An Elementary Note: Operations on Contours in the Complex Plane

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I. Algebra of Contours (See [1])

Zeno (or equivalent parametric) contours are defined algorithmically as a distribution of points $\lim_{n\to\infty}\{z_{k,n}\}_{k=0}^{n}$ by the iterative procedure: $z_{k,n}=z_{k-1,n}+\eta_{k,n}\cdot\varphi\left(z_{k-1,n},\frac{k}{n}\right)$ which arises from the following composition structure:

$$G(z)=\lim_{n\to\infty}G_{n,n}(z)\ ,\ G_{k,n}(z)=g_{k,n}\left(G_{k-1,n}(z)\right)\ ,\ g_{k,n}(z)=z+\eta_{k,n}\cdot\varphi(z,\tfrac{k}{n})\ ,\ G_{1,n}(z)=g_{1,n}(z)\ .$$

Usually $\eta_{k,n}=\frac{1}{n}$, providing a partition of the *unit time interval*. $\gamma(z)$ is the continuous arc from z to $G(z)$ that results as $n\to\infty$ (*Euler's Method* is a special case of a Zeno contour).

When $\varphi(z,t)$ is well-behaved an equivalent closed form of the contours , $z(t)$, has the property $\frac{dz}{dt}=\varphi(z,t)$, with vector field $f(z,t)=\varphi(z,t)+z$. *Siamese contours* are streamlines or pathlines joined at their origin and arising from different vector fields. It will be assumed in most of what follows that all contours in a particular discussion have identical initial points.

Contours will be abbreviated, using the iterative algorithm, as

$$\gamma: z_{k,n}=z_{k-1,n}+\eta_{k,n}\varphi(z_{k-1,n},\tfrac{k}{n})\ =\ z_{k-1,n}+\eta_{k,n}\left(f(z_{k-1,n},\tfrac{k}{n})-z_{k-1,n}\right)$$

Or
$$\gamma:\ \frac{dz}{dt}=\varphi(z,t)$$

A parametric form of $\gamma(z):\ z=z(t)$, exists when the equation $\frac{dz}{dt}=\varphi(z,t)$ admits a closed solution. For example $\gamma: z(t)=z_0e^{t+2it^2}\ \Leftrightarrow\ \varphi(z,t)=z(1+4it)$.

A ***Contour Sum*** $\gamma=\gamma_1\overset{\leftrightarrow}{\oplus}\gamma_2$ is defined as follows: $\gamma:\ \frac{dz}{dt}=\varphi(z,t)=\varphi_1(z,t)+\varphi_2(z,t)$

For $\gamma_1:\ \frac{dz}{dt}=\varphi_1(z,t)$ and $\gamma_2:\ \frac{dz}{dt}=\varphi_2(z,t)$

And a ***Scalar Product***:

$$\gamma = \alpha \odot \gamma_1 : \ \frac{dz}{dt} = \alpha \cdot \varphi_1(z,t),$$

Combined to show a distributive feature:

$$\gamma = \alpha \odot (\gamma_1 \overset{\leftrightarrow}{\oplus} \gamma_2) = (\alpha \odot \gamma_1) \overset{\leftrightarrow}{\oplus} (\alpha \odot \gamma_2) \ : \frac{dz}{dt} = \alpha(\varphi_1(z,t) + \varphi_2(z,t)) = \alpha\varphi_1(z,t) + \alpha\varphi_2(z,t)$$

A ***Contour Product*** is defined: $\gamma = \gamma_1 \overset{\leftrightarrow}{\otimes} \gamma_2 : \frac{dz}{dt} = \varphi_1(z,t) \cdot \varphi_2(z,t)$, from which one derives

$$\gamma = (\alpha \odot \gamma_1) \overset{\leftrightarrow}{\otimes} \gamma_2 = (\alpha \odot \gamma_2) \overset{\leftrightarrow}{\otimes} \gamma_1$$

Define ***Contour Composition:*** $\gamma = \gamma_1 \circ \gamma_2 : \quad z_{k+1,n} = z_{k,n} + \eta_{k,n} \cdot \varphi_1(\varphi_2(z,t),t)$ or $\frac{dz}{dt} = \varphi_1 \circ \varphi_2$

A *norm* $\|\gamma\|$ of a contour in a z_0- based space can be formulated as $\|\gamma\| = \underset{t \in [0,1]}{Sup} |\varphi(z_0,t)|$,

giving rise to a *metric*: $\quad d(\gamma_1,\gamma_2) = \|\gamma_1 - \gamma_2\| = \left\|\gamma_1 \overset{\leftrightarrow}{\oplus} (-1) \cdot \gamma_2\right\|$.

