An Expository Note: Woven Contours in the Complex Plane & Images

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Codependent, coupled, or woven contours: "Weave" two contours in the complex plane:

(1) $z_{k,n}=z_{k-1,n}+\eta_{k,n}\varphi_1(\zeta_{k-1,n},\tfrac{k-1}{n})$ and $\zeta_{k,n}=\zeta_{k-1,n}+\eta_{k,n}\varphi_2(z_{k-1,n},\tfrac{k-1}{n})$,

where $\mathbb{P}_n[0,1]=\{t_{k,n}\}_{k=0}^{n}$, $0=t_{0,n}<t_{1,n}<\cdots<t_{n,n}=1$, $\eta_{k,n}=t_{k,n}-t_{k-1,n}$, $\|\mathbb{P}_n\|\to 0$,

a discrete system analogous to

$$\frac{dz}{dt}=\varphi_1(\zeta,t) \text{ and } \frac{d\zeta}{dt}=\varphi_2(z,t) .$$

(See the appendix for a sketch of a proof that solutions exist under certain conditions)

In this note we are concerned with computing and displaying the moduli of

(2) $\lambda_1(\zeta_0)=z(1)-z_0=\int_0^1\psi_1(\zeta_0,t)dt$ and $\lambda_2(z_0)=\zeta(1)-\zeta_0=\int_0^1\psi_2(z_0,t)dt$

over rectangular regions in $\mathbb{C}$,

$$z_0,\zeta_0\in S[a_1,a_2;b_1,b_2] = \{\omega: a_1<re(\omega)<a_2, b_1<\mathrm{Im}(\omega)<b_2\}$$

both in a varied color scheme and a single tinted color scheme. The definite "integrals" in (2) are defined in the single contour case as follows:

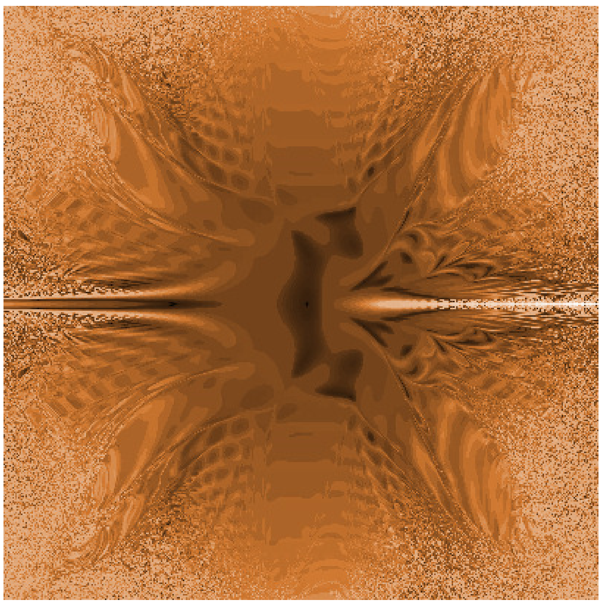
Virtual Integrals: The contour , $z_{k,n}=z_{k-1,n}+\eta_{k,n}\cdot\varphi(z_{k-1,n},\tfrac{k-1}{n})$, $(\frac{dz}{dt}=\varphi(z,t))$, can be expanded linearly: $z_{n,n}=z_0+\eta_n\psi(z_0,\tfrac{0}{n})+\eta_n\psi(z_0,\tfrac{1}{n})+\cdots+\eta_n\psi(z_0,\tfrac{n-1}{n})\approx z_0+\int_0^1\psi(z_0,t)dt$, with $\psi(z_0,\tfrac{k}{n}) \doteq \varphi(z_{k,n},\tfrac{k}{n})$, $\eta_{k,n}=\eta_n=\frac{1}{n}$ and $n\to\infty$. I call this a *virtual integral* due to the fact that the integrand normally cannot be precisely stated although the value of the integral can be easily computed. However, when $z(t)$ can be derived in closed form, $\psi(z_0,t)=\varphi(z(t),t)$ and

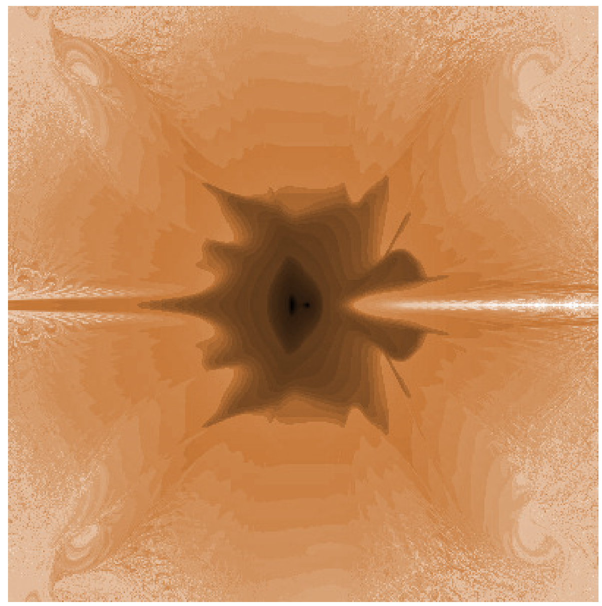
$$\lambda(z_0)=\int_0^1\psi(z_0,t)dt = \int_0^1\varphi\big(z(t),t\big)dt = z(1)-z_0,\ z(0)=z_0$$

Example 1: $\frac{dz}{dt} = \varphi_1(\zeta, t) = uCos(v) + ivSin(u)$, ζ=u+iv and

$$\frac{d\zeta}{dt} = \varphi_2(z,t) = z^2 = (x^2 - y^2) + i2xy,\ z = x + iy\ ,\ \ S[-10,10;-10,10]$$

$\lambda(\zeta_0) = \int_0^1 \psi_1(\zeta_0, t)dt$: $\lambda(z_0) = \int_0^1 \psi_2(z_0, t)dt$: n=30





Mystical Moths

The patterns show, for example, where the virtual integrals $\lambda(\zeta_0) = \int_0^1 \psi_1(\zeta_0, t)dt$ and $\lambda(z_0) = \int_0^1 \psi_2(z_0, t)dt$ are either zero or close to zero – the darkest areas. Very light areas correspond to very large moduli – infinite perhaps. Black to red to green to blue in ascending moduli.

Example 2 : $\dfrac{dz}{dt} = \varphi_1(\zeta,t) = \zeta + \dfrac{1}{\zeta}$, $\dfrac{d\zeta}{dt} = \varphi_2(z,t) = xSin(y) + iyCos(x)$ $\Rightarrow \lambda(\zeta_0) = \int_0^1 \psi_1(\zeta_0,t)dt$

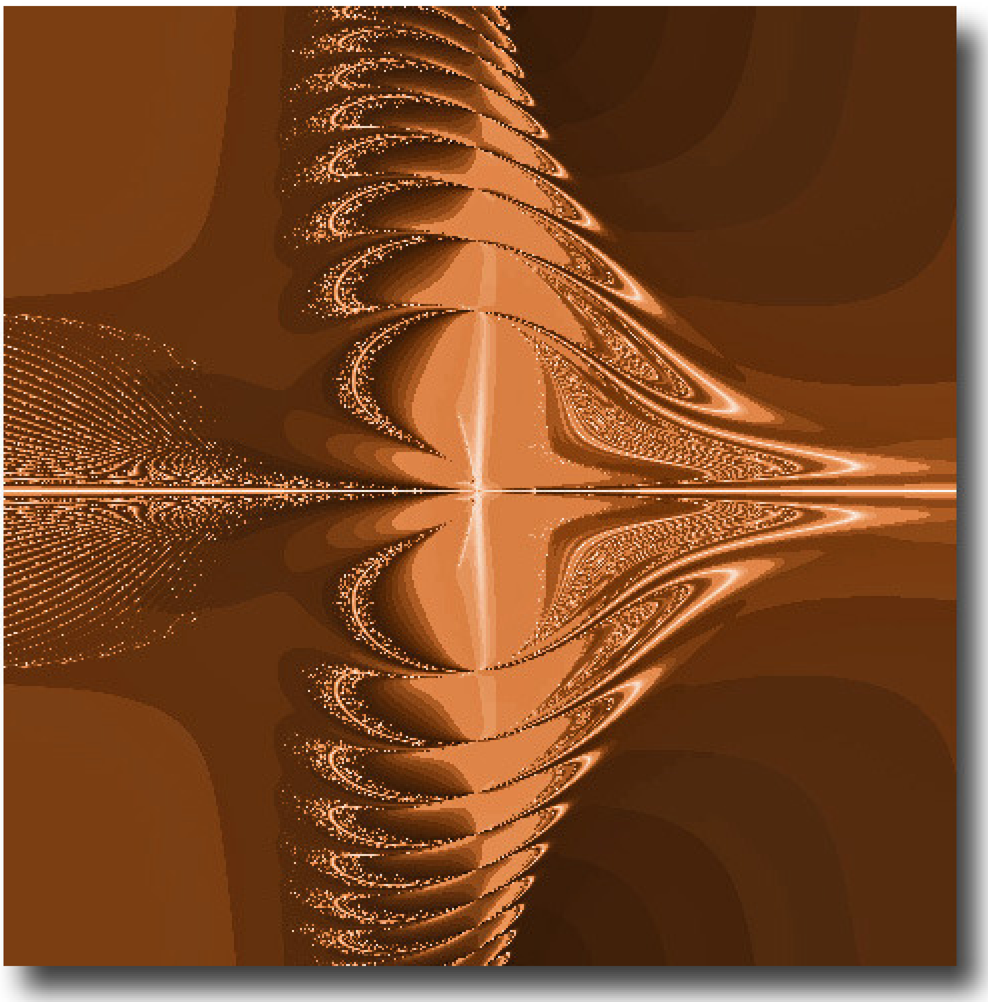
S[-.1,.1;-.15,.2], n=30 and Detailed portion: S[-.01,.01;-.015,.015]

