A Note on Expanding Functions into Infinite Compositions

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(Preliminary Discussion)

Infinite compositions of analytic functions may occur in two forms:

I Inner or right compositions: $\underset{k=1}{\overset{n}{\mathcal{R}}} t_k(z) = t_1 \circ t_2 \circ \cdots \circ t_n(z)$, $T(z) = \lim_{n\to\infty} \underset{k=1}{\overset{n}{\mathcal{R}}} t_k(z)$.

II Outer or left compositions: $\underset{k=1}{\overset{n}{\mathcal{L}}} t_k(z) = t_n \circ t_{n-1} \circ \cdots \circ t_1(z)$, $T(z) = \lim_{n\to\infty} \underset{k=1}{\overset{n}{\mathcal{L}}} t_k(z)$.

Convergence theory of each of these may be found in [1] and [2]. Here, the emphasis will be on finding algorithms for converting closed form expressions into infinite expansions. Simple functional equations that relate a function of nz to an expression of the same function of z sometimes lead directly to such expansions. Consider the example:

Example Ia $Tan(2z) = \dfrac{2Tan(z)}{1 - Tan^2(z)}$. We follow the procedure (Kojima [2]):

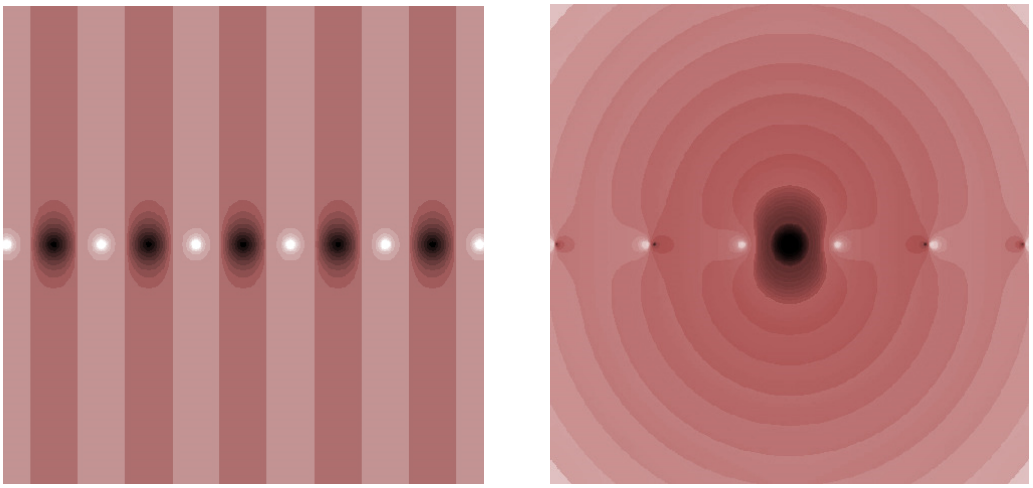
$$T(z) = Tan(z) = 2\frac{z}{1-z^2} \circ Tan(\tfrac{z}{2}) = \frac{z}{1-\frac{1}{4}z^2} \circ 2z \circ Tan(\tfrac{z}{2}) = \frac{z}{1-\frac{1}{4}z^2} \circ 4\frac{z}{1-z^2} \circ Tan(\tfrac{z}{4})$$
$$= \frac{z}{1-\frac{1}{4}z^2} \circ \frac{z}{1-\frac{1}{4^2}z^2} \circ 4z \circ Tan(\tfrac{z}{4}) = \cdots$$

Which leads to $T(z) = \underset{k=1}{\overset{n}{\mathcal{R}}} t_k(z) = t_1 \circ t_2 \circ \cdots \circ t_n(r_n(z))$ with

$$t_k(z) = \frac{z}{1-\frac{1}{4^k}z^2} \quad , \quad r_n(z) = 2^n Tan(z/2^n) \to z$$

Computer experiments lead to the conjecture $Tan(z) = \lim_{n\to\infty} \underset{k=1}{\overset{n}{\mathcal{R}}} t_k(z) = \underset{k=1}{\overset{\infty}{\mathcal{R}}} t_k(z)$.

