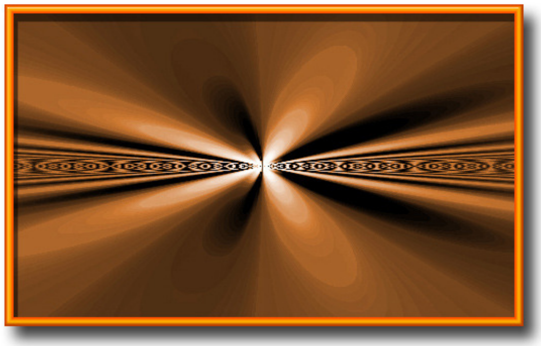
Infinite Compositions of Complex Functions & Weak Emergence



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Abstract: The graphs of infinite compositions of complex functions in $\mathbb{C}$ demonstrate the quality of *weak emergence* – largely unpredictable imagery arising from complicated mathematical structures.

Infinite compositions of complex functions may occur in two general forms:

I Inner compositions: $F_n(z)=\underset{k=1}{\overset{n}{\mathcal{R}}} f_k(z)=f_1\circ f_2\circ\cdots\circ f_n(z)$, $F(z)=\lim_{n\to\infty}\underset{k=1}{\overset{n}{\mathcal{R}}} f_k(z)$, <u>or</u>

$F_n(z)=\underset{k=1}{\overset{n}{\mathcal{R}}} f_{k,n}(z)=f_{1,n}\circ f_{2,n}\circ\cdots\circ f_{n,n}(z)$, $F(z)=\lim_{n\to\infty}\underset{k=1}{\overset{n}{\mathcal{R}}} f_{k,n}(z)$.

II Outer compositions: $G_n(z)=\underset{k=1}{\overset{n}{\mathcal{L}}} g_k(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z)$, $G(z)=\lim_{n\to\infty}\underset{k=1}{\overset{n}{\mathcal{L}}} g_k(z)$, <u>or</u>

$G_n(z)=\underset{k=1}{\overset{n}{\mathcal{L}}} g_{k,n}(z)=g_{n,n}\circ g_{n-1,n}\circ\cdots\circ g_{1,n}(z)$, $G(z)=\lim_{n\to\infty}\underset{k=1}{\overset{n}{\mathcal{L}}} g_{k,n}(z)$.

For continued fractions written in fixed-point format: $f_k(z)=\dfrac{\alpha_k(\zeta)\cdot\beta_k(\zeta)}{\alpha_k(\zeta)+\beta_k(\zeta)-z}$, $z=0$

generates a function $F(\zeta)$.

Frequently, $g_n(z)=z+\eta_n\cdot\varphi(\zeta)$ or $\mathrm{f}_n(z)=z+\eta_n\cdot\varphi(\zeta)$, where $\zeta\neq z$ <u>or</u> $\zeta=z$.

Two elementary results follow:

Theorem 1. Suppose $g_n(z) = z + \rho_n \cdot \varphi(z)$ where there exist $R > 0$ and $M > 0$ such that $|z| < R \Rightarrow |\varphi(z)| < M$. Furthermore, suppose $\rho_k \geq 0$, $\sum_1^\infty \rho_k < \infty$ and $R > M \cdot \sum_{k=1}^\infty \rho_k$. Then there exists $0 < R^* < R$ such that

$$G_n(z) \equiv g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \;\to\; G(z) \text{ for } \{z : |z| < R^*\}.$$

Sketch of proof: Assume momentarily that R^* exists. Then

$|g_n(z)| \leq |z| + \rho_n M < R^* + \rho_n M$

$|g_{n+1} \circ g_n(z)| \leq |z| + M(\rho_n + \rho_{n+1}) < R^* + M(\rho_n + \rho_{n+1})$

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$|g_{n+m} \circ g_{n+m-1} \circ \cdots \circ g_n(z)| \leq |z| + M\sum_{k=n}^{n+m} \rho_k < R^* + M\sum_{k=n}^{n+m} \rho_k \leq R^* + M\sum_{k=1}^{\infty} \rho_k$

with $R^* < R - M\sum_{k=1}^\infty \rho_k$. Next,

$|G_{n+1}(z) - G_n(z)| = |g_{n+1}(G_n) - G_n| < \rho_{n+1} M$

$|G_{n+2}(z) - G_n(z)| \leq |g_{n+2}(G_{n+1}) - G_{n+1}| + |g_{n+1}(G_n) - G_n| < (\rho_{n+2} + \rho_{n+1})M$

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$|G_{n+m}(z) - G_n(z)| < M \cdot \sum_{k=n}^\infty \rho_k \to 0$ as $\mathrm{n} \to \infty$

There is no requirement that φ be analytic or even continuous, merely that it contract as described. #

Theorem 2: Suppose $f_n(z) = z + \rho_n \cdot \varphi(z)$ where there exist $R > 0$ and $M > 0$ such that $|z| < R$ and $|\zeta| < \mathrm{R} \Rightarrow |\varphi(z)| < M$ and $|\varphi(z) - \varphi(\zeta)| < r|z - \zeta|$. Furthermore, suppose $\rho_k \geq 0$, $\sum_1^\infty \rho_k < \infty$ and $R > M \cdot \sum_{k=1}^\infty \rho_k$. Then there exists $0 < R^* < R$ such that

$$F_n(z) \equiv f_1 \circ f_2 \circ \cdots \circ f_n(z) \;\; \to \;\; \mathrm{F}(z) \;\; \text{for} \;\; \{z : |z| < R^*\}.$$

Sketch of proof: (similar to the previous proof)

$$|f_{n+m}(z) - z| < M\rho_{n+m} \Rightarrow |f_{n+m}(z)| < R^* + M\rho_{n+m} < R$$

$$|f_{n+m-1} \circ f_{n+m}(z) - z| \leq |f_{n+m-1}(f_{n+m}(z)) - f_{n+m}(z)| + |f_{n+m}(z) - z| < M\rho_{n+m-1} + M\rho_{n+m}$$

$$\Rightarrow |f_{n+m-1} \circ f_{n+m}(z)| < R^* + M(\rho_{n+m-1} + \rho_{n+m}) < R$$

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$$|f_n \circ f_{n+1} \circ \cdots \circ f_{n+m}(z) - z| < M \cdot \sum_{k=n}^{n+m} \rho_k$$

Next, for $R^* < R - M\sum_{k=1}^\infty \rho_k$,

$$|F_{n+m}(z) - F_n(z)| < \prod_{k=1}^n (1 + r\rho_k) \cdot |f_{n+m} \circ f_{n+m-1} \circ \cdots \circ f_n(z) - z| < M \cdot \prod_{k=1}^n (1 + r\rho_k) \cdot \sum_{k=n+1}^{n+m} \rho_k = M \cdot S_n(m)$$

With $S_n(m) \to 0$ as $n \to \infty$. #

The following two theorems may be of use when the individual functions $f_k(z)$ or $g_k(z)$ do not approach z as $k \to \infty$, but rather display a contractive quality.

Theorem 3*:* (Lorentzen , 1990) Let $\{f_n\}$ be a sequence of functions *analytic* on a simply-connected domain D. Suppose there exists a compact set $\Omega \subset D$ such that for each n, $f_n(D) \subset \Omega$.
Then $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$ converges uniformly in D to a constant function F(z) = λ.

Theorem 4 (Gill , 1991) Let $\{g_n\}$ be a sequence of functions analytic on a simply-connected dom continuous on the closure of D. Suppose there exists a compact set $\Omega \subset D$ such that $g_n(D) \subset \Omega$ f Define $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$. Then $G_n(z) \to \alpha$ uniformly on the closure of D *if and only* sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in Ω converge to the number α.

Now, suppose $g_{k,n}(z) = z + \eta_{k,n} \cdot \varphi(z, \frac{k}{n}),\ \varphi \in C(S \times I),\ z \in S \Rightarrow g_{k,n} \in S$.

Convergence is not as easily established if $\{\eta_{k,n}\}_{k=1}^{n}$ are the sub- intervals of a partition of [0,1], $\pi_n = < t_{0,n}, t_{1,n}, t_{2,n}, \cdots, t_{n,n} >$, with $t_{0,n} = 0$, $t_{n,n} = 1$ and $\eta_{k,n} = t_{k,n} - t_{k-1,n}$ with the norm of the partition $\|\pi_n\| = \max_{k \le n} \eta_{k,n}$ where $\|\pi_n\| \to 0$ as $n \to \infty$. For simplicity let $\eta_{k,n} := \frac{1}{n}$.

