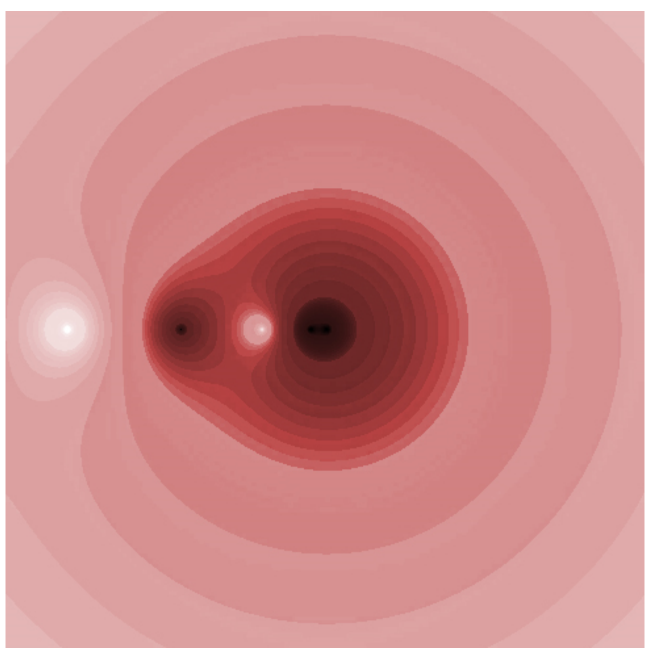
Example 5: $F(2z)=2F(z)+e^z\cdot F^3(z) \Rightarrow F(z_0)=\mathop{\mathcal{R}}_{k=1}^{\infty}\left[z+e^{\frac{z_0}{2^k}}\cdot\frac{z^3}{2^{k+1}}\right]\circ z_0$, -40<x,y<40 n=20



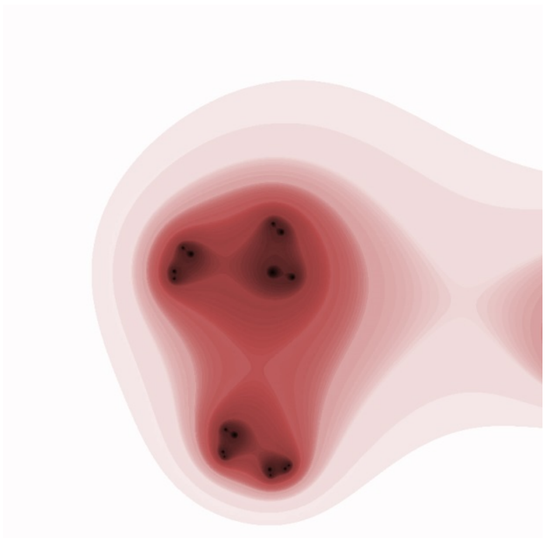
Example 6: $F(3z)=3F(z)+z^2F^2(z) \;\Rightarrow\; F_p(z_0)=\mathop{\mathcal{R}}_{k=1}^{p}\left[z+\left(\frac{z_0}{3^k}\right)^2\cdot\frac{z^2}{3^{k+1}}\right]\circ z_0$, -45<x,y<45 n=30



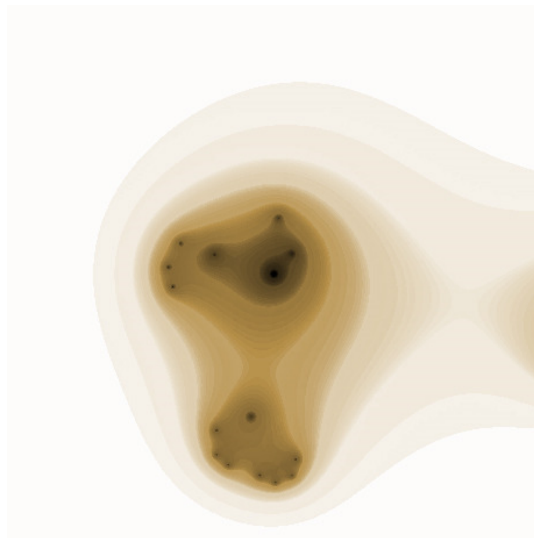
Example 7: $F(2z)=2F(z)+\frac{z^3}{F(z)} \;\Rightarrow\; F_p(z_0)=\mathop{\mathcal{R}}_{k=1}^{p}\left[z+\frac{z_0^3}{4^k z}\right]\circ z_0$ -50<x,y<50 n=20