Quantum Bugs

Example 3:

$$\frac{dz}{dt}=\varphi_1(\zeta,t)=\zeta+\frac{1}{\zeta}\ ,\ \frac{d\zeta}{dt}=\varphi_2(z,t)=(x\,/\,y)Sin(y)+i(y\,/\,x)Cos(x)\ \Rightarrow \lambda(\zeta_0)=\int_0^1 \psi_1(\zeta_0,t)dt$$

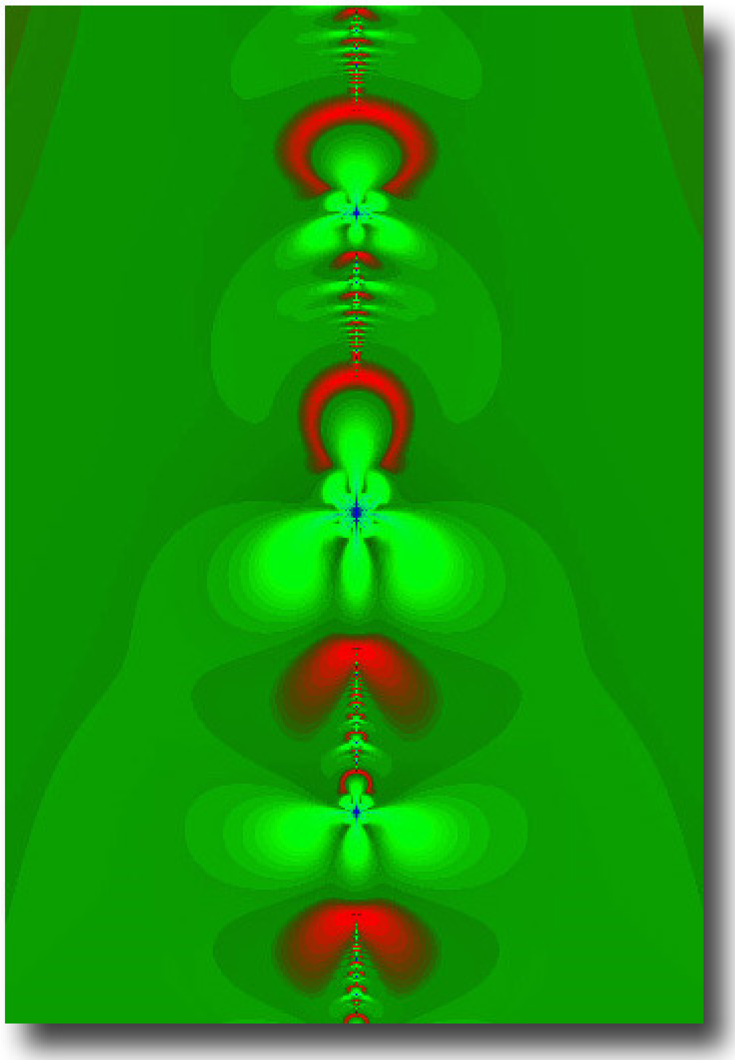
S[-.5,.5;-.5,.5]   n=30



Alpha Centauri Ship at Warp Speed

Example 4: $\frac{dz}{dt} = \varphi_1(\zeta,t) = \zeta + \frac{1}{\zeta}$, $\frac{d\zeta}{dt} = \varphi_2(z,t) = xe^{Sin(y)} + iye^{Cos(x)} \Rightarrow \lambda(\zeta_0) = \int_0^1 \psi_1(\zeta_0,t)dt$

S[-.045,.045;-.065,.065]   n=50

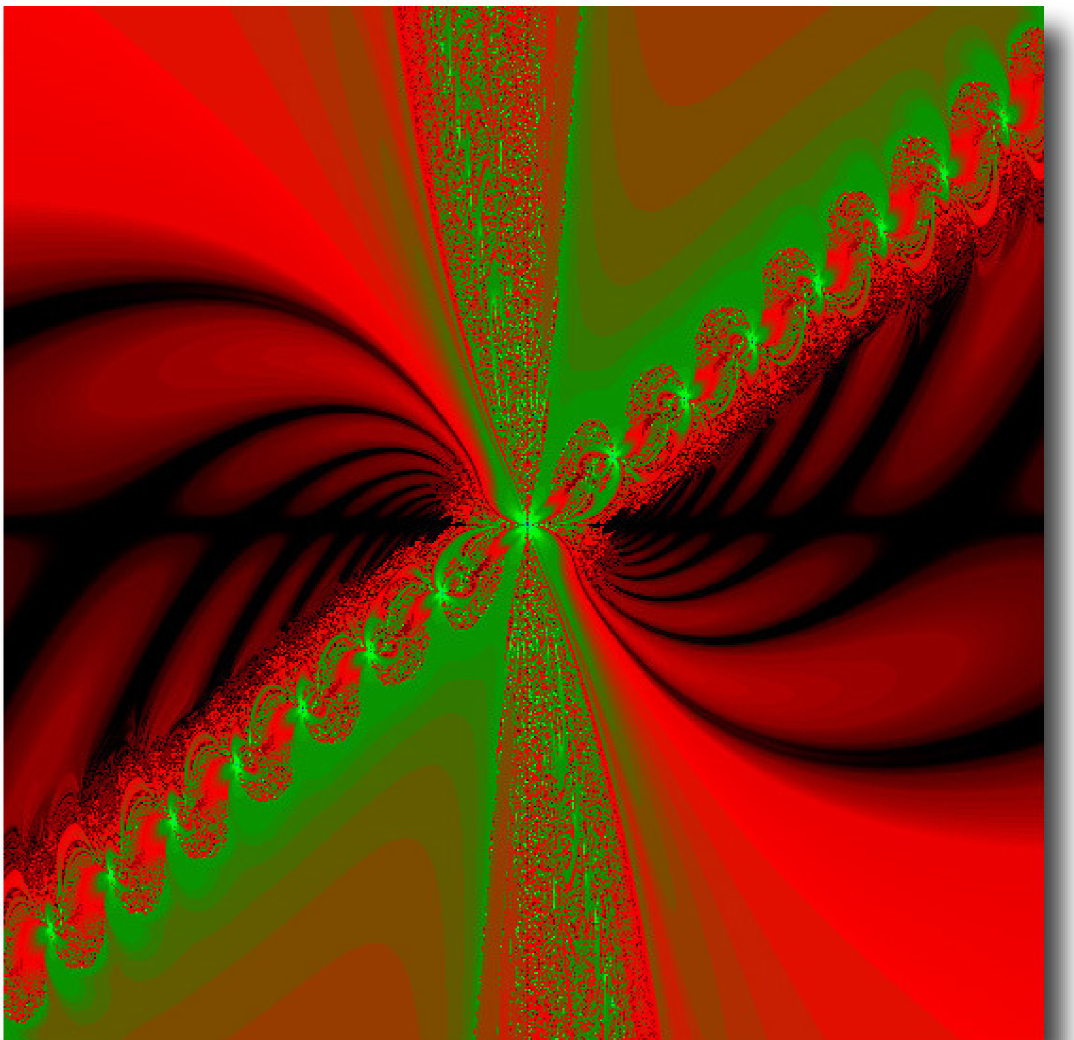


Angel Ladder

Example 5:

$$\frac{dz}{dt}=\varphi_1(\zeta,t)=\zeta+\frac{1}{\zeta}\ ,\ \frac{d\zeta}{dt}=\varphi_2(z,t)=xSin(y/x)+iyCos(y/x)\Rightarrow\lambda(\zeta_0)=\int_0^1\psi_1(\zeta_0,t)dt$$