Set $G_{k,n}(z) = g_{k,n} \circ g_{k-1,n} \circ \cdots \circ g_{1,n}(z),\ G_n(z) = G_{n,n}(z)$. Here, a contour in $\mathbb{C}$,
[$\lim_{n\to\infty} \{z_{k,n}\}_{k=1}^{n},\quad z_{k+1,n} = z_{k,n} + \frac{1}{n}\varphi(z_{k,n}, \frac{k}{n})$] results with initial point $z = z_0$, and

$$\lambda_n(z) = G_{n,n}(z) - z = \frac{1}{n} \cdot \sum_{k=1}^{n} \varphi(G_{k-1,n}(z), \tfrac{k}{n}),\ \ G_{0,n}(z) = z \quad .$$

For convenience set $\psi\left(z, \frac{k}{n}\right) \equiv \varphi\left(G_{k,n}(z), \frac{k}{n}\right)$, with $\int_0^1 \psi(z,\tau)\, d\tau$ defined :

$$G_{n,n}(z) - z = \frac{1}{n}\psi\left(z, \frac{1}{n}\right) + \frac{1}{n}\psi\left(z, \frac{2}{n}\right) + \frac{1}{n}\psi\left(z, \frac{3}{n}\right) + \cdots + \frac{1}{n}\psi\left(z, \frac{n}{n}\right) \to \int_0^1 \psi(z,\tau)\, d\tau = \lambda(z)$$

The existence of this function (and the "virtual" integral) is equivalent to the convergence of the contour, which *may be* describable in closed parametric form when $\frac{dz}{dt} = \varphi(z)$ or $\frac{dz}{dt} = \varphi(z,t)$ has an *exact* solution, $Z(t)$: $\lambda(z_0) := \int_0^1 \psi(z_0, t)\, dt = \int_0^1 \varphi(Z(t), t)\, dt = Z(1) - Z(0)$

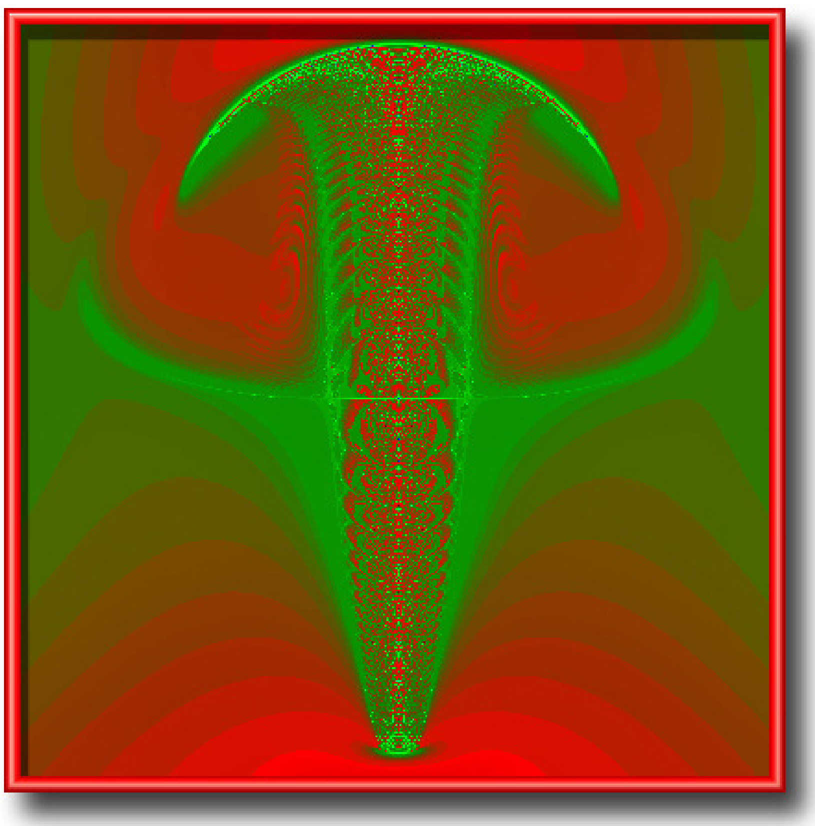
In a very rough sense this is true because

$$z_{k+1,n} = z_{k,n} + \frac{1}{n}\cdot\varphi(z_{k,n},\tfrac{k}{n}) \;\Rightarrow\; \frac{\Delta z_{k,n}}{\frac{1}{n}} = \varphi(z_{k,n},\tfrac{k}{n}) \;\Rightarrow\; \frac{dz}{dt} = \varphi(z,t)$$

The *Picard–Lindelöf theorem* implies unique solutions of these differential equations – under certain restrictions - so that the contours terminate properly .

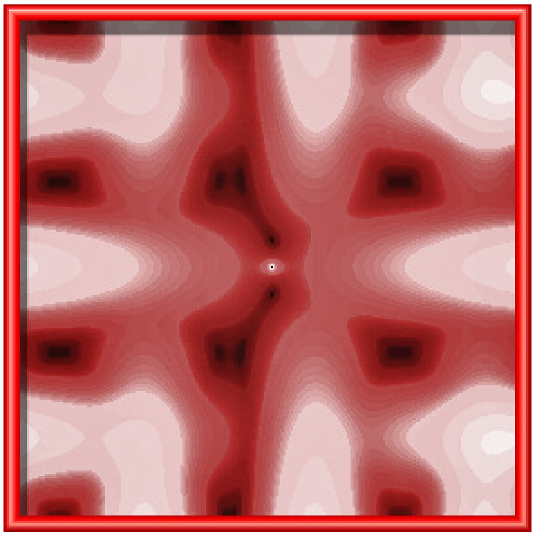
The virtually unpredictable images that follow are examples of ***weak emergence*** emanating from infinite compositions far more complicated than simple iteration. They are topographical graphs in which the contour lines represent constant moduli. Dark indicates very small moduli and light very large moduli.

Example : $\dfrac{dz}{dt} = \varphi_z = \zeta + \dfrac{1}{\zeta}\;,\;\; \dfrac{d\zeta}{dt} = \varphi_\zeta = xSin(y) + iyCos(x) \;\Rightarrow \lambda(\omega_0) = \displaystyle\int_0^1 \psi_z(\omega_0,t)dt$

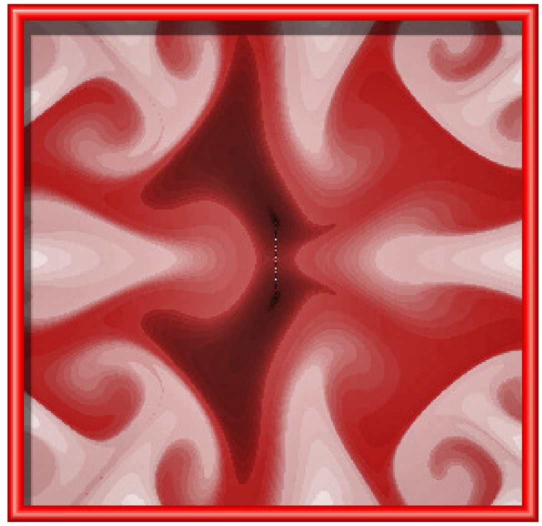


Plague doctor

Example : $\varphi(z,\frac{k}{n})=xCos(y)+iySin(x)+\frac{k}{n}\cdot\frac{1}{z}$. For $g_{k,n}(z)=z+\frac{1}{n}\cdot\varphi(z_0,\frac{k}{n})$, as $n\to\infty$ a standard Riemann contour integral is defined for each z_0 . However, $g_{k,n}(z)=z+\frac{1}{n}\cdot\varphi(z,\frac{k}{n})$ produces a *virtual Riemann integral* for each z , from the sum $\lambda_n(z)=\frac{1}{n}\cdot\sum_{k=1}^{n}\varphi(G_{k-1,n}(z),\frac{k}{n})$.