II. Operations on Contours

Set $\gamma_I : \frac{dz}{dt} = 1$, the *identity contour* $z(t) = z_0 + t$.

Define a linear operator

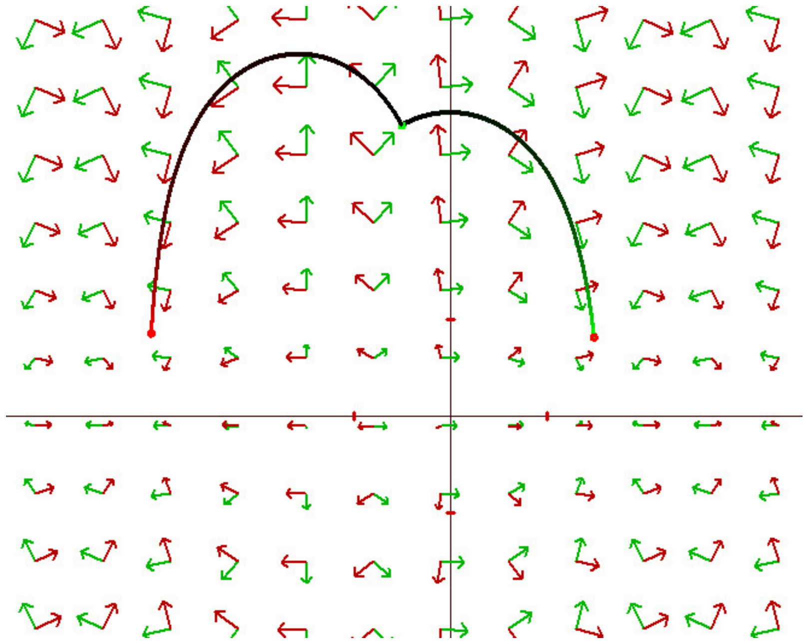
$$T(\gamma) = \alpha \cdot \gamma: \quad \frac{dz}{dt} = \alpha \cdot \varphi(z,t) \qquad \text{so that} \qquad T(\gamma_1 \overset{\leftrightarrow}{\oplus} \gamma_2) = T(\gamma_1) \overset{\leftrightarrow}{\oplus} T(\gamma_2)$$

If we assume $\varphi(z)$ is analytic the following are linear operators as well:

$$\int \gamma: \quad \frac{dz}{dt} = \int \varphi(z) \quad \text{(in the sense of antiderivative)}$$

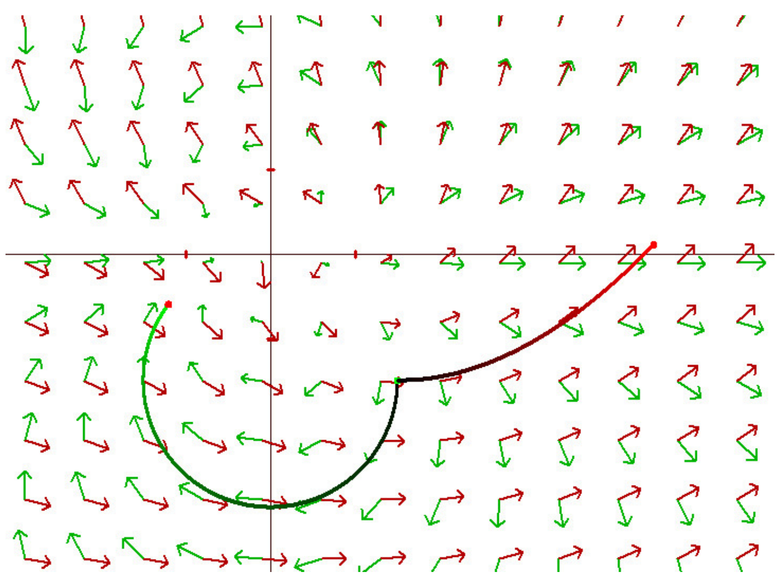
and $\qquad D\gamma: \quad \frac{dz}{dt} = D_z \varphi(z)$

Example 1 : γ: $\dfrac{dz}{dt} = Cos(z) \Rightarrow \int\gamma$: $\dfrac{dz}{dt} = Sin(z)$. $z = -.5 + 3i$, γ green, $\int\gamma$ red.