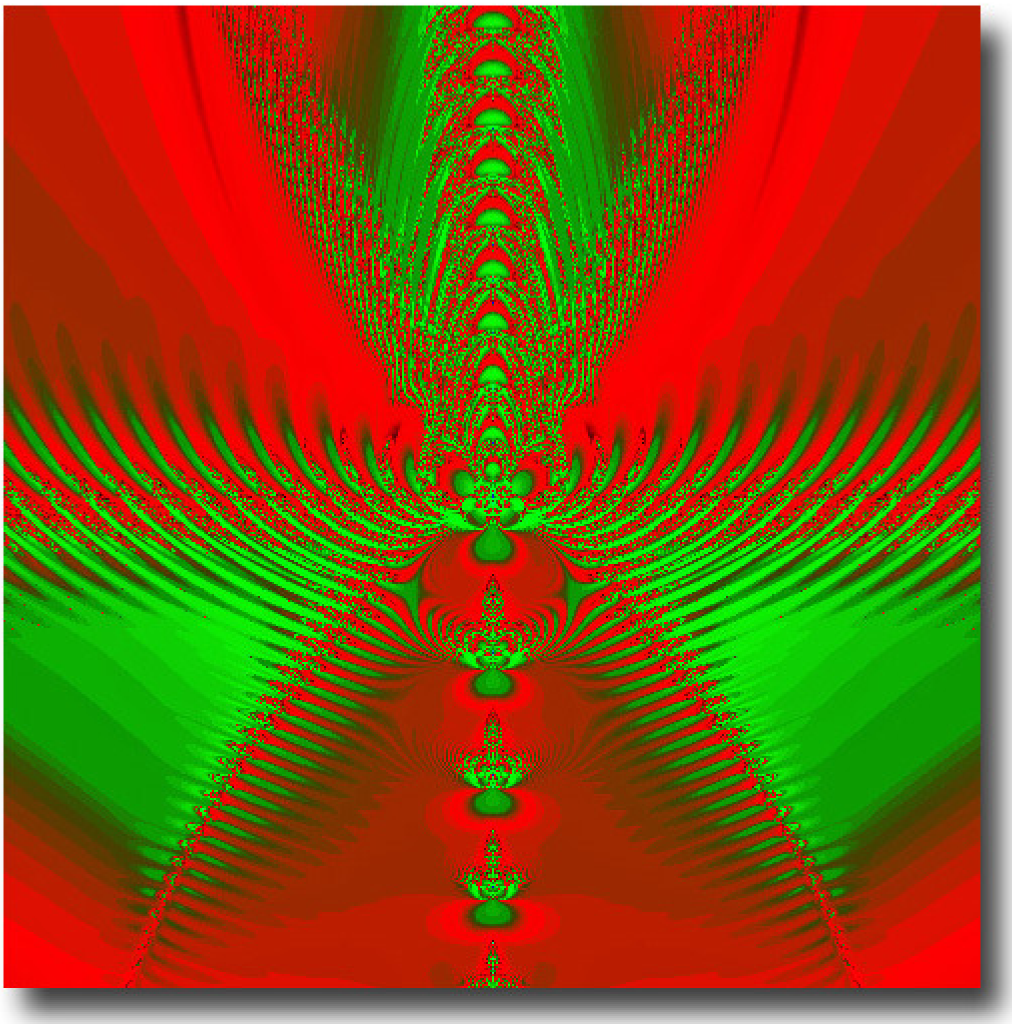
S[-.1,.1;-.1,.1]   n=50



Curtain Call

Example 6: $\dfrac{d\zeta}{dt} = \varphi_2(z,t) = x(Sin(y) + Cos(x)) + iy(Sin(y) + Cos(x))$, $\dfrac{dz}{dt} = \varphi_1(\zeta,t) = \zeta + \dfrac{1}{\zeta}$

$\Rightarrow \lambda(\zeta_0) = \int_0^1 \psi_1(\zeta_0,t)dt$ S[-.1,.1;-.1,.1] n=50



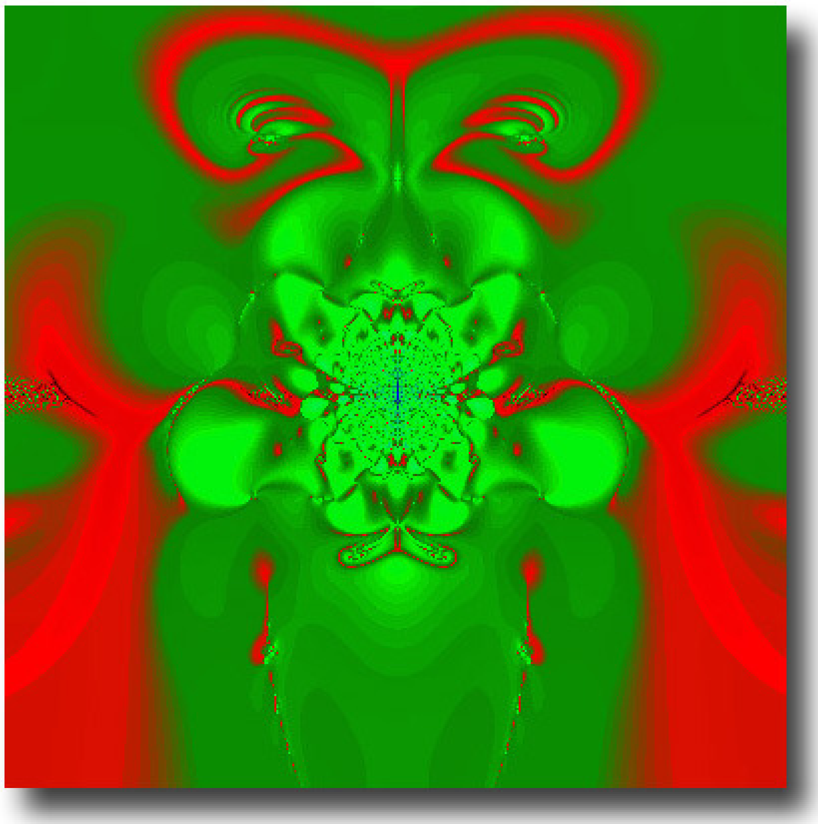
Treasure of the Czars

Detail: First image below center

Holy Relic of the Czars (tiny Christian cross in the center)

Example 7: $\frac{d\zeta}{dt} = \varphi_2(z,t) = x(Sin(y)+Cos(x)) + iy(Sin(y)-Cos(x))$, $\frac{dz}{dt} = \varphi_1(\zeta,t) = \zeta + \frac{1}{\zeta}$

$\Rightarrow \lambda(\zeta_0)=\int_0^1 \psi_2(\zeta_0,t)dt$, S[-.01,.01;-.01,.01] n=50