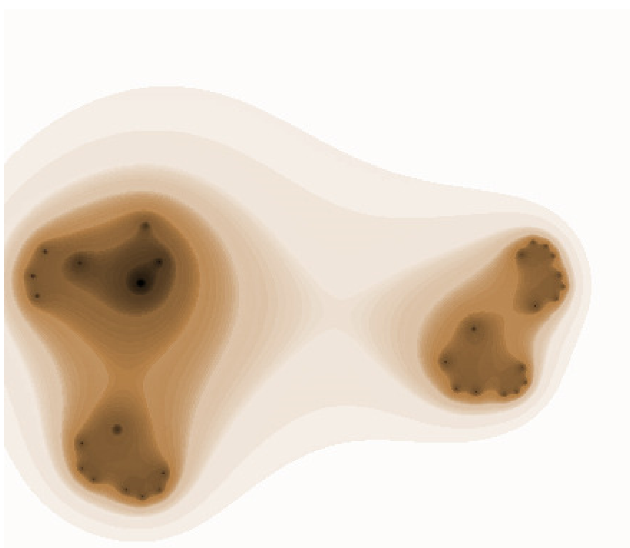
Example 8: $F(2iz) = 2iF(z) + F^2(z) \Rightarrow F(z) = \overset{\infty}{\underset{k=1}{\mathcal{R}}} \left[z + \frac{z^2}{(2i)^{k+1}} \right]$ -40<x,y<40 n=30



$|F(z)|$

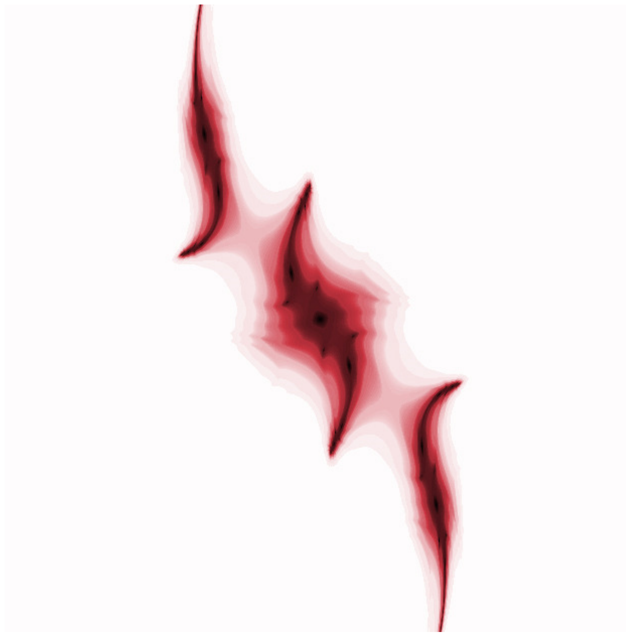


$|F(z) - z|$

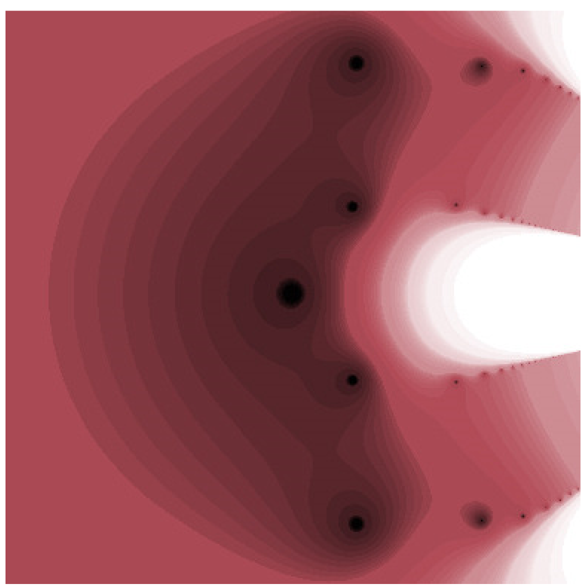


-20<x<70 , -40<y<40

Example 9: $F(2iz) = 2iF(z) + F^3(z) \Rightarrow F(z) = \underset{k=1}{\overset{\infty}{\mathcal{R}}}\left[z + \frac{z^3}{(2i)^{2k+1}} \right]$ -40<x,y<40 n=30