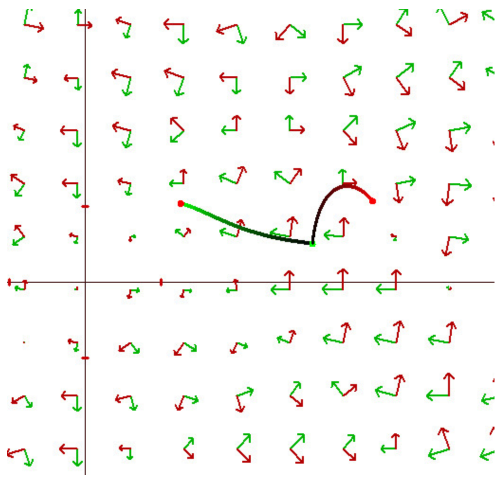
Contour exponentiation: γ: $\dfrac{dz}{dt} = \varphi(z) \Rightarrow \gamma^{\alpha}$: $\dfrac{dz}{dt} = \varphi(z)^{\alpha}$

Example 2 : γ: $\dfrac{dz}{dt} = z^2 \Rightarrow \gamma^{1+i}$: $\dfrac{dz}{dt} = z^{2(1+i)}$, $z_0 = 1.5 - 1.5i$



Orthogonality of contours: $\gamma: \dfrac{dz}{dt} = \varphi(z)$, γ^*: $\dfrac{dz}{dt} = -i\varphi(z) \Rightarrow \gamma \perp \gamma^*$

Example 3 : $\gamma : \dfrac{dz}{dt} = \varphi = xCos(x+y) + iySin(x-y)$



Consider a contour γ: $z_{k,n} = z_{k-1,n} + \mu_{k,n}\varphi_1(z_{k-1,n})$, and a second function $\varphi_2(z)$.

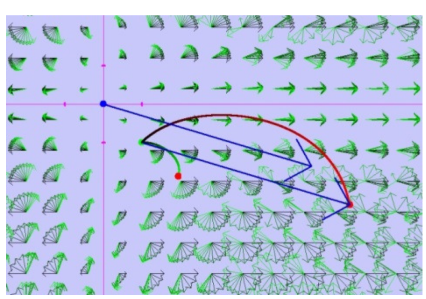
Define an *Integration Operator* $\gamma\varphi_2$: $\zeta_{k,n} = \zeta_{k-1,n} + \mu_{k,n}\varphi_2(z_{k-1,n})\varphi_1(z_{k-1,n})$. Then

$$\zeta_{n,n} - \zeta_{0,n} = \sum_{k=1}^{n}(\zeta_{k,n} - \zeta_{k-1,n}) = \sum_{k=1}^{n}\varphi_2(z_{k-1,n})(z_{k,n} - z_{k-1,n}) \to \int_{\gamma(z)} \varphi_2(z)dz = \int_0^1 \varphi_2(z(t))\varphi_1(z(t))dt$$

The result: The integral of the second function along the contour described by the first.

This is analogous to $\dfrac{dz}{dt} = \varphi_1(z)$ and $\dfrac{d\zeta}{dt} = \varphi_2(z)\cdot\varphi_1(z)$

A more complete description can be found in [2]. The vector $\overrightarrow{\zeta_{n,n} - \zeta_{0,n}}$ provides a graphical depiction of this integral.



Define a *Composition Operator* $\varphi_2 \circ \gamma$: $\zeta_{k,n} = \zeta_{k-1,n} + \mu_{k,n}\varphi_2 \circ \varphi_1(z_{k-1,n})$.