Azael the Demon

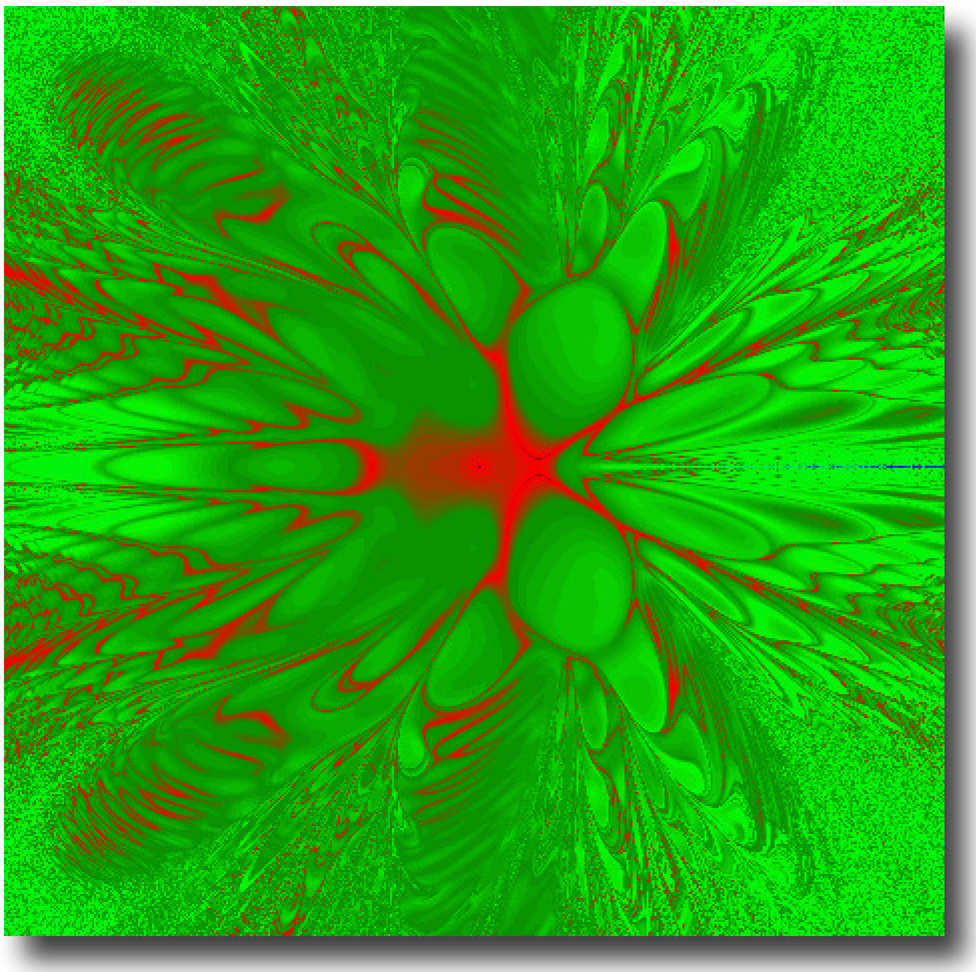
Example 8: $\dfrac{d\zeta}{dt}=\varphi_2(z,t)=x(Sin(x)+Cos(y))+iy(Sin(x)-Cos(y))$, $\dfrac{dz}{dt}=\varphi_1(\zeta,t)=\zeta+\dfrac{1}{\zeta}$

$\Rightarrow\lambda(\zeta_0)=\int_0^1\psi_2(\zeta_0,t)dt$ S[-.01,.01;-.01,.01] n=50

Beelzebub the Demon

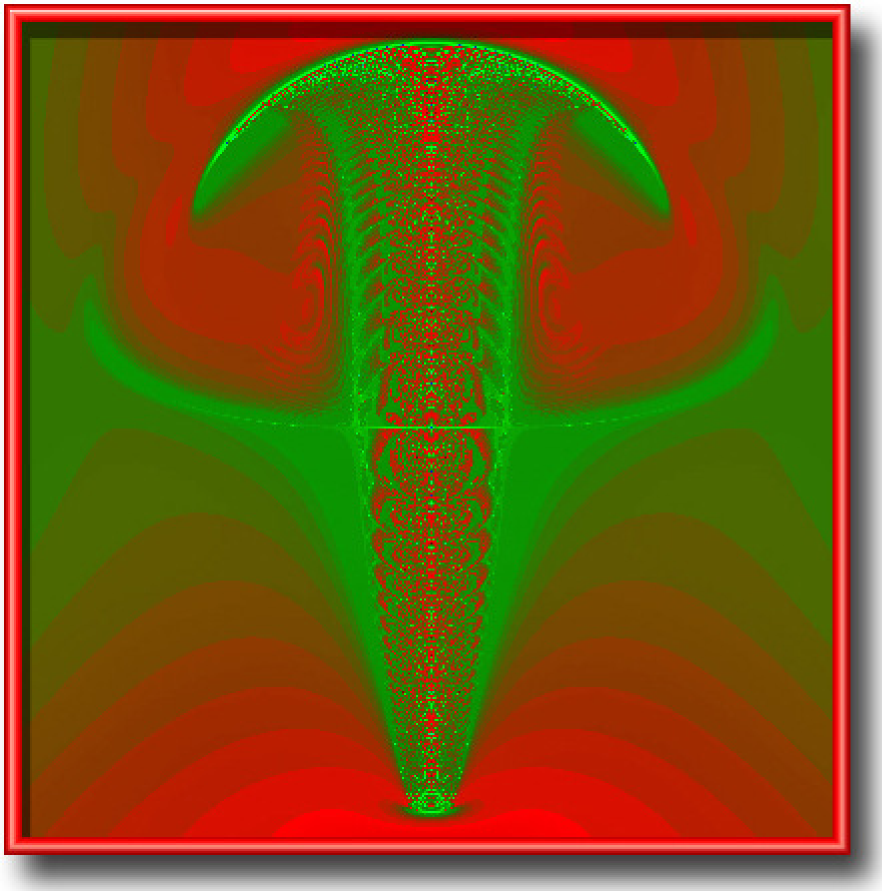
Example 9: $\dfrac{d\zeta}{dt} = \varphi_2(z,t) = x(Cos(y) - Sin(x)) + iy(Cos(y) - Sin(x))$, $\dfrac{dz}{dt} = \varphi_1(\zeta,t) = \zeta^2$

-8<x,y<8 n=50 $\Rightarrow \lambda(z_0) = \int_0^1 \psi_1(z_0,t)dt$:



Fairy Land

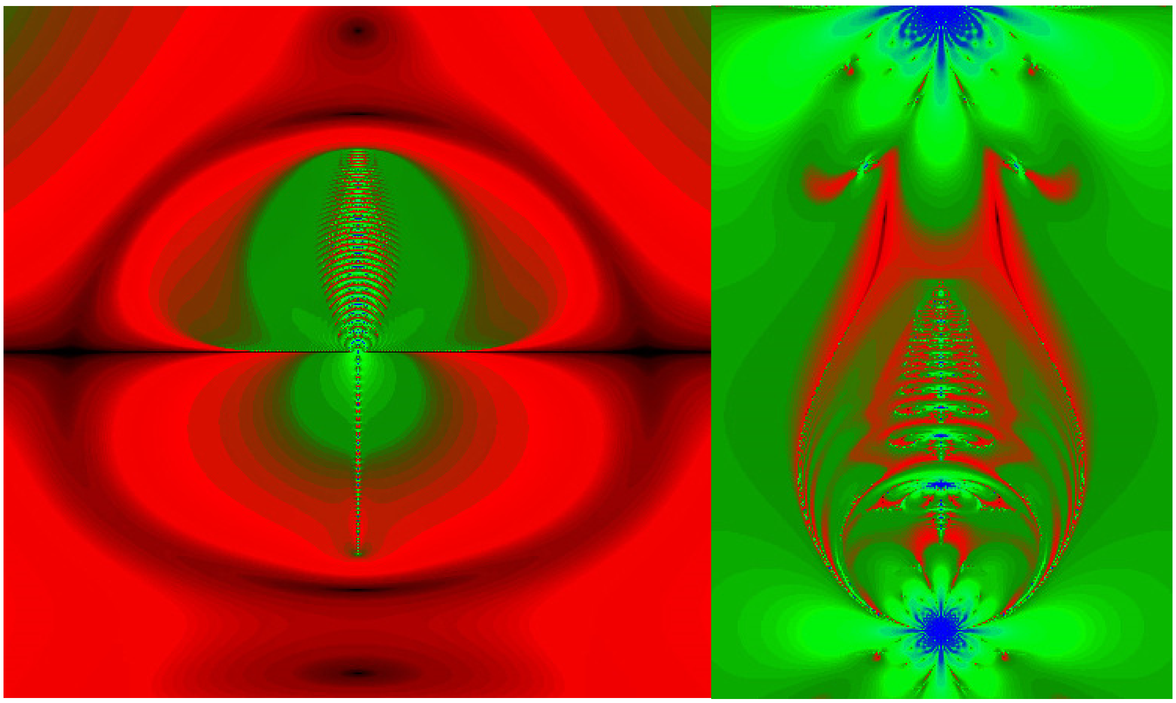
Example 10: $\dfrac{dz}{dt} = \varphi_1(\zeta, t) = \zeta + \dfrac{1}{\zeta}$, $\dfrac{d\zeta}{dt} = \varphi_2(z, t) = xSin(y) + iyCos(x)$. (A variation)



Plague Doctor

Example 11: $\frac{dz}{dt} = \varphi_z = \zeta + \frac{1}{\zeta}$, $\frac{d\zeta}{dt} = \varphi_\zeta = xSin(y) + iyCos(x)$, $\frac{d\omega}{dt} = \varphi_\omega = \varphi_z \varphi_\zeta$,