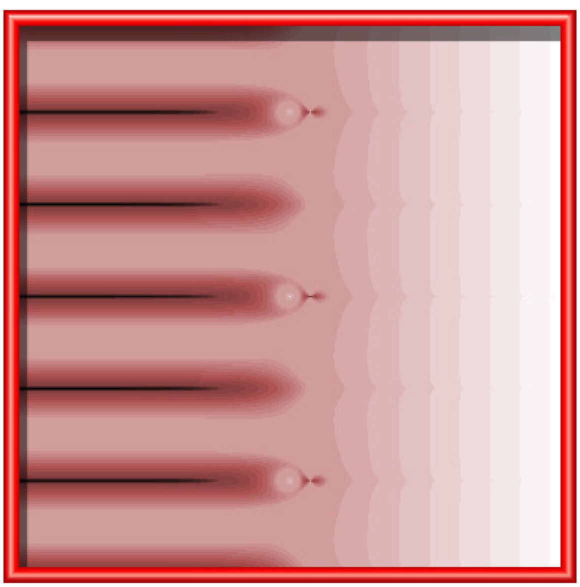
2. Riemann Integral [-10,10]

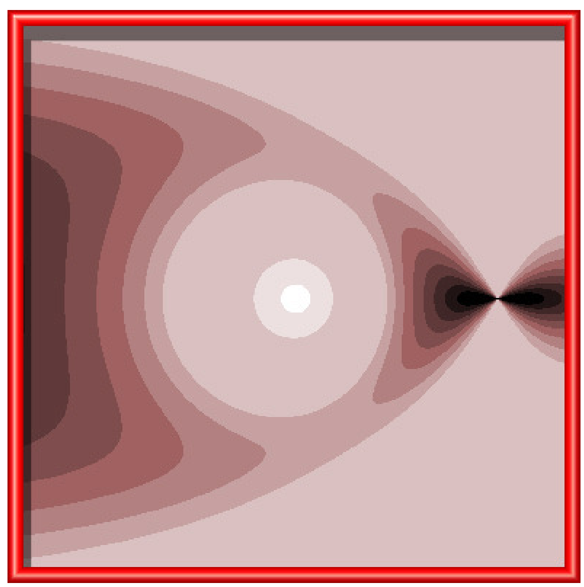


3. Virtual Riemann Integral

Example : $\varphi(z,t)=e^z$. Here $Z(z,t)=z-Ln(1-e^z t)$ and $\psi(z,t)=e^z/(1-e^z t)$.

Thus $\lambda(z)=Ln\left(\frac{1}{1-e^z}\right)$. [-10<x,y<10] and [-1<x,y<1] $\lambda(z)=\int_0^1\psi(z,t)dt$:

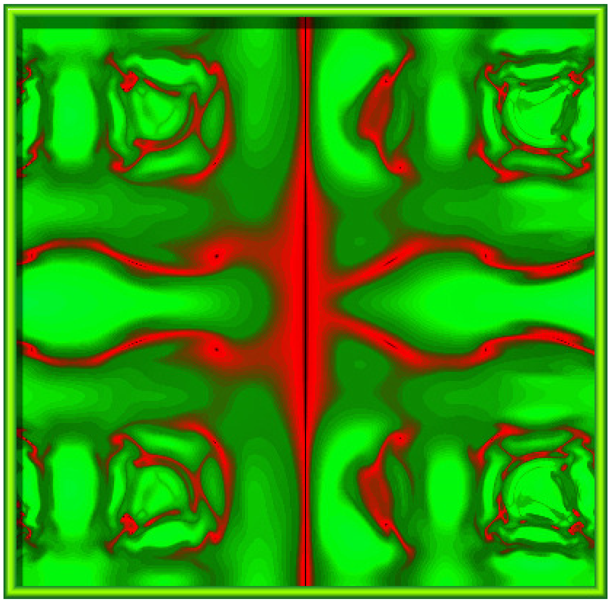




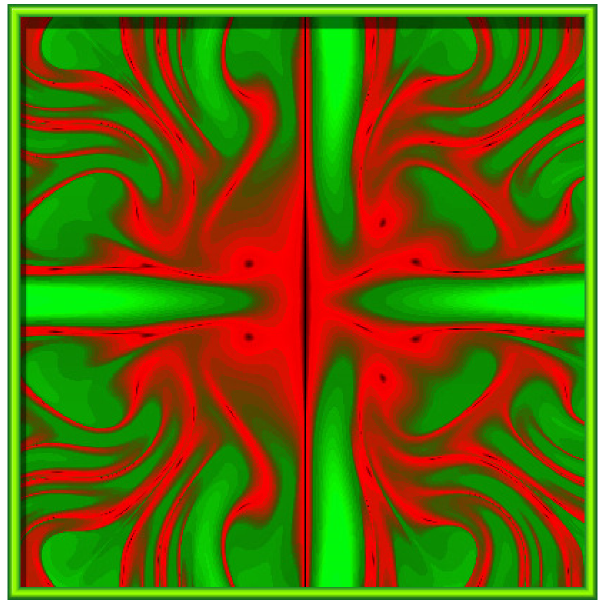
Minnows in a stream

Example : $\varphi(z,t)=xCos(y(t+1))+iySin(x(t+1))$, [-10,10] , n=30

(a) $\eta_{k,n}=\dfrac{1}{k^2}$, (b) $\eta_{k,n}=\dfrac{1}{n}$, $g_{k,n}(z)=z+\eta_{k,n}\cdot\varphi(z,\tfrac{k}{n})$: $\lambda_n(z)=\sum_{k=1}^{n}\eta_{k,n}\cdot\varphi(G_{k-1,n}(z),\tfrac{k}{n})$



(a)



(b)

Demonic possession

Consider $g_{k,n}(z)=z\left(1+\tfrac{1}{n}\varphi(z,\tfrac{k}{n})\right)$, $1<k\le n$ and $g_{1,n}(z)=\left(1+\tfrac{1}{n}\varphi(z,\tfrac{1}{n})\right)$. Then

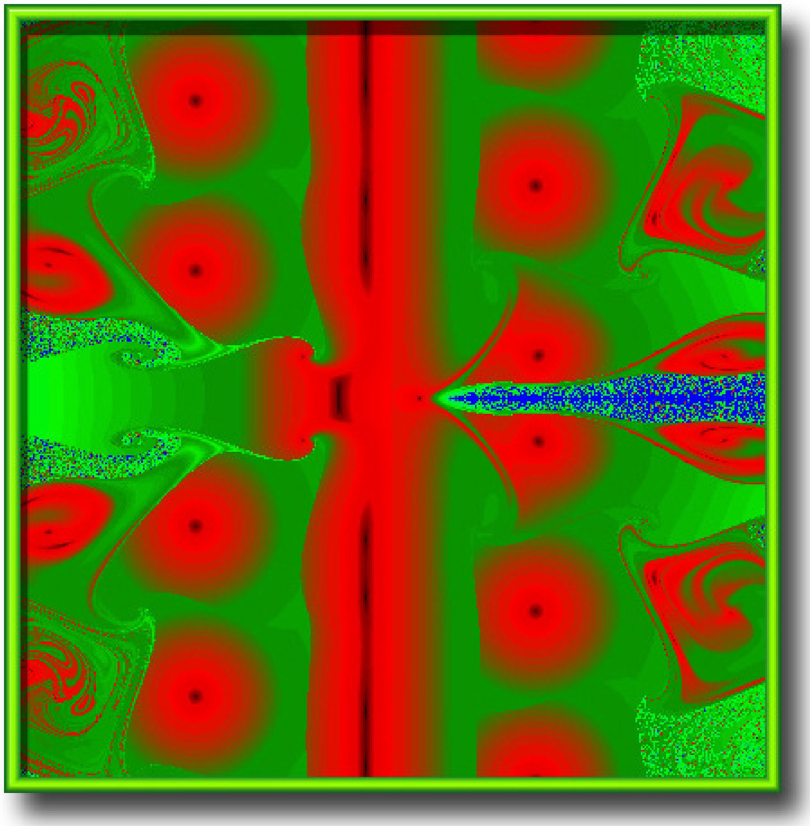
$G_{n,n}(z)=g_{n,n}\circ g_{n-1,n}\circ\cdots\circ g_{1,n}(z)=\prod_{k=1}^{n}\left(1+\tfrac{1}{n}\varphi(G_{k-1,n}(z),\tfrac{k}{n})\right)$, an ***infinite product*** (as $n\to\infty$) .