Image (I) Tan(z) (n=5) and Tan(z) – z (n=20) [-8<x,y<8]



Convergence in a neighborhood of $z = 0$ can be seen by applying the following

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Theorem 2.6 [1] Suppose $f_n(z) = z\left(1+\eta_n(z)\right)$, with η_n analytic for $|z| \le R_1$ and

$|\eta_n(z)| < \varepsilon_n$, $\sum \varepsilon_n < \infty$. Choose $0 < r < R_1$, and define $R = R(r) = \dfrac{R_1 - r}{\prod_{k=1}^{\infty}(1+\varepsilon_k)}$ Then

$F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z) \rightarrow F(z)$ uniformly for $|z| \le R$ and $|F'(z)| \le \prod_{k=1}^{\infty}(1+\beta_k) < \infty$ where

$\beta_k = \dfrac{R_1}{r}\varepsilon_k$.

Here $|\eta_n(z)| = \dfrac{1}{4^n}\dfrac{|z|^2}{\left|1 - \frac{1}{4^n}z^2\right|}$. The actual region of convergence is larger.

Outer of left expansions: Although there are many examples showing the convergence of an inner or right composition to the function of which it is an expansion (see, e.g., analytic theory of continued fractions [3]), there are perhaps no previous non-trivial examples showing the same for outer or left compositions.

Example IIa Set $g_k(z) = t_k^{-1}(z)$, $\gamma_n(z) = r_n^{-1}(z)$ (Example Ia). Then $g_k(z) \to z$, $\gamma_n(z) \to z$ and

$$Arc\tan(z) = \gamma_n \circ g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \approx g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$$

with $g_k(z) = \frac{2 \cdot 4^k}{z}\left(\sqrt{1 + \frac{1}{4^k} z^2} - 1\right)$. Set $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$.

Hence, the conjecture $G_n(z) \to Arc\tan(z)$, or

$$Arc\tan(z) = \mathop{\mathcal{L}}_{k=1}^{\infty} \frac{2 \cdot 4^k}{z}\left(\sqrt{1 + \frac{1}{4^k} z^2} - 1\right),$$ P.V. of course, which is supported by computer examples. This may be written in a simpler form:

$$Arc\tan(z) = \mathop{\mathcal{L}}_{k=0}^{\infty} \left(\frac{2z}{1 + \sqrt{1 + \zeta_k z}} \right) \quad , \quad \zeta_k = \frac{z}{4^k} \quad .$$

Convergence in a neighborhood of $z = 0$ can be verified by employing the following

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Theorem 2.8 [1] Let $\{g_n\}$ be a sequence of complex functions defined on $S=(|z|<M)$. Suppose there exists a sequence $\{\rho_n\}$ such that $\sum_{k=1}^{\infty} \rho_n < \infty$ and $|g_n(z) - z| < C\rho_n$ if $|z| < M$. Set $\sigma = C\sum_1^{\infty} \rho_k$ and $R_0 = M - \sigma > 0$. Then, for every $z \in S_0 = (|z| < R_0)$, $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \to G(z)$, uniformly on compact subsets of S_0 .

Here $|g_n(z) - z| = \frac{1}{4^n} \frac{|z|^3}{\left|1 + \sqrt{1 + \frac{1}{4^n} z^2}\right|}$. The actual convergence region is larger.

Example Ib $F(z) = e^z - 1$. $F(2z) = F(z)(F(z) + 2)$ gives

$$F(z) = z(z+2) \circ F(z/2) = \left(\frac{z^2}{4} + z\right) \circ 2z \circ F(z/2) = \left(\frac{z^2}{4} + z\right) \circ 2F(z/2)$$

$$= \left(\frac{z^2}{4} + z\right) \circ \left(2z^2 + 4z\right) \circ F(z/4) = \left(\frac{z^2}{4} + z\right) \circ \left(\frac{z^2}{8} + z\right) \circ 4z \circ F(z/4) = \cdots$$

With $r_n(z) = 2^n F\left(\frac{z}{2^n}\right) \to z$. We have the following

$$e^z = 1 + \underset{k=1}{\overset{\infty}{\mathcal{R}}}\left(\frac{z^2}{2^{k+1}} + z\right)$$

Theorem 2.6[1] can be used to show convergence in a neighborhood of $z = 0$ with $\eta_n(z) = \frac{z}{2^{n+1}}$.