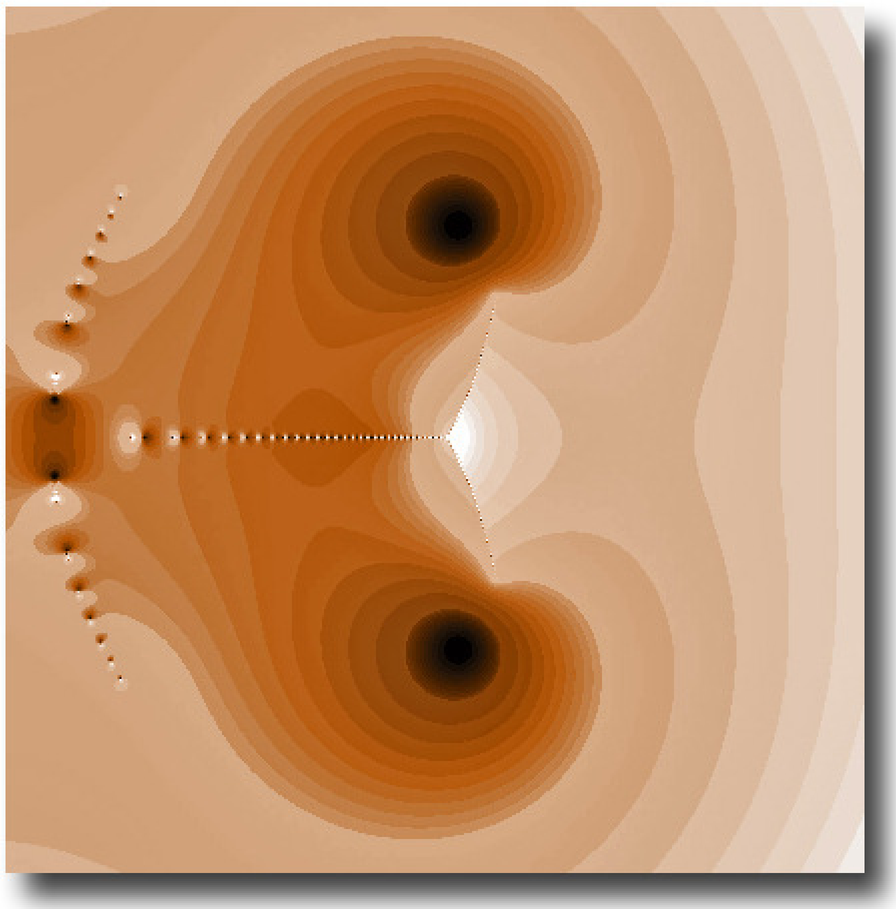
S[-.8,.8;-.8,.8] and S[-.01,.01;-.03,0] n=50 $\lambda(\alpha) = \int_0^1 \psi_\omega(\alpha, t)dt$:



Quantum Tunnel

Example 12: $\dfrac{dz}{dt}=\varphi_1(\zeta)=\dfrac{1}{\zeta}$, $\dfrac{d\zeta}{dt}=\varphi_2(z)=\dfrac{z^2}{2}$, $\dfrac{d\omega}{dt}=\varphi_\omega(z,\zeta)=\dfrac{\varphi_1+\varphi_2}{2}$,

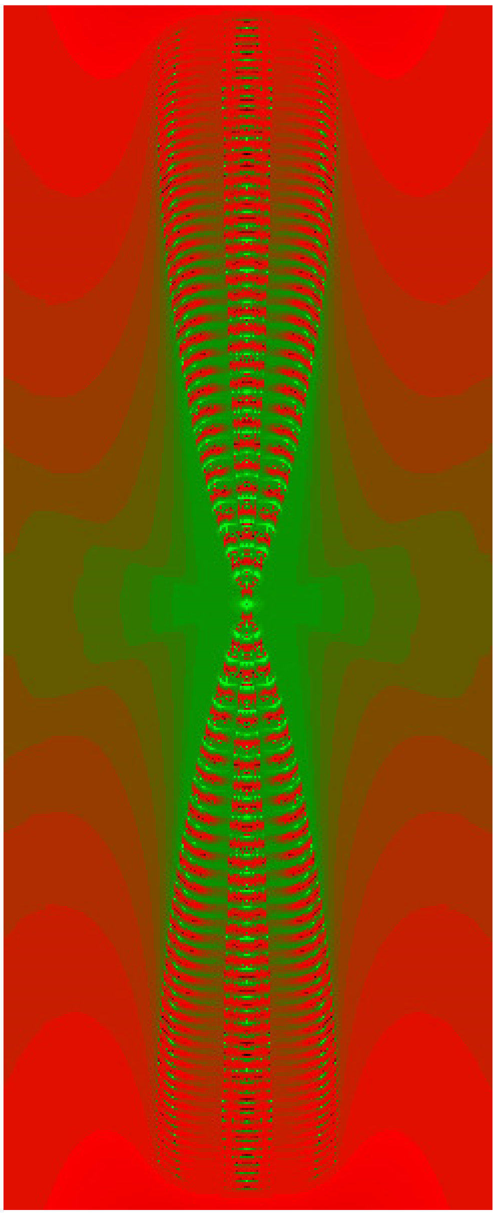
$\lambda(\alpha)=\int_0^1 \psi_\omega(\alpha,t)dt$: S[-1.5,1.5;-1.5,1.5] n=50