Setting $P_{k,n}(z)=\tfrac{1}{n}\varphi(G_{k-1,n}(z),\tfrac{k}{n})$,

$\prod_{k=1}^{n-1}\left(1+P_{k,n}(z)\right)=1+P_{1,n}(z)+P_{2,n}(z)+\cdots+P_{k-1,n}(z)\ +\ \mathrm{R}_n(z)$, with the Riemann-like sum

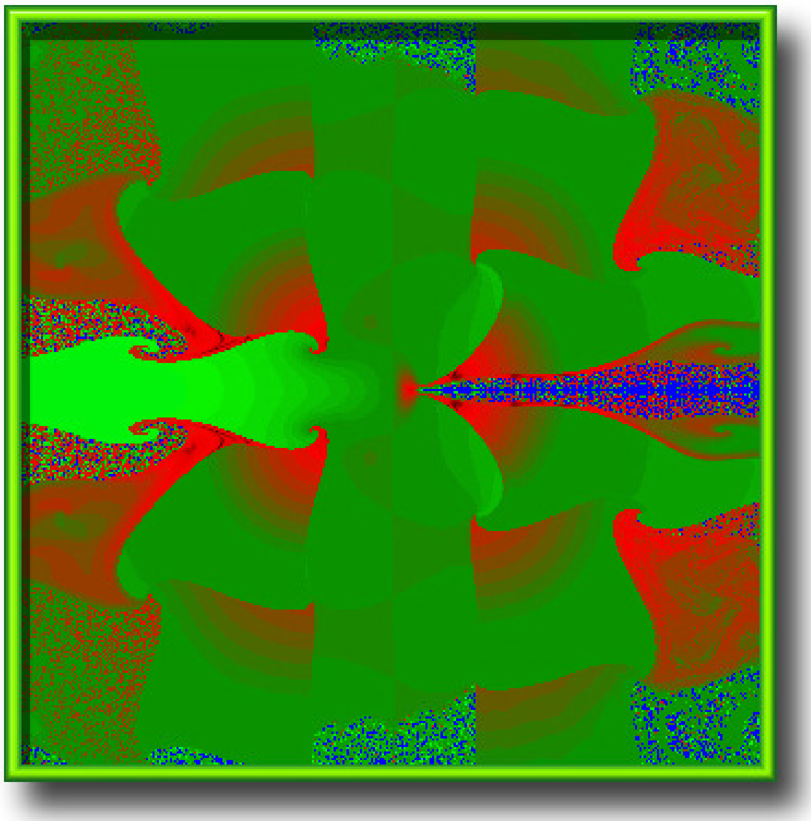
$$P_{1,n}(z)+P_{2,n}(z)+\cdots+P_{k-1,n}(z)\ \sim\ \int_0^1\pi(z,t)dt$$

Example : $\varphi(z) = xCos(y) + i \cdot ySin(x)$, $\prod_{k=1}^{100}\left(1+P_{k,n}(z)\right)$, [-15,15]



Mystical hives

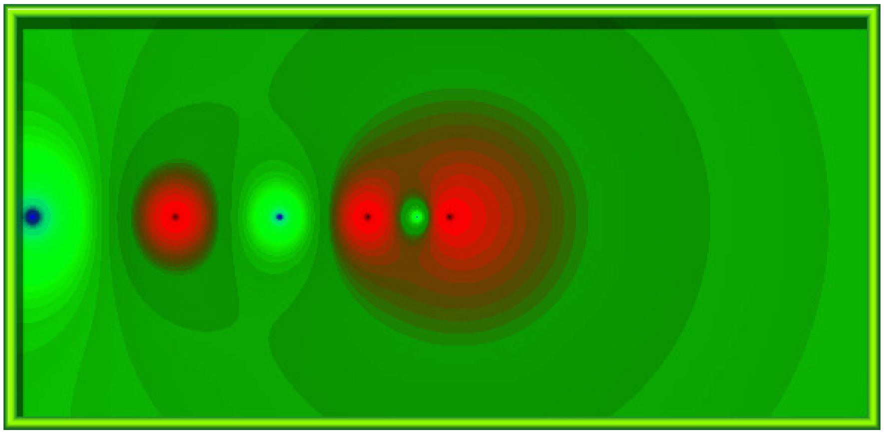
Example : $\varphi(z) = xCos(y) + i \cdot ySin(x)$, $\tilde{\lambda}(z) = \int_0^1 \pi(z,t)dt - z$, [-15,15]



Picasso's dilemma

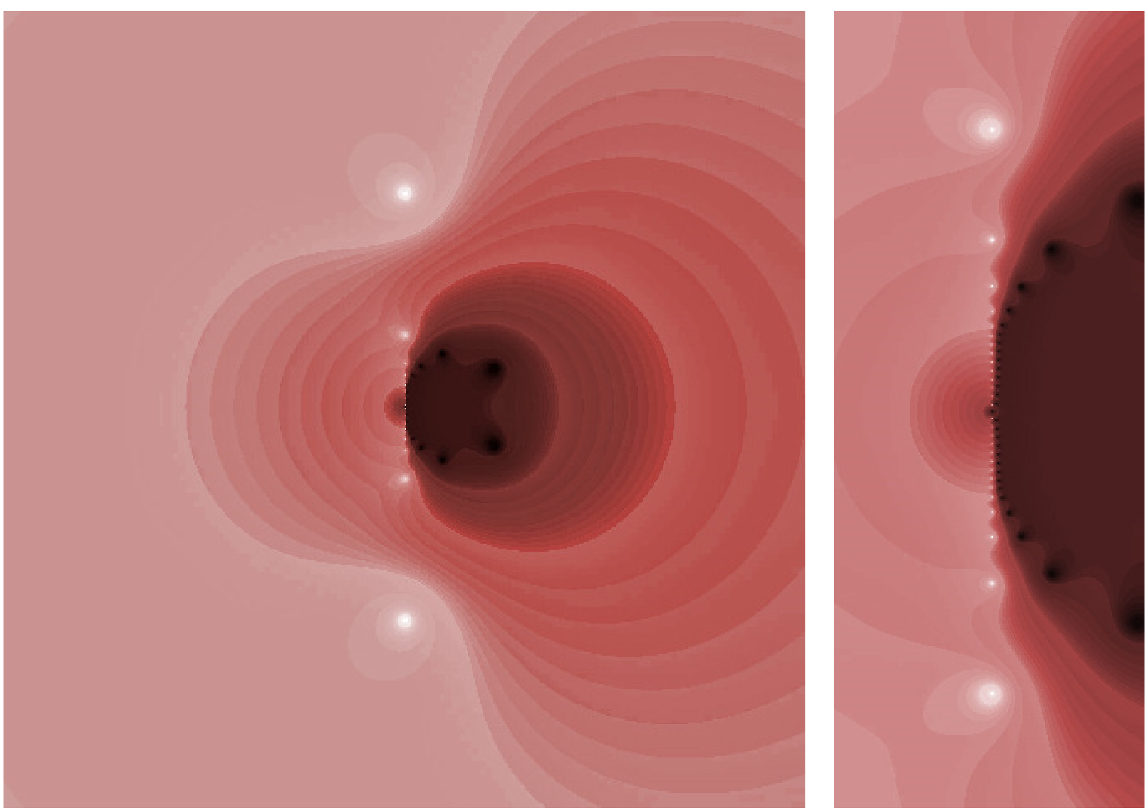
Continued fractions of an elementary nature. Nth approximates.

Example : $F(z)=\frac{z}{1+}\,\frac{z}{2+}\,\frac{z}{3+\cdots}$, [-20<x<20] , n=40



Anxious billiards

Example : $F_n(z)=\frac{z^2}{\frac{1}{n}+}\,\frac{z^2}{\frac{2}{n}+}\,\frac{z^2}{\frac{3}{n}+}\,\cdots\,\frac{z^2}{\frac{n}{n}}$,   [-10,10] and [-1,1], n=50. $W_n(z)=F_n(z)-z$:



Wormhole

Set $g_k(\zeta)=\dfrac{\rho(G_{k-1}(z))}{1+\rho(G_{k-1}(z))-\zeta}$, $z\,\&\,\zeta\in S$, $g_k(\zeta)\in S$ for a suitably well-behaved $\rho(z)$.

Form $G_k(z)=\underset{j=1}{\overset{k}{\mathcal{L}}}g_j(0)$, $\lim_{n\to\infty}G_n(z)=G(z)$.