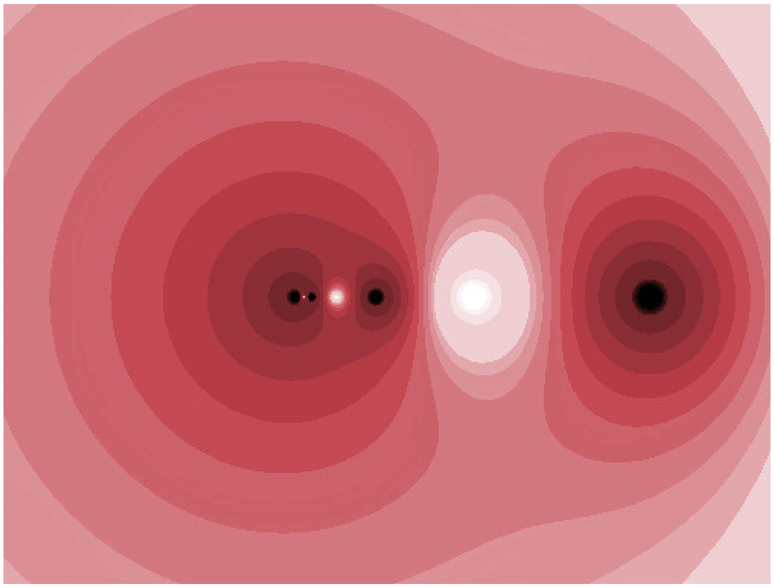


Example 10: $F(3z) = 3F(z) + z^2 e^{F(z)} \Rightarrow F(z_0) = \underset{k=1}{\overset{\infty}{\mathcal{R}}}\left[z + \frac{z_0^2}{3^{k+1}} \cdot e^{\frac{z}{3^k}} \right] \circ z_0$ -20<x,y<20 n=20

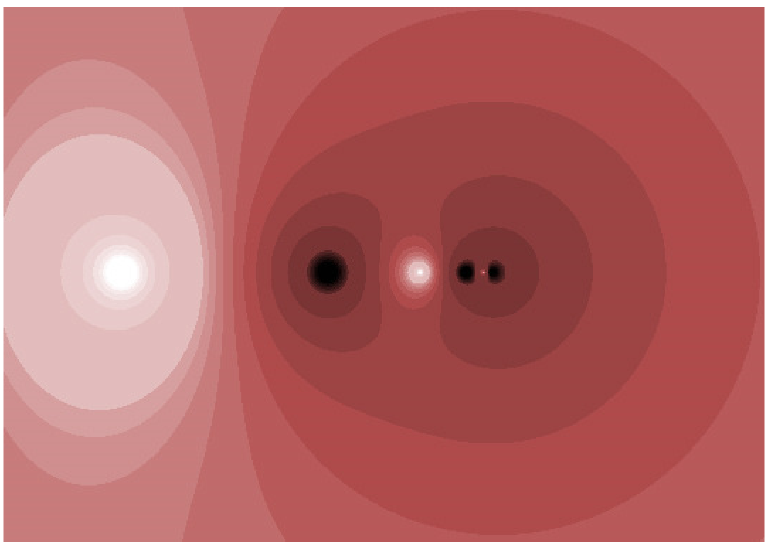


Example 11: $F(2z) = \dfrac{2z}{1-F(z)} \quad \Rightarrow \quad F(z_0) = \mathcal{R}_{k=1}^{\infty}\left[\dfrac{z_0}{1 - z/2^k}\right] \circ z_0$ -50<x<80, -50<y<50 n=30.

Another way to look at this expansion is as a continued fraction: $F(z) = \dfrac{z}{1-}\ \dfrac{z/2}{1-}\ \dfrac{z/4}{1-}\ \dfrac{z/8}{1-} \ldots$



Example 12: $F(2z) = \dfrac{z+F(z)}{1+F(z)} \quad \Rightarrow \quad F(z_0) = \mathcal{R}_{k=1}^{\infty}\left[\dfrac{z_0 + z}{2 + z/2^{k-1}}\right] \circ z_0$ -70<x<40 , -40<y<40 n=30



[1] J. Gill, *A Note on Expanding Functions into Infinite Compositions,* Comm Anal Th Cont Frac, Vol XX (2014)