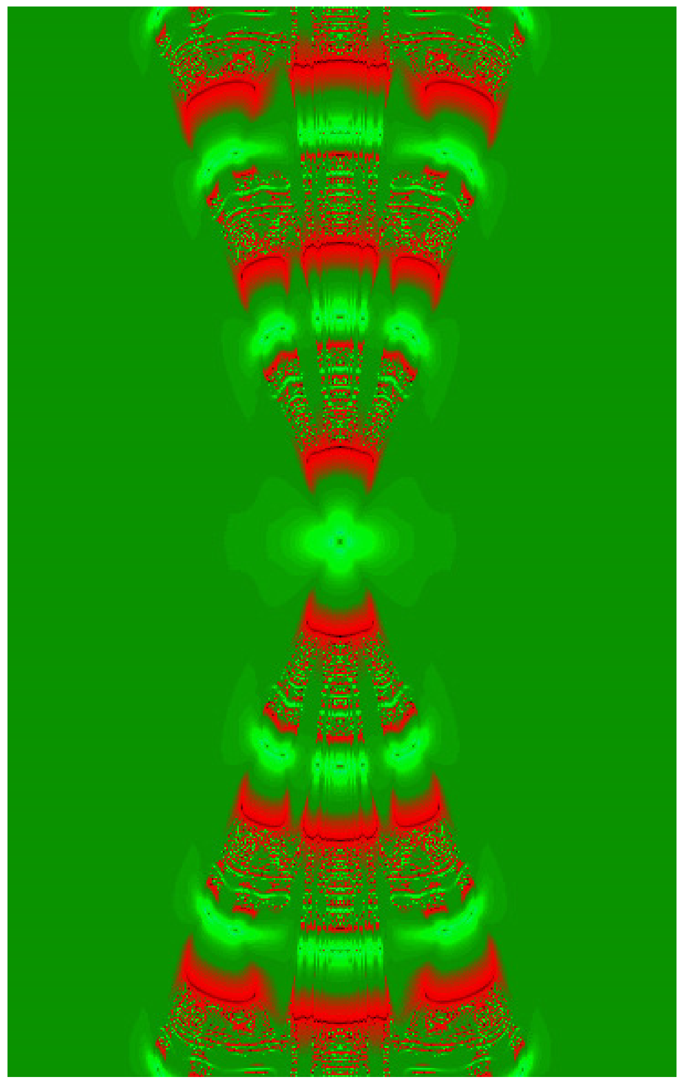


Gravity's Slingshot

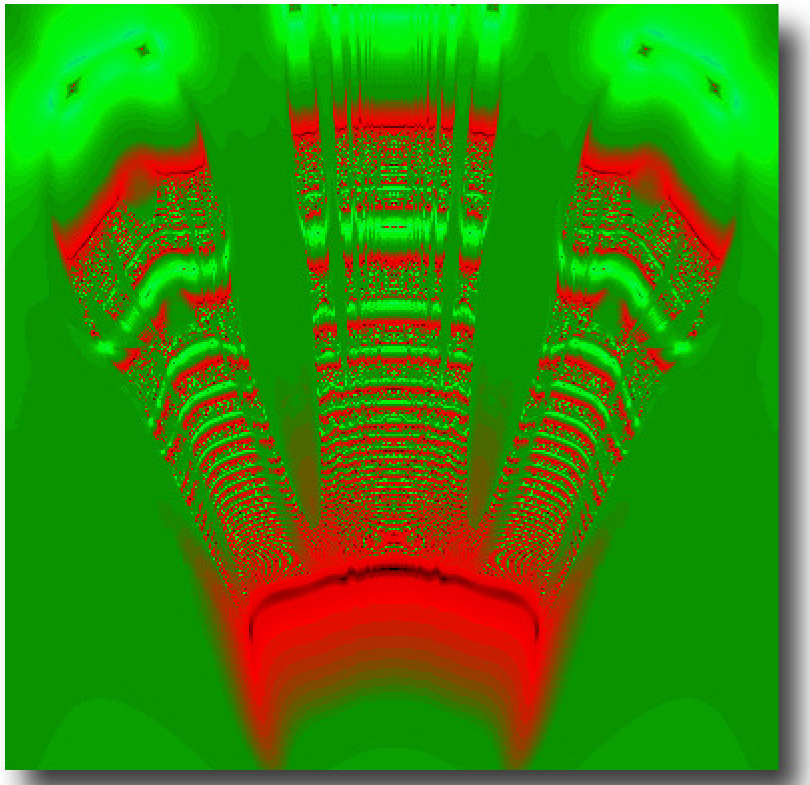
Example 13: $\dfrac{dz}{dt}=\varphi_z=\zeta+\dfrac{1}{\zeta}$, $\dfrac{d\zeta}{dt}=\varphi_\zeta=xCos(y/x)+iySin(x/y)$, $\dfrac{d\omega}{dt}=\varphi_\omega=\frac{1}{2}(\varphi_z+\varphi_\zeta)$

S[-1.5,1.5;-2.5,2.5] n=50 Plus details $\lambda(\alpha)=\int_0^1 \psi_\omega(\alpha,t)dt$:





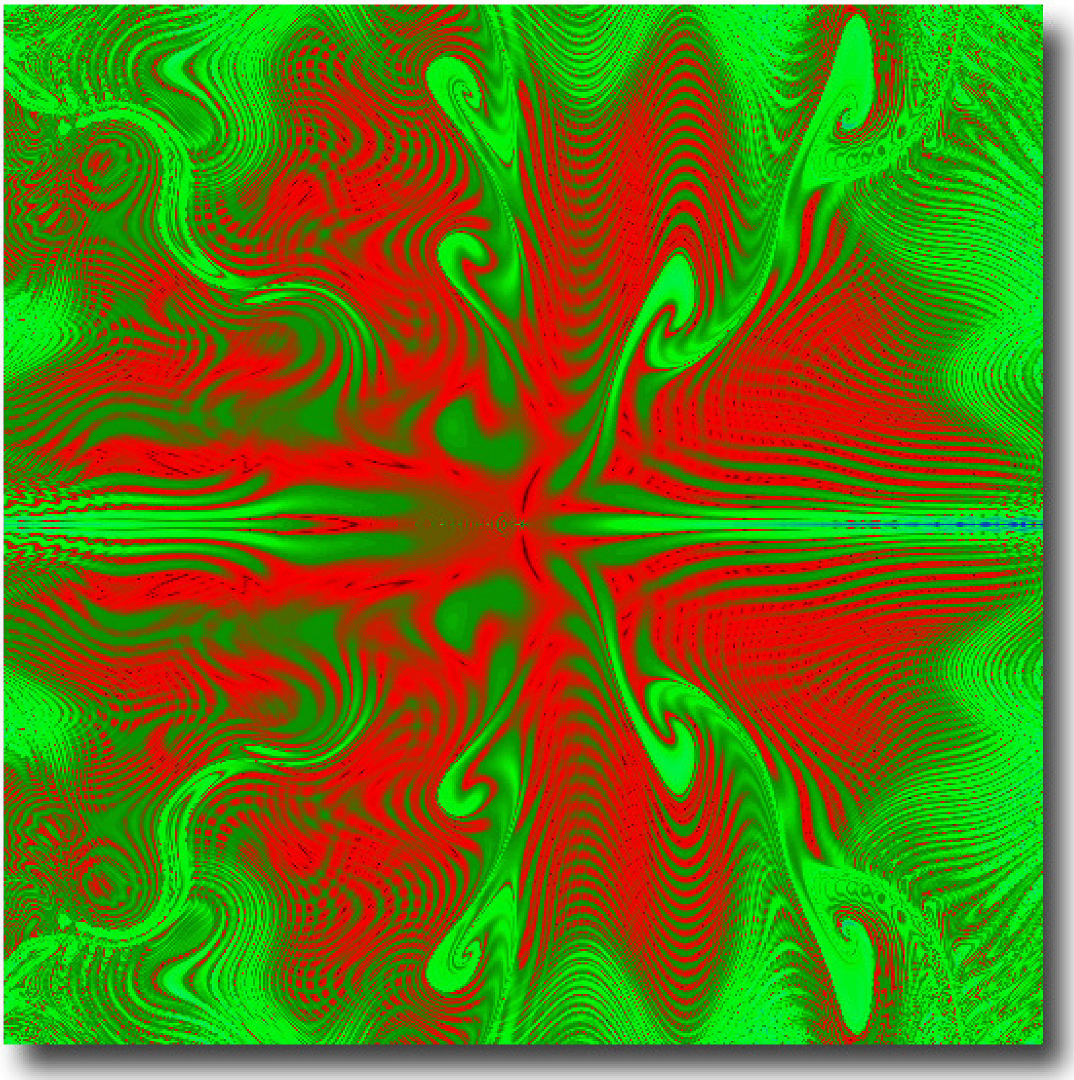
Judgment Day



Heaven Bound

Example 14: $\frac{dz_2}{dt} = \varphi_2 = x_1 Cos(y_1) + iy_1 Sin(x_1),\ \frac{dz_3}{dt} = \varphi_3 = z_2^2,\ \frac{dz_1}{dt} = \varphi_1 = z_3 + \frac{1}{z_3}$

S[-18,18;-18,18] n=50 $\lambda(\alpha) = \int_0^1 \psi_2(\alpha,t)dt$:

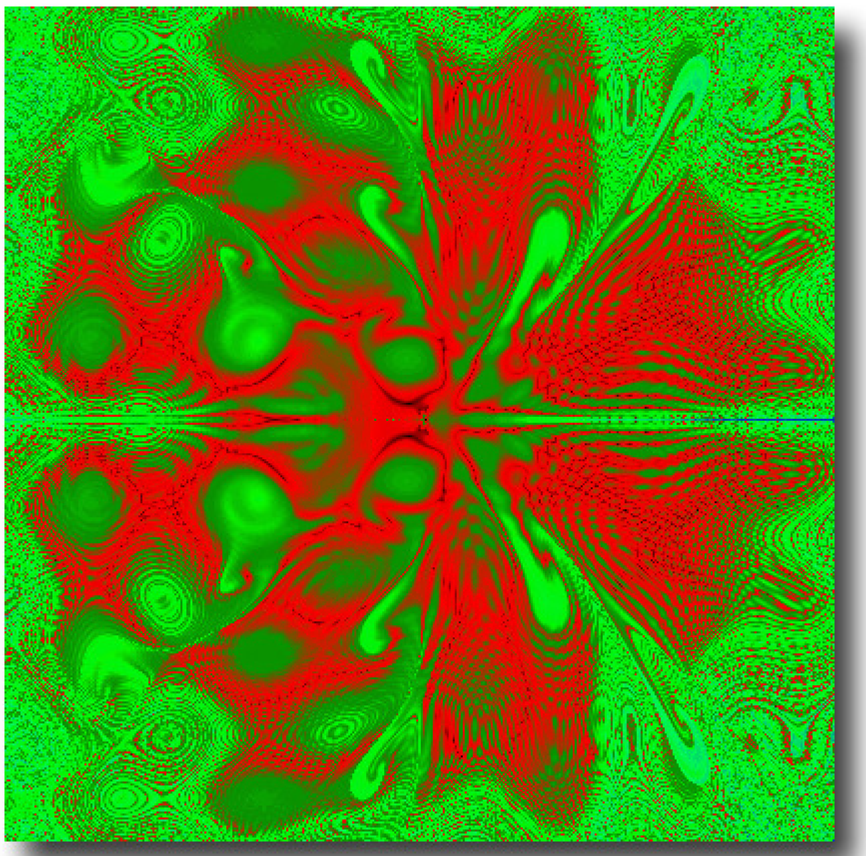


Planck's Mind

Example 15:

$$\frac{dz_2}{dt} = \varphi_2 = x_1(Cos(y_1) + Sin(x_1)) + iy_1(Cos(y_1) + Sin(x_1)),\ \frac{dz_3}{dt} = \varphi_3 = z_2^2,\ \frac{dz_1}{dt} = \varphi_1 = z_3 + \frac{1}{z_3}$$