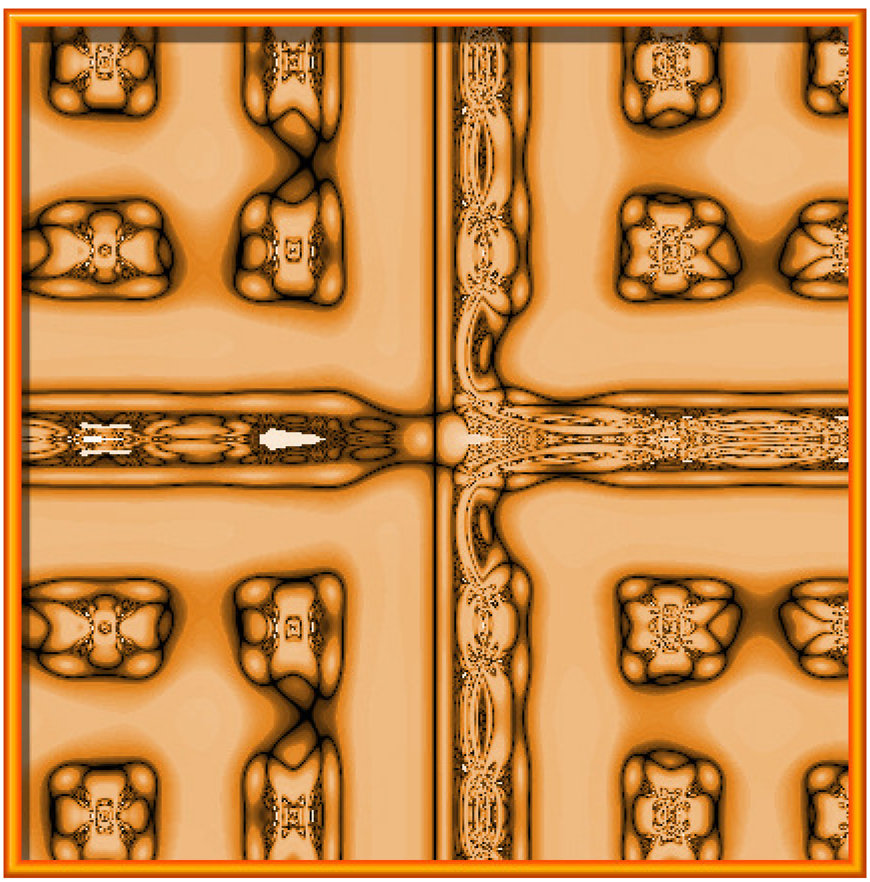
We have **Euler's *equivalent (reverse & self-generating) continued fraction***:

$$G_n(z)=\frac{\rho(G_{n-1}(z))}{1+\rho(G_{n-1}(z))-}\ \frac{\rho(G_{n-2}(z))}{1+\rho(G_{n-2}(z))-}\ \cdots\ \frac{\rho(G_1(z))}{1+\rho(G_1(z))-}\ \frac{\rho(z)}{1+\rho(z)-\zeta},\quad \zeta=0$$

Abbreviate $\rho_k=\rho(G_{k-1}(z))$. Then $G_n(z)=\dfrac{S_n(z)}{1+S_n(z)}$, $S_n(z)=\sum_{k=1}^{n}\left(\prod_{j=k}^{n}\rho_j\right)=a_1(n)+a_2(n)+\cdots+a_n(n)$

If $a_k(n)\to a_k$, $|a_k(n)|\le M_k$, $\sum M_k<\infty$, Tannery's Theorem will insure convergence of $S_n(z)$.

Example : $\rho(z)=\rho(x+iy)=xCos(y)+iySin(x)$, [-15,15], n=30, $S_n(z)$:



Dreams of gold

A variation on the contour generator produces a ***self-generating series*** whose terms are ***continued fractions***:

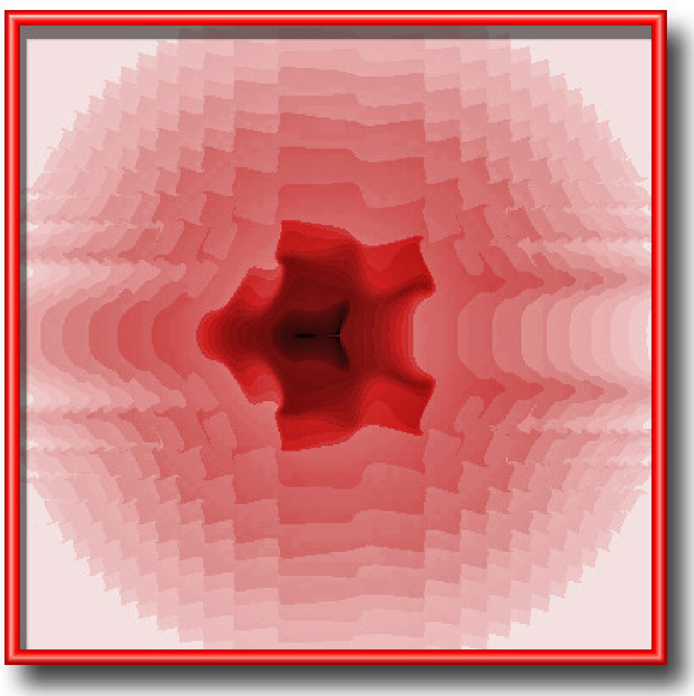
$$g_{k,n}(z)=z+\frac{\eta_n\varphi(z,\frac{k}{n})}{z},\quad \eta_n=\frac{1}{n}.$$

$$G_n(z)-z=\eta_n\left(\frac{\varphi(z,\frac{1}{n})}{z}\right)+\eta_n\left(\frac{\varphi(G_{1,n}(z,\frac{2}{n}))}{G_{1,n}(z,\frac{2}{n})}\right)+\cdots+\eta_n\left(\frac{\varphi(G_{n-1,n}(z,\frac{n-1}{n}))}{G_{n-1,n}(z,\frac{n-1}{n})}\right)$$

$$=\eta_n\psi(z,\tfrac{1}{n})+\eta_n\psi(z,\tfrac{2}{n})+\cdots+\eta_n\psi(z,\tfrac{n-1}{n})\sim\int_0^1\psi(z,t)dt$$

E.g., $$G_3(z)=z+\frac{\eta_3\varphi(z,\frac{1}{3})}{z}+\frac{\eta_3\varphi(G_{1,3}(z))}{z+\frac{\eta_3\varphi(z,\frac{1}{3})}{z}}+\frac{\eta_3\varphi(G_{2,3}(z))}{z+\frac{\eta_3\varphi(z,\frac{1}{3})}{z}+\frac{\eta_3\varphi(G_{1,3}(z))}{z+\frac{\eta_3\varphi(z,\frac{1}{3})}{z}}}$$

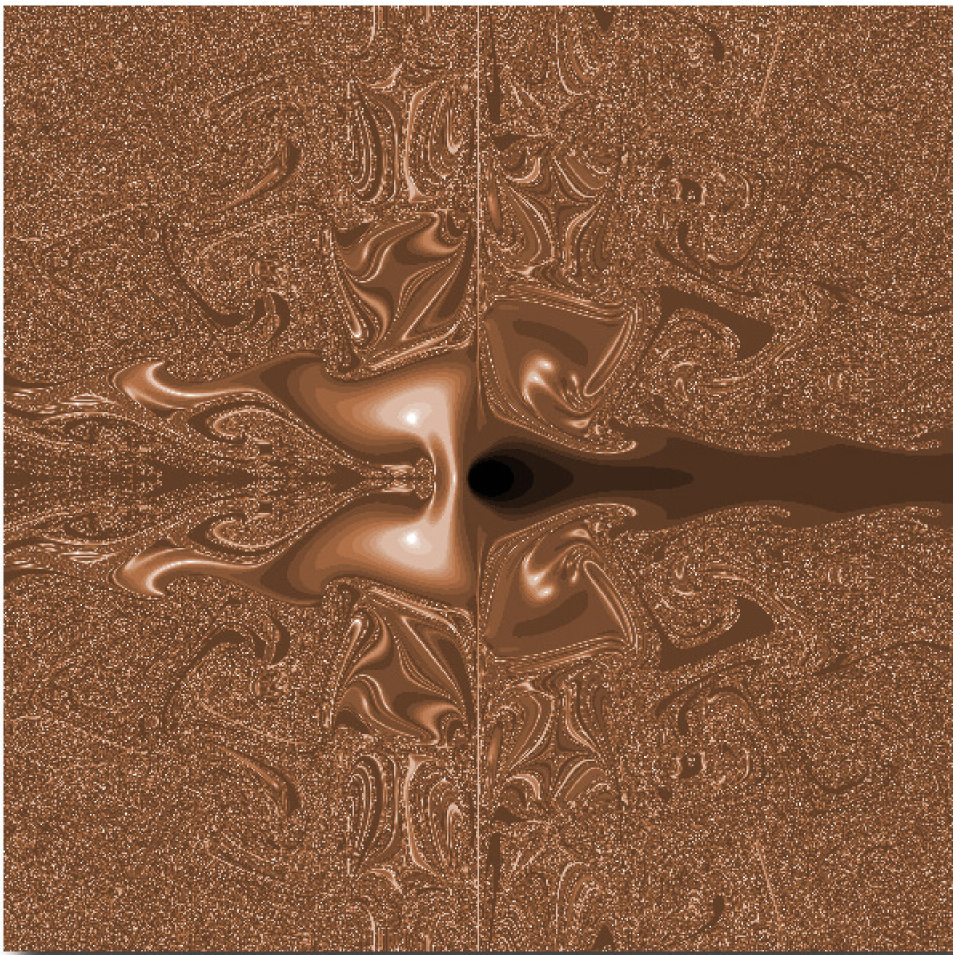
Example : $\varphi(z)=xSin(y^2)+iyCos(x^2)$, [-10,10] , n=40 , $\lambda^*(z)=z+\int_0^1\psi(z,t)dt$:



Double vision

Example : $F_{40}(x+iy)=\underset{k=1}{\overset{40}{\mathcal{R}}}\left(\dfrac{x+iy}{1+\frac{1}{2^k}\left(xCos(y)+iySin(x)\right)}\right),\ -20<x,y<20$

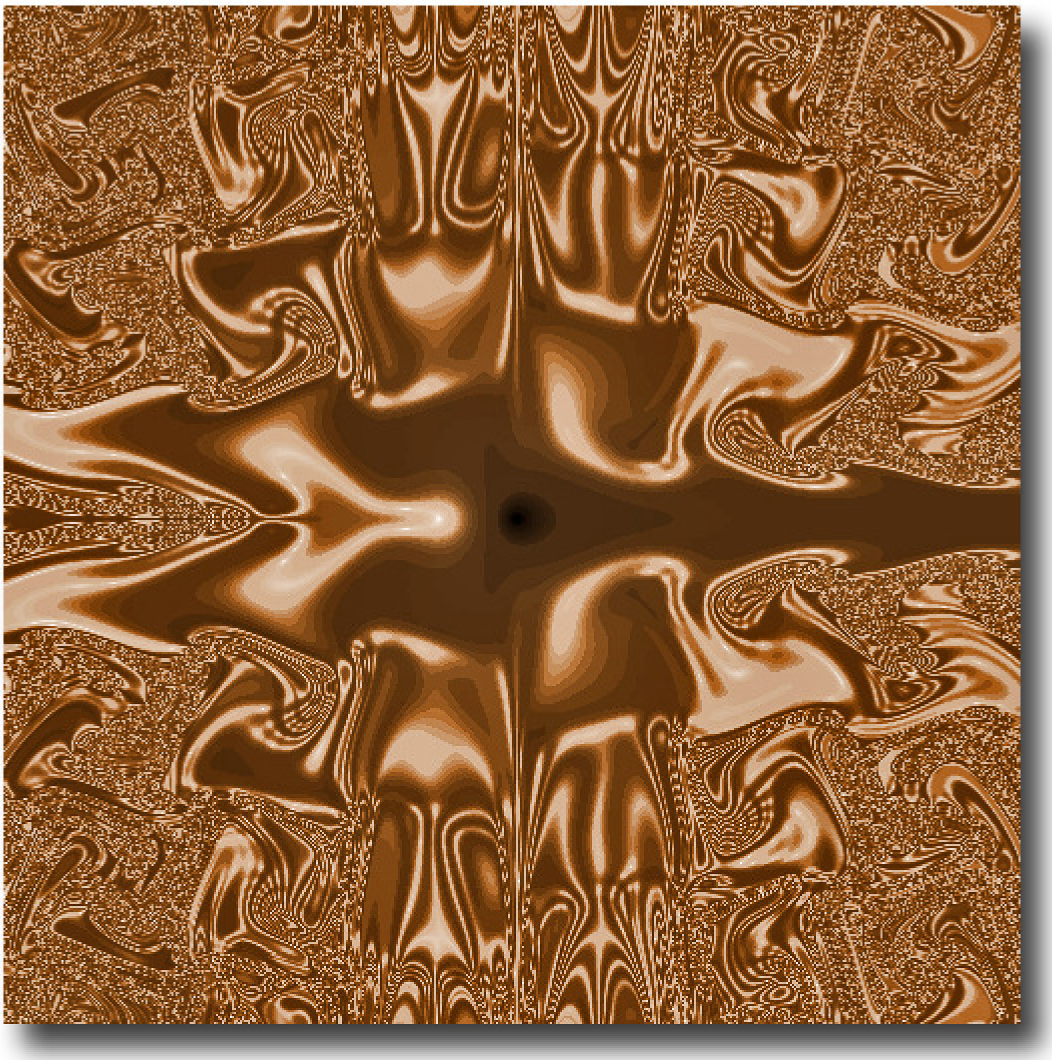
$f_k(z)=z+\eta_k\cdot\varphi(z,\eta_k),\ \ \eta_k=\dfrac{1}{2^k}.$



Reproductive universe (1)

Example : $F_{40}(x+iy)=\mathop{\mathcal{R}}_{k=1}^{40}\left(\dfrac{x+iy}{1+\frac{1}{4^k}\left(xCos(y)+iySin(x)\right)}\right)$, [-20,20],

$f_k(z)=z+\eta_k\cdot\varphi(z,\eta_k),\;\; \eta_k=\dfrac{1}{4^k}$:



Reproductive universe (2)

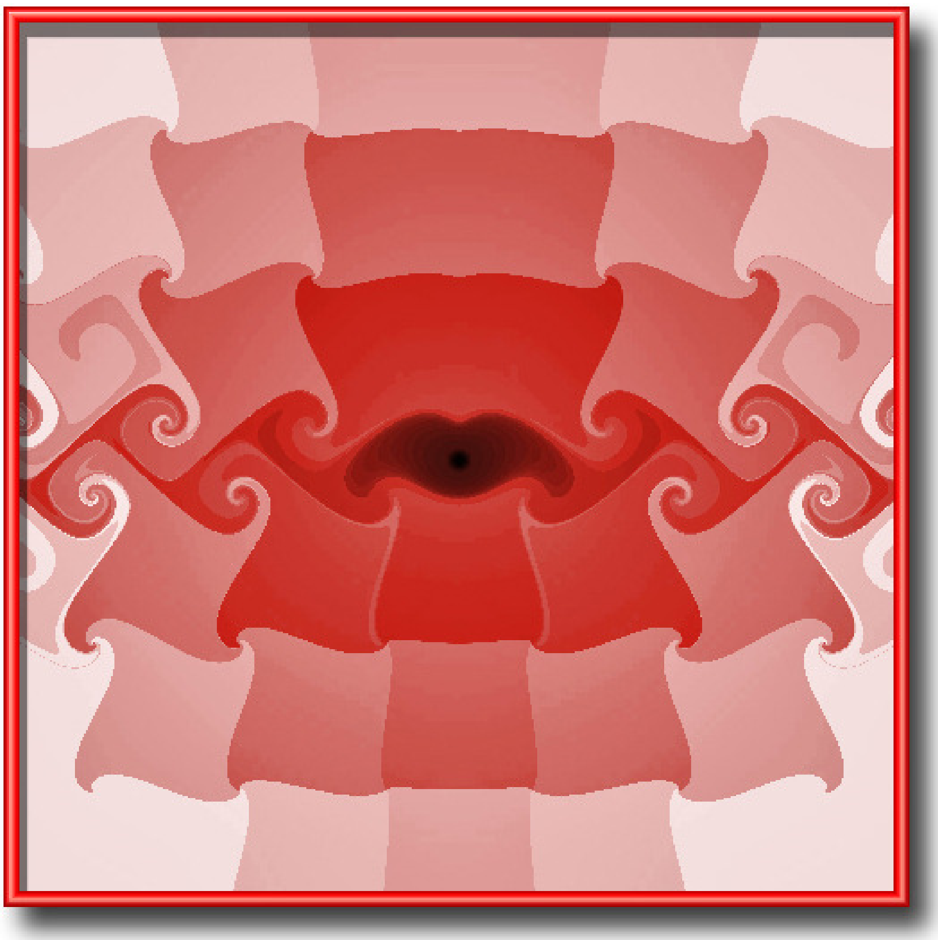
Example : $F_{40}(x+iy)=\overset{40}{\underset{k=1}{\mathcal{R}}}\left(\dfrac{x+iy}{1+\frac{1}{k^2}\left(xCos(y)+iySin(x)\right)}\right)$, [-20,20],