Of course, the radius of convergence of the composition is actually infinite.

Example IIb In the previous example $t_k(z) = \frac{z^2}{2^{k+1}} + z$, so that

$g_k(z) = t_k^{-1}(z) = \dfrac{2z}{1+\sqrt{1+4\left(\frac{1}{2^{k+1}}\right)z}}$. Thus $Ln(z+1) \approx G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$ and

$$Ln(z+1) = \underset{k=1}{\overset{\infty}{\mathcal{L}}}\left(\frac{2z}{1+\sqrt{1+4\left(\frac{1}{2^{k+1}}\right)z}}\right).$$ $n = 50$ gives ten decimal place accuracy for $Ln(6-8i)$.

Image (2) Ln(1+z) - z , n=10, -8<x,y<8 and Ln(1+z)

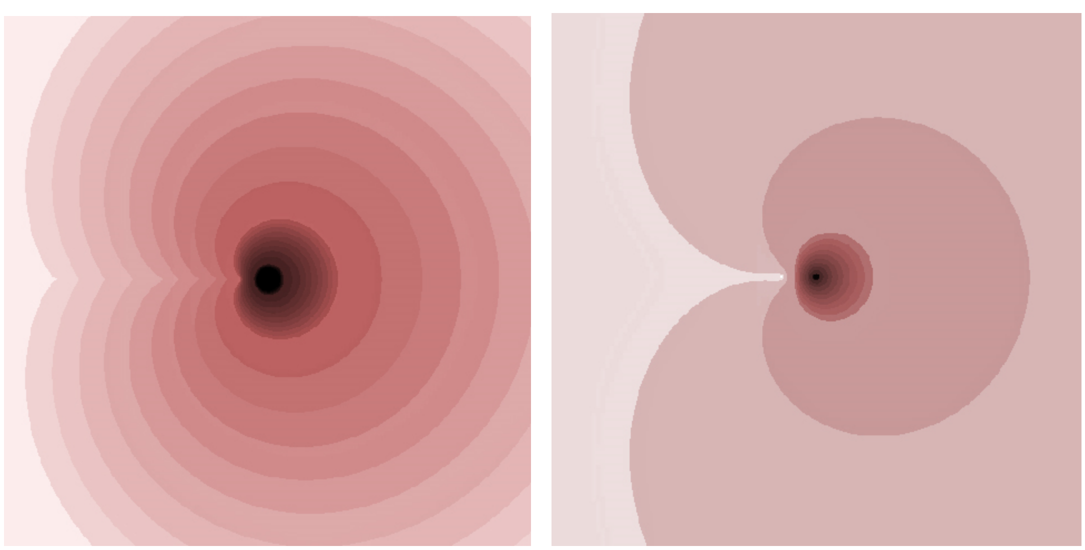


Image (3) $F(z)=\underset{k=1}{\overset{\infty}{\mathcal{R}}}\left(\frac{z^2}{10^k}+\frac{1}{z}\right)$ n=5 -8<x,y<8 $G_{10}(z)=\underset{k=0}{\overset{10}{\mathcal{L}}}\left(\frac{z^2}{10^k}+\frac{1}{z}\right)$ non-convergent