An argument similar to the previous leads to

$$\zeta_{n,n} - \zeta_{0,n} = \sum_{k=1}^{n} \varphi_2(\Delta z_{k,n} / \mu_{k,n})\, \mu_{k,n} \to \int_0^1 \varphi_2(dz/dt)\, dt = \int_0^1 \varphi_2(\varphi_1(z(t)))dt .$$

Analogous to $\frac{dz}{dt} = \varphi_1(z)$ and $\frac{d\zeta}{dt} = \varphi_2(\varphi_1(z))$

Define a *Contour Mixing Operation*

$$\mathfrak{M}(\gamma_1, \gamma_2):\ z_{k,n} = z_{k-1,n} + \mu_n \varphi_1(\zeta_{k-1,n}) \ \text{ and } \ \zeta_{k,n} = \zeta_{k-1,n} + \mu_n \varphi_2(z_{k-1,n}) \ ,$$

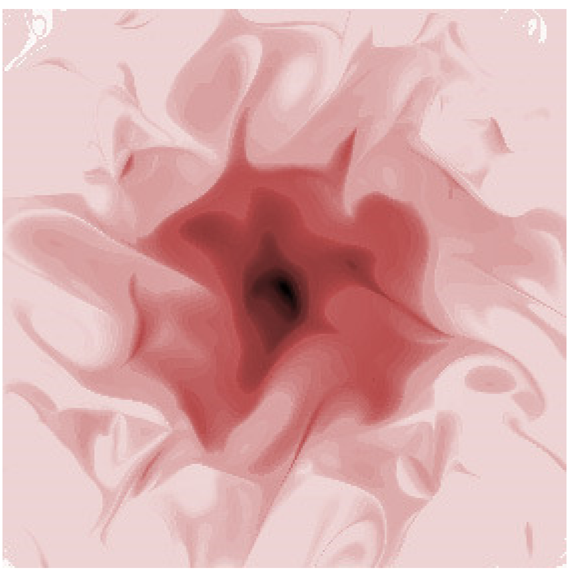
analogous to the system

$$\frac{dz}{dt} = \varphi_1(\zeta) \ \text{ and } \ \frac{d\zeta}{dt} = \varphi_2(z)\ , \quad z = x + iy\ ,\ \zeta = \tau + i\upsilon$$

Example 4 : $\frac{dz}{dt} = \left(\tau Sin(\tau+\upsilon) + \upsilon Cos(\tau-\upsilon)\right) + i\left(\tau Cos(\tau+\upsilon) - \upsilon Sin(\tau-\upsilon)\right)$ and

$\frac{d\zeta}{dt} = xCos(y) + iySin(x)$, $v_{\mathfrak{M}}(\omega) = \int_0^1 z(\omega,t)\zeta(\omega,t)dt$, $\omega = z(0) = \zeta(0)$

Topographical image of the ω-plane:



Further mixing of contours:

$$\gamma_1 : z_{k,n} = z_{k-1,n} + \mu_n \varphi_1(z_{k-1,n}) \quad \text{and} \quad \gamma_2 : \zeta_{k,n} = \zeta_{k-1,n} + \mu_n \varphi_2(z_{k-1,n}) \zeta_{k-1,n}$$

Expanding γ_2 two different ways:

(1) $\zeta_{n,n} = \zeta_0 + \frac{1}{n} \sum_{k=1}^{n} \varphi_2\left(z_{k-1,n}\right) \zeta_{k-1,n} = \zeta_0 + \frac{1}{n} \sum_{k=1}^{n} \varphi_2\left(z(\tfrac{k-1}{n})\right) \zeta(\tfrac{k-1}{n})$

(2) $\zeta_{n,n} = \zeta_0 \prod_{k=1}^{n} \left(1 + \frac{1}{n} \varphi_2\left(z(\tfrac{k-1}{n})\right)\right)$

Using the notation of virtual integral, (1) becomes

$$\zeta_{n,n} \approx \zeta_0 + \int_0^1 \psi_2(z,t) \zeta(t) dt$$

And simple manipulations produce the following form of (2)

$$\zeta_{n,n} \simeq \zeta_0 e^{\int_0^1 \psi_2(z,t) dt}$$

Combined, these two expressions give

$$\lambda(\alpha) \; = \; \int_0^1 \psi_2(\alpha,t) \zeta(t) dt \;\; \simeq \;\; \alpha \left(e^{\int_0^1 \psi_2(\alpha,t) dt} - 1 \right)$$

Observe that the contour along which integrals are evaluated is γ_1.