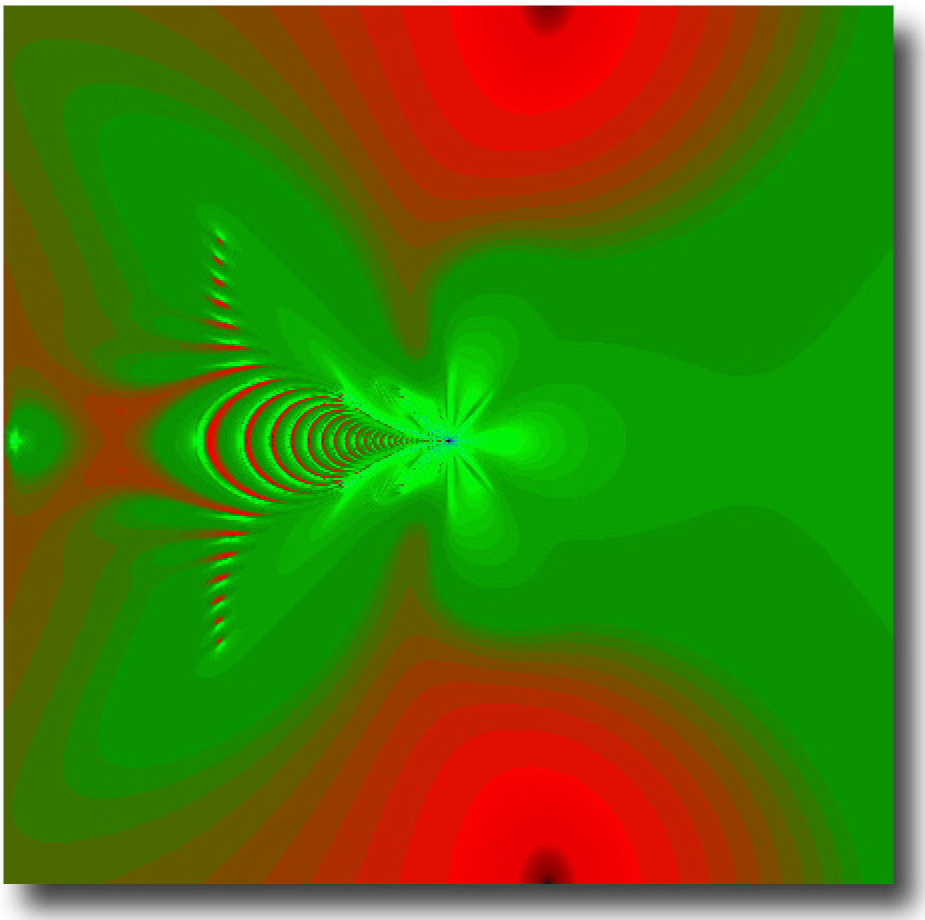
S[-18,18;-18,18] n=50 $\lambda(\alpha) = \int_0^1 \psi_2(\alpha, t)dt$:



Dirac's Dilemma

Example 16: $\dfrac{dz_2}{dt} = \varphi_2 = x_1 Cos(y_1) + iy_1 Sin(x_1)$, $\dfrac{dz_3}{dt} = \varphi_3 = z_2^2$, $\dfrac{dz_1}{dt} = \varphi_1 = z_3 + \dfrac{1}{z_3}$

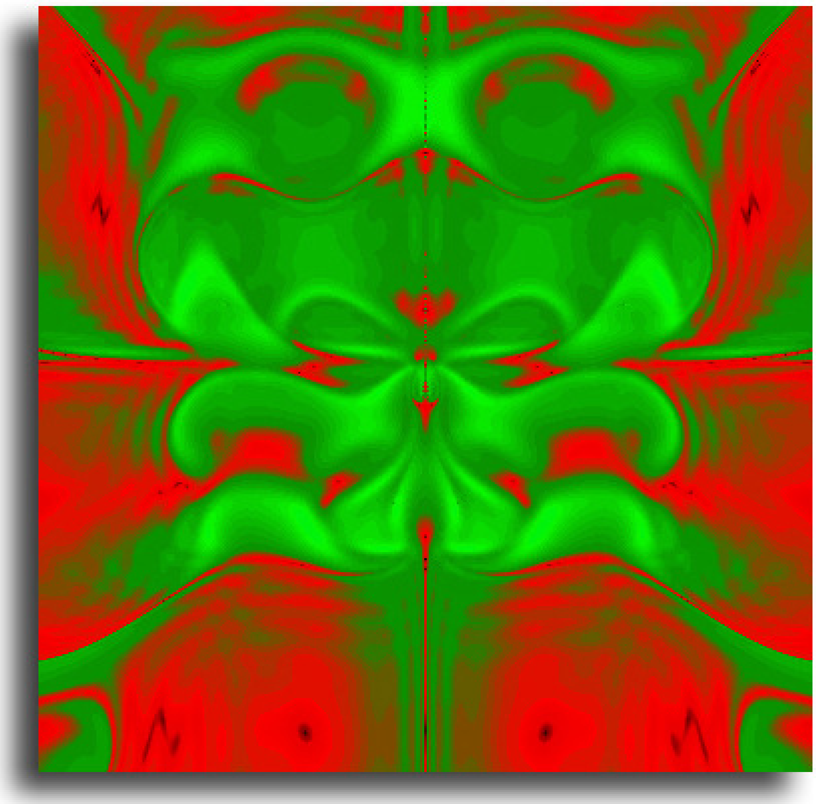
$\varphi = \frac{1}{3}(\varphi_1 + \varphi_2 + \varphi_3)$, S[-.75,.75;-.75,.75] n=50 $\lambda(\alpha) = \int_0^1 \psi(\alpha, t)dt$:



Cosmic Sting

Example 17: $\frac{dz_2}{dt} = \varphi_2 = 10Cos(x_1) + i10Sin(y_1)$, $\frac{dz_3}{dt} = \varphi_3 = z_2^2$, $\frac{dz_1}{dt} = \varphi_1 = z_3 + \frac{1}{z_3}$

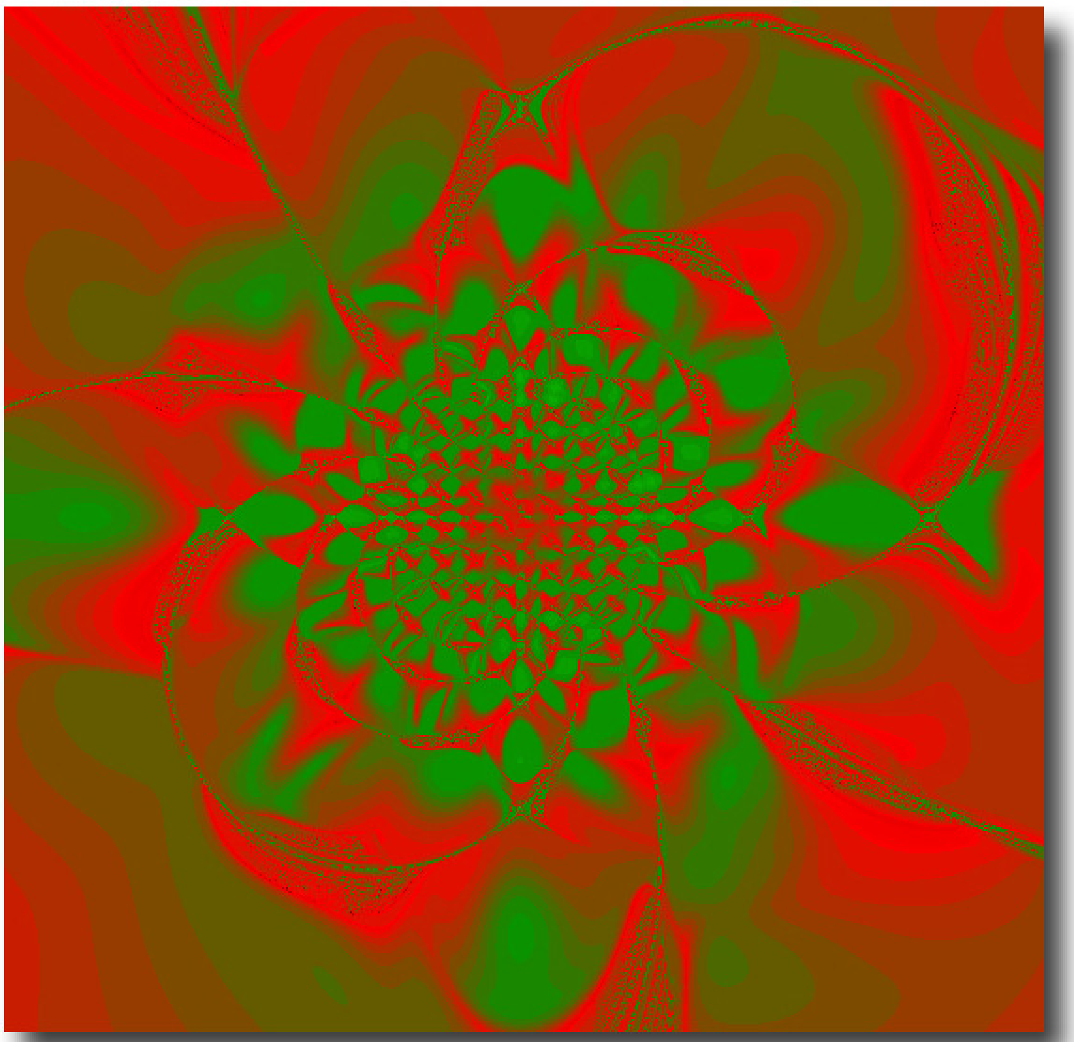
S[-8,8;-8,8]   n=50   (Rotated) $\lambda(\alpha) = \int_0^1 \psi_2(\alpha,t)dt$:



Mask of Zeno

Example 18: $\dfrac{d\zeta}{dt} = \varphi_1 = x(Cos(y) - Sin(x)) + iy(Cos(x) - Sin(y)),\ \dfrac{dz}{dt} = \varphi_2 = \zeta + \dfrac{1}{\zeta}$

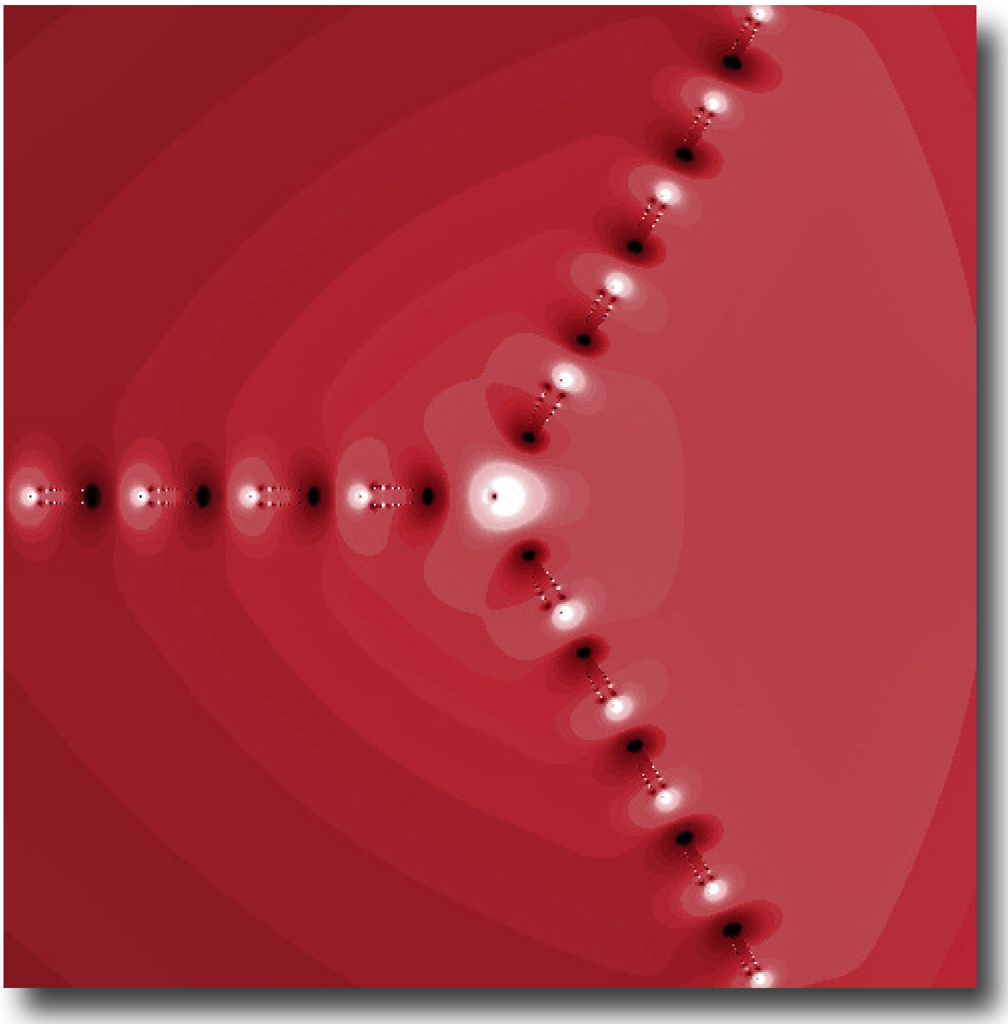
S[-.004,.004;-.004,.004] n=50 70,000X $\lambda(\alpha) = \int_0^1 \psi_1(\alpha,t)dt$



Whirlpool in the Styx

Example 19: $\dfrac{dz_2}{dt} = \varphi_2 = \frac{1}{3} z_1^2$, $\dfrac{dz_1}{dt} = z_2 + \dfrac{1}{z_2} = \varphi_1$, S[-.05,.05;-.05,.05], n=50

$\lambda(\alpha) = \int_0^1 \psi_2(\alpha, t)\, dt$:



Electrodynamics

Appendix (a <u>sketch</u> of a proof – from *Picard/Lindelof theorem*):

Given $\frac{dz}{dt} = \varphi_1(\zeta, t)$ and $\frac{d\zeta}{dt} = \varphi_2(z, t)$ let Φ be the space of vectors $\begin{pmatrix} z(t) \\ \zeta(t) \end{pmatrix}$ with

$\|z(t)\| = \sup_{t \in I} |z(t)|$, $I = [0,1]$, and $\left\| \begin{pmatrix} z(t) \\ \zeta(t) \end{pmatrix} \right\| = \|z(t)\| + \|\zeta(t)\|$.

Express

$$z(t) = z_0 + \int_0^t \varphi_1(\zeta(s), s)dt \quad \text{and} \quad \zeta(t) = \zeta_0 + \int_0^t \varphi_2(z(s), s)dt .$$

Define functionals

$$\mathbb{F}_1\zeta = z_0 + \int_0^t \varphi_1(\zeta(s), s)dt \quad \text{and} \quad \mathbb{F}_2 z = \zeta_0 + \int_0^t \varphi_2(z(s), s)dt$$

And $\mathbb{F}\begin{pmatrix} z(t) \\ \zeta(t) \end{pmatrix} = \begin{pmatrix} \mathbb{F}_1\zeta \\ \mathbb{F}_2 z \end{pmatrix} \in \Phi$.

Then, for $\mathrm{X}_k = \begin{pmatrix} z_k(t) \\ \zeta_k(t) \end{pmatrix}$ show that

$$\|\mathbb{F}(\mathrm{X}_1) - \mathbb{F}(\mathrm{X}_2)\| < \rho \|\mathrm{X}_1 - \mathrm{X}_2\|, \ \rho < 1$$

Assume uniform Lipshitz conditions

$$|\varphi_1(\zeta_1, t) - \varphi_1(\zeta_2, t)| < k_1|\zeta_1 - \zeta_2| \quad \text{and} \quad |\varphi_2(z_1, t) - \varphi_2(z_2, t)| < k_2|z_1 - z_2|$$

It is easily seen that for small t

$$\|\mathbb{F}(\mathrm{X}_1) - \mathbb{F}(\mathrm{X}_2)\| < \cdots < tk_1|\zeta_1 - \zeta_2| + tk_2|z_1 - z_2| < \rho\left(|\zeta_1 - \zeta_2| + |z_1 - z_2|\right) \le \rho\|\mathrm{X}_1 - \mathrm{X}_2\| .$$

Thus the iterative sequence $\mathbb{F}^{(n)}(\mathrm{X}_0) \to \mathrm{X}$ converges locally (Banach Fixed Point theorem).