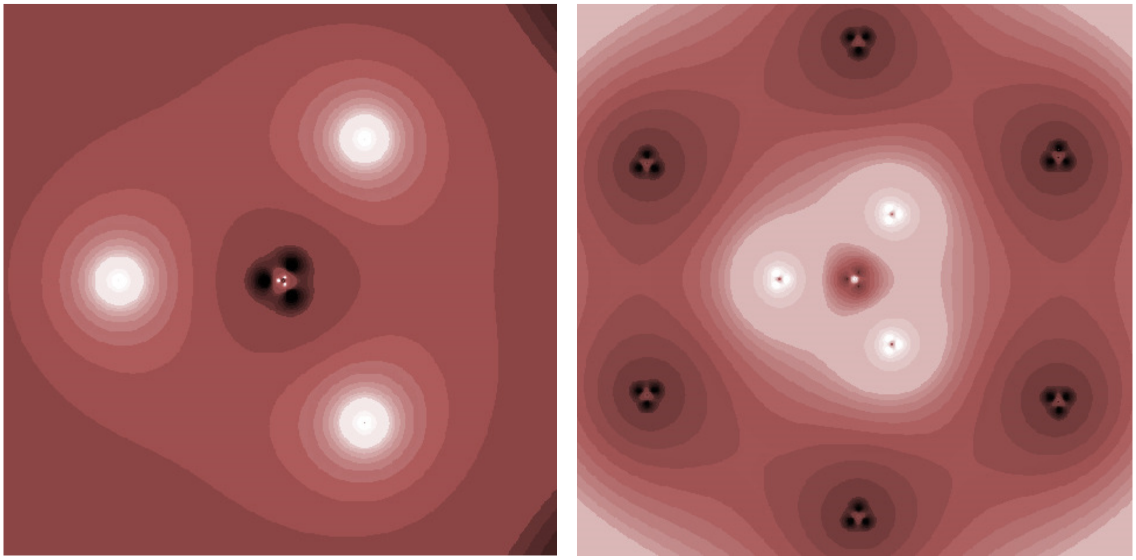
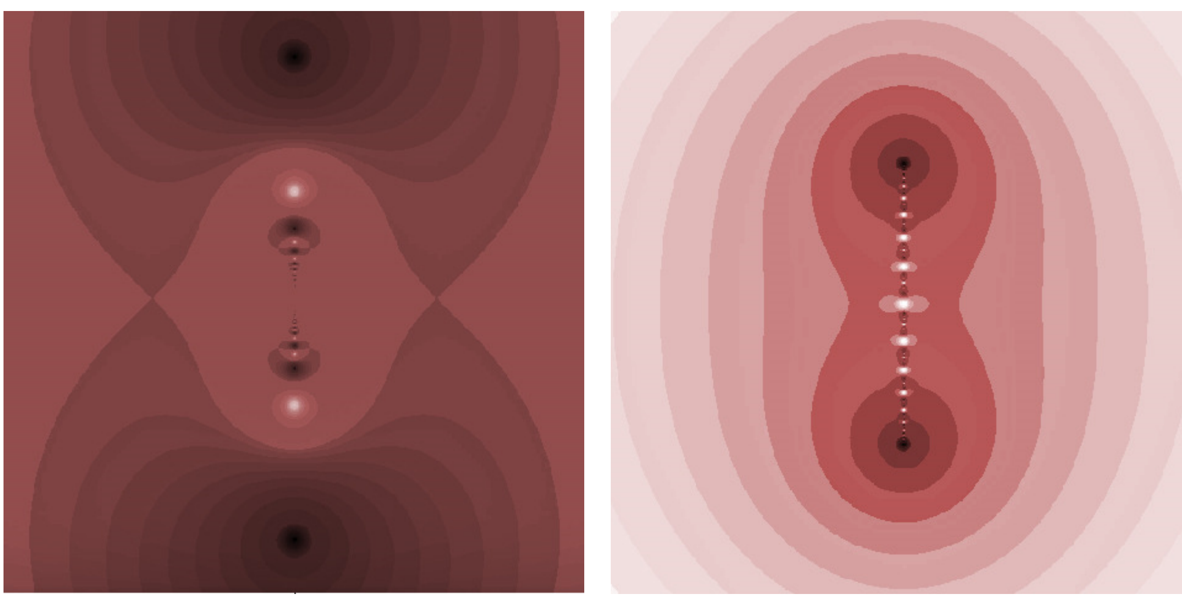


Image (4) $F(z)=\underset{k=1}{\overset{50}{\mathcal{R}}}\left(\left(\frac{k}{k+1}\right)z+\frac{1}{2^k z}\right)$ **and** $G(z)=\underset{k=0}{\overset{50}{\mathcal{L}}}\left(\left(\frac{k}{k+1}\right)z+\frac{1}{2^k z}\right)$ **-8<x,y<8**