This is analogous to

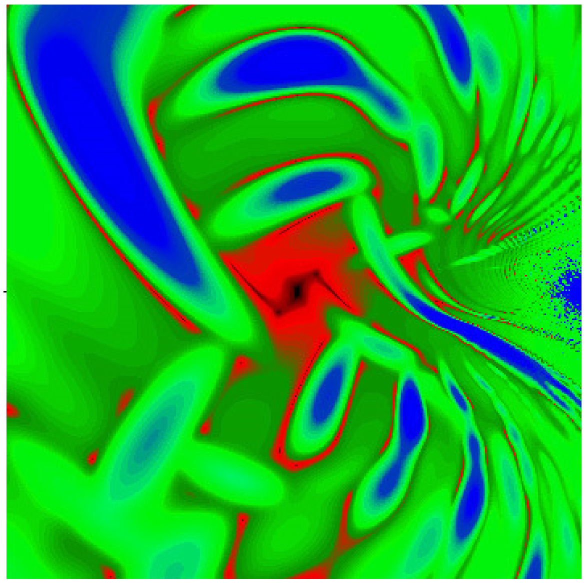
$$\frac{d\zeta(t)}{dt} = \varphi_2(z(t)) \cdot \zeta(t) \, , \; \frac{dz}{dt} = \varphi_1(z) \;\; \Rightarrow \;\; \zeta(t) = \zeta_0 e^{\int_0^t \varphi_2(z(t)) dt}$$

Recall, under ideal conditions: $\psi(\alpha,t) = \varphi(z(t))$, $\alpha = z(0)$.

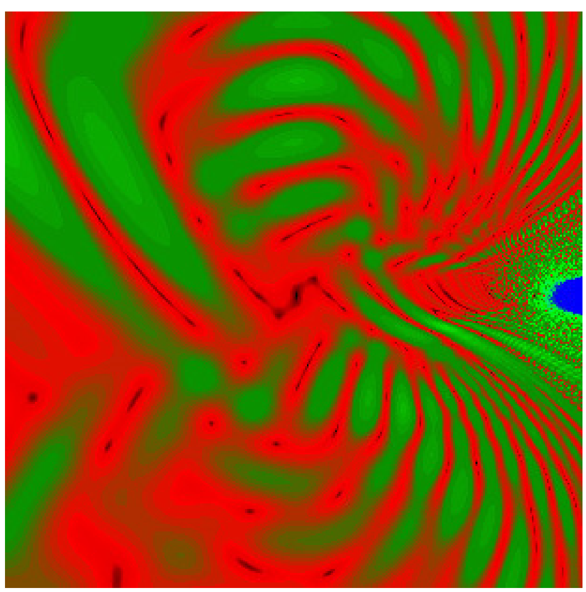
A general scenario: $\frac{d\zeta}{dt} = \varphi_2(z,\zeta)$, $\frac{dz}{dt} = \varphi_1(z)$

Example 5a : $\varphi_1 = \frac{1}{10}z^2$, $\varphi_2 = xCos(x+y) + iySin(x-y)$, α - plane

$$\lambda(\alpha) = \alpha \cdot \exp\left(\int_0^1 \psi_2(\alpha,t)dt\right) \ , \ \frac{dz}{dt} = \varphi_1(z) \ , \ z_0 = \alpha$$



Example 5b : $\frac{d\zeta}{dt} = \varphi_2(z(t))$, $\frac{dz}{dt} = \varphi_1(z) \Rightarrow \lambda(\alpha) = \int_0^1 \psi_2(\alpha,t)dt$, $\psi_2(\alpha,t) = \varphi_2(z(t))$



[1] J. Gill, *A Space of Siamese Contours in Time-dependent Complex Vector Fields*, Comm. Anal. Th. Cont. Frac., Vol XX (2014)

[2] J. Gill, *A Note on Integrals & Hybrid Contours in the Complex Plane*, Comm. Anal. Th. Cont. Frac., Vol XX (2014)