$f_k(z)=z+\eta_k\cdot\varphi(z,\eta_k),\ \ \eta_k=\dfrac{1}{k^2}$

Reproductive universe (3)

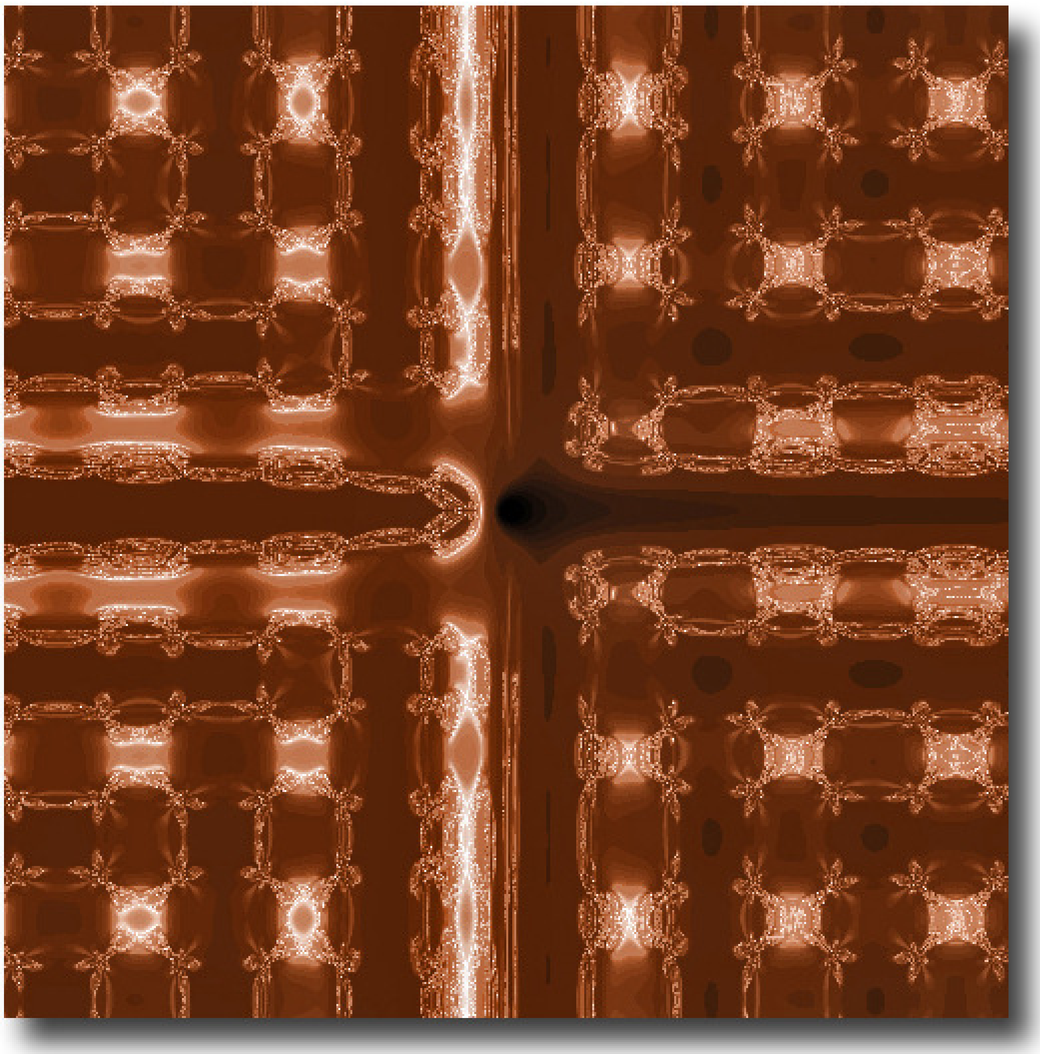
Example : $G_{40}(x+iy)=\underset{k=1}{\overset{40}{\mathcal{L}}}\left(\dfrac{x+iy}{1+\frac{1}{40}\left(Cos(y)+iSin(x)\right)}\right)$, [-20,20], n=40

$f_k(z)=z+\eta_k\cdot\varphi(z,\eta_k),\ \ \eta_k=\dfrac{1}{n}$. This is a virtual integral.



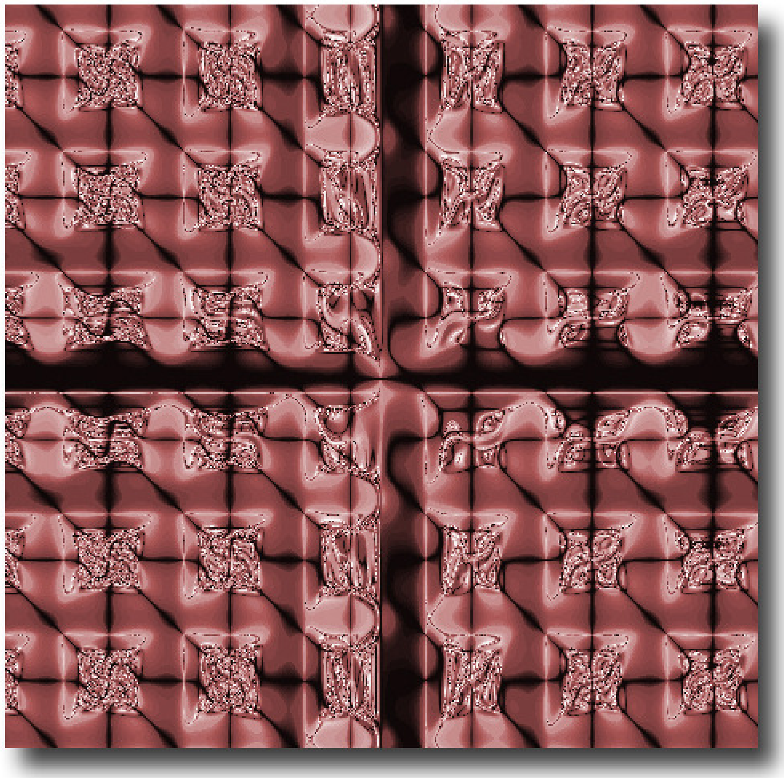
Kissimee

Example : $G_{40}(x+iy)=\underset{k=1}{\overset{40}{\mathcal{L}}}\left(\frac{x+iy}{1+\frac{1}{2^k}\left(xCos(y)+iySin(x)\right)}\right)$, [-20,20], $g_k(z)=z+\frac{1}{2^k}\cdot\varphi(z)$:



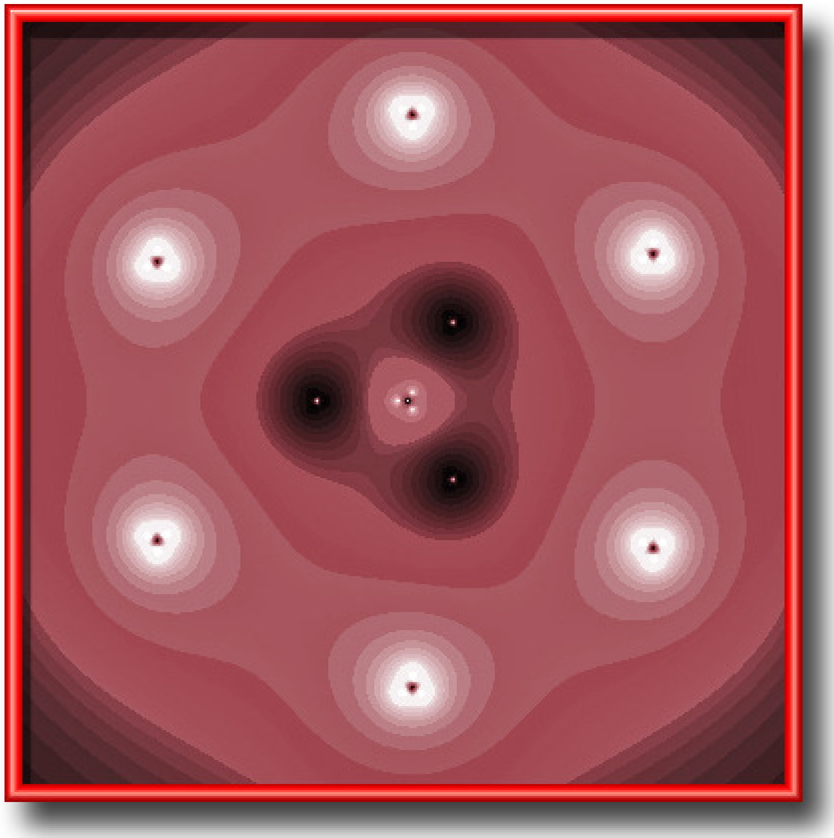
Metropolis from 30k

Example : $$G_{20}(x+iy)=\mathop{\mathcal{L}}_{k=1}^{20}\left[\frac{xSin(y)+iyCos(x)}{1+\frac{1}{2^k}\left(xe^{4Cos(y)}+iye^{4Sin(x)}\right)}\right]\ ,\quad -20<x,y<20\ .$$