Example Ic $F(z) = Sin(z)$

$$Sin(z) = 2z\sqrt{1-z^2} \circ Sin(z/2) = z\sqrt{1-\frac{1}{4}z^2} \circ 2Sin(z/2) = z\sqrt{1-\frac{1}{4}z^2} \circ 4z\sqrt{1-z^2} \circ Sin(z/4)$$

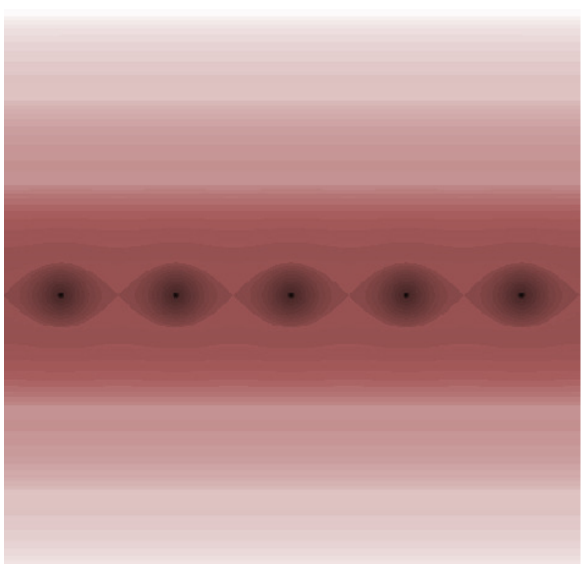
$$= z\sqrt{1-\frac{1}{4}z^2} \circ z\sqrt{1-\frac{1}{4^2}z^2} \circ 4Sin(z/4) = \cdots$$

with $r_n(z) = 2^n Sin(z/2^n) \to z$. We have

$$Sin(z) = \pm \underset{k=1}{\overset{\infty}{\mathcal{R}}} \left(z\sqrt{1-\frac{1}{4^k}z^2} \right)$$

Where the positive sign is valid in Q1 and Q4 and the negative sign in Q2 and Q3. For $Sin(1+4i)$ the value is accurate to ten decimal places for $n = 20$.

Image (5) Sin(z) n=10



Continued Fractions (CFs) are a special case of inner composition (I), involving two complex variables.

One type is $F(z) = \cfrac{a_1(z)}{1+\cfrac{a_2(z)}{1+\cfrac{a_3(z)}{1+\ddots}}}$, defined by $t_n(z;\zeta) = \frac{a_n(z)}{1+\zeta}$ and

$T_1(z;\zeta) = t_1(z;\zeta)$, $T_n(z;\zeta) = T_{n-1}(z;t_n(z;\zeta))$. Then

$$F(z)=\lim_{n\to\infty}T_n(z;\zeta) \text{ or } \mathop{\mathcal{R}}_{n=1}^{\infty}\left(t_n(z;\zeta)\right)_{\zeta=0}.$$

Although $\zeta=0$ normally, other values of the variable ζ frequently lead to the same value of $F(z)$. The essential difference between the examples cited previously and CFs is that the former represent compositions on z that lead to functions $F(z)$, whereas the latter evolve from compositions on an "auxiliary" variable ζ, leading to $F(z)$.

Example Id $t_n(z;\zeta)=\dfrac{a_n(z)}{1+\zeta}$ where $|\zeta|<R \left(R\le\dfrac{1}{2}\right)$ and $|a_n(z)|<\rho R(1-R)$, $0<\rho<1$, with $a_n(z)$ analytic for $z\in S$. Then $|t_n(z;\zeta)|<\rho R$ and these functions contract uniformly.

Therefore $F(z)=\mathop{\mathcal{R}}_{n=1}^{\infty}\left(\dfrac{a_n(z)}{1+\zeta}\right)_{\zeta}$, analytic for $|\zeta|<R$ and $z\in S$ (see Contraction Theorems in [1]).

III Implicit functions and Zeno contours: Consider an expression defining a function implicitly:

$$\Phi\left(\zeta,f(\zeta)\right)=0 \text{ or } \Phi\left(\zeta,z\right)=0 ,\ z=f(\zeta).$$

The following definition is from [4]:

Zeno contour: Let $g_{k,n}(z)=z+\eta_{k,n}\varphi(z)$ where $z\in S$ and $g_{k,n}(z)\in S$ for a convex set S in the complex plane. Require $\lim_{n\to\infty}\eta_{k,n}=0$, where (usually) $k=1,2,...,n$. Set $G_{1,n}(z)=g_{1,n}(z)$, $G_{k,n}(z)=g_{k,n}\left(G_{k-1,n}(z)\right)$ and $G_n(z)=G_{n,n}(z)$ with $G(z)=\lim_{n\to\infty}G_n(z)$, when that limit exists.

The *Zeno contour* is a graph of this iteration. Normally, for a vector field, $\mathbb{F}=F$, $\varphi(z)=F(z)-z$, and under the right conditions $G(z)=\alpha$, an attractive fixed point of F.

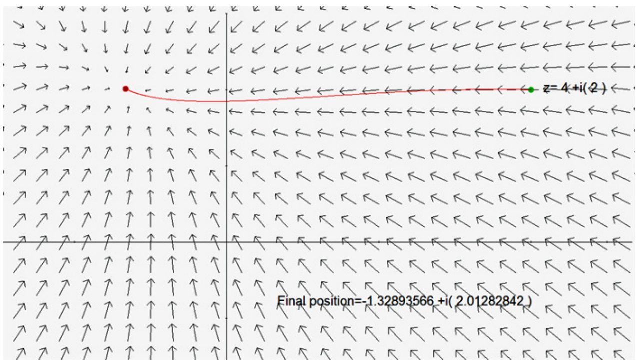
In the context of this discussion $g_{k,n}(z)=z+\eta_{k,n}\left(F\left(\zeta,z\right)-z\right)$, and if $\left|\dfrac{\partial F}{\partial z}\right|<\rho<1\ \forall\zeta\in S$, a suitable domain, then $G_n(z)\to\alpha(\zeta)=f(\zeta)$ a fixed point for each value of ζ, starting with an initial value z in some neighborhood of the fixed points [4]. Thus, from the notation II,

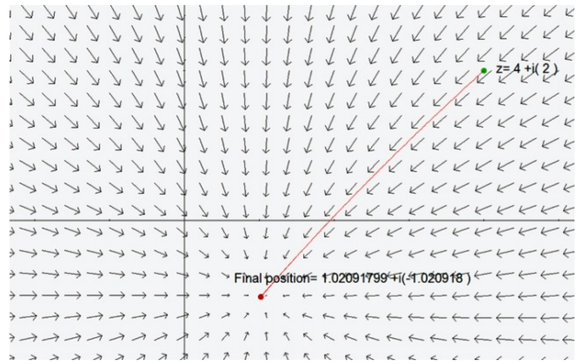
III $$\mathop{\mathcal{L}}_{k=1}^{n}g_{k,n}(z)=g_{n,n}\circ g_{n-1,n}\circ\cdots\circ g_{1,n}(z) \quad\text{and}\quad G(z)=\lim_{n\to\infty}\mathop{\mathcal{L}}_{k=1}^{n}g_{k,n}(z).$$

Example IIIa: $\Phi(\zeta,z)=\zeta Cos\left(\frac{\zeta z}{10}\right)+z=0$. Then $F(\zeta,z)=-\zeta Cos\left(\frac{\zeta z}{10}\right)$ and the Zeno contour terminates at $z=f(\zeta)$ for ζ in a neighborhood of the origin and initial values of z near the fixed points. For example,

$$f(1-2i)\approx -1.3289+i(2.0128) \quad \text{and} \quad f(-1+i)\approx 1.0209+i(-1.0209),$$

starting with $z_0 = 4+2i$.





References

[1] J. Gill, Convergence of infinite compositions of complex functions, Comm. Anal. Th. Cont. Frac., Vol XIX (2012)

[2] S. Kojima, Convergence of infinite compositions of entire functions, arXiv:1009.2833v1

[3] L. Lorentzen, H. Waadeland, *Continued Fractions with Applications*, North Holland (1992)

[4] J. Gill, Zeno contours and attractors, Comm. Anal. Th. Cont. Frac., Vol XIX (2012)