Diamonds on velvet

Example : $$G_{20}(x+iy)=\mathop{\mathcal{L}}_{k=1}^{20}\left(\frac{z^2}{10^k}+\frac{1}{z}\right)\ ,\quad -10<x,y<10$$



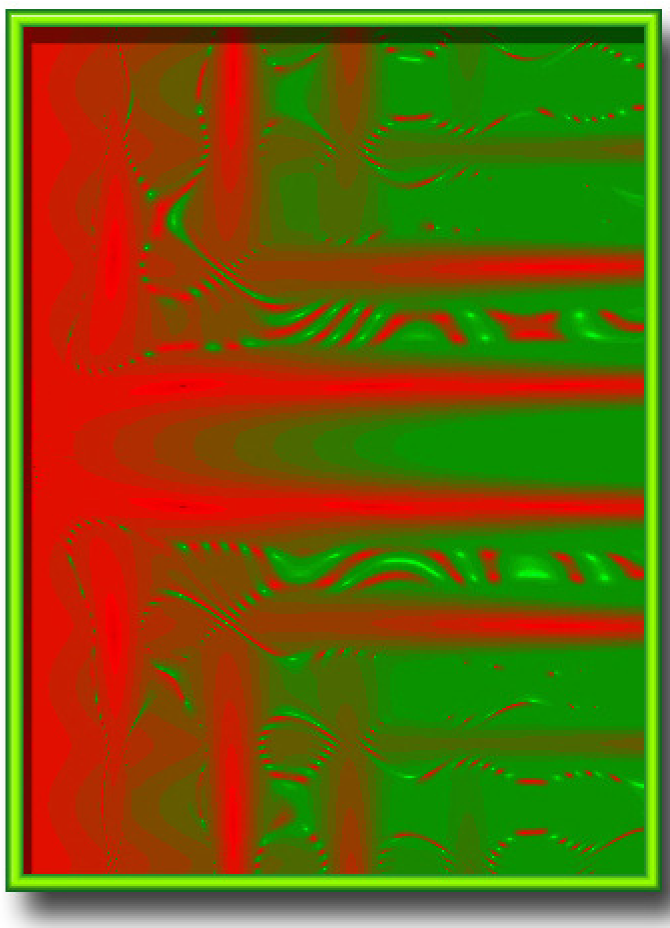
Nuclear waste of the mind

Fixed-point continued fractions have the form

$$\frac{\alpha_1\beta_1}{\alpha_1+\beta_1-}\ \frac{\alpha_2\beta_2}{\alpha_2+\beta_2-}\ \cdots$$

where α_k, β_k are the fixed points of the function $f_k(\zeta)=\dfrac{\alpha_k\beta_k}{\alpha_k+\beta_k-\zeta}$. The *nth approximant* of the continued fraction then can be written $F_n(\zeta)=f_1\circ f_2\circ\cdots\circ f_n(\zeta)$. If $\alpha_k=\alpha_k(z)$ and $\beta_k=\beta_k(z)$, we have $F_n(z,\zeta)=f_1\circ f_2\circ\cdots\circ f_n(z,\zeta)$. Normally, $\zeta=0$. These continued fractions converge if $|\alpha_k(z)|<\rho|\beta_k(z)|$, $\rho<1$.

Example : $\alpha_k(z)=xCos(\frac{y}{x})+iySin(x)$, $\beta_k(z)=xCos(y)+iySin(\frac{x}{y})$, 0<x<18, n=30

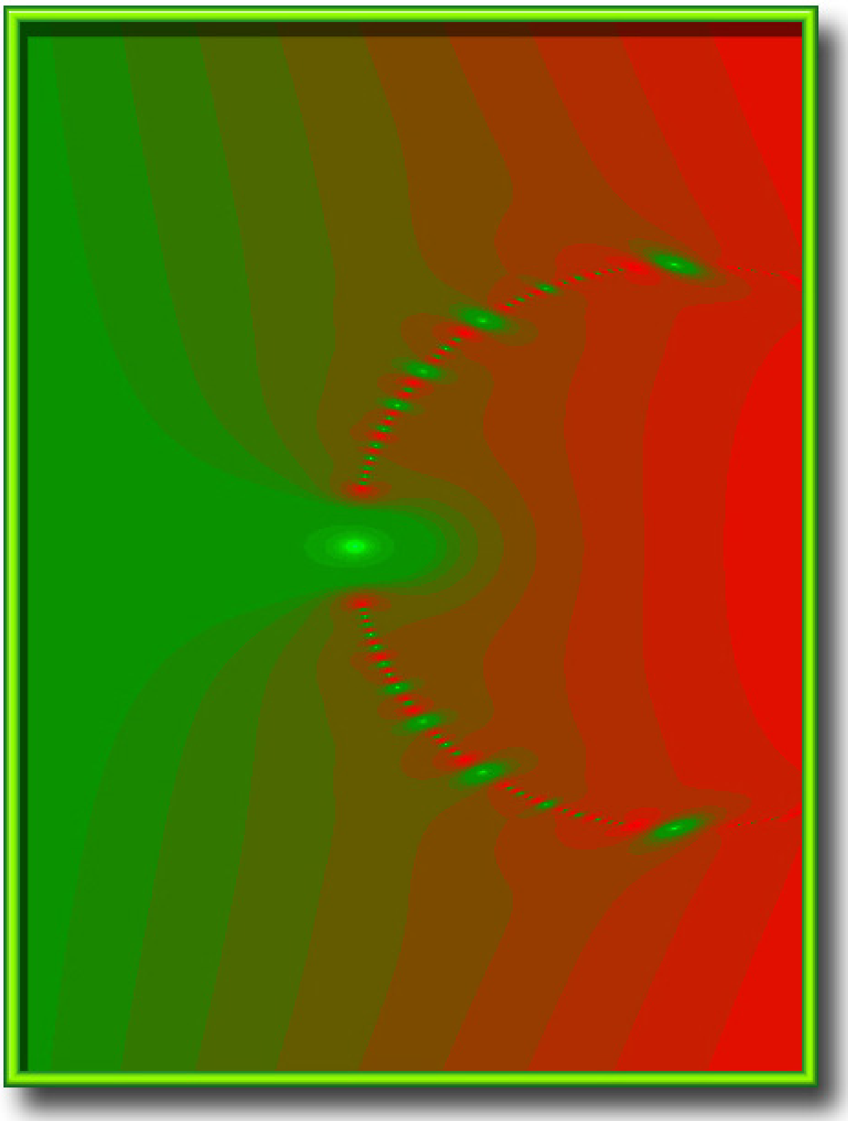


Strange thicket

Set $T_{k,n}(\zeta) = f_k \circ f_{k+1} \circ \cdots \circ f_n(\zeta)$, then form the (virtual) Riemann sum,

$$T_n(0) \ = \ \frac{1}{n} \cdot \sum_{k=1}^{n} T_{k,n}(0) \ \rightarrow \ \int_0^1 T(z,t)dt = \lambda(z)$$

Example : $\alpha_k \equiv xe^{Cos(y/x)} + i \cdot ye^{Sin(x)}$, $\beta_k \equiv z^2$, -4<x<-1, n=30 $\lambda(z)$:



The Ring of the Nibelung

A *self-generating continued fraction*:

Consider

$$CF_n(z) = \frac{\rho(z)}{\delta_1 \ +} \ \frac{\rho(G_1(z))}{\delta_2 \ +} \ \frac{\rho(G_2(z))}{\delta_3 \ +} \ \cdots \ \frac{\rho(G_n(z))}{\delta_{n+1}} \ ,$$

Where $G_1(z) = \frac{\rho(z)}{\delta_1}$, $G_2(z) = \frac{\rho(z)}{\delta_1 + \frac{\rho(G_1(z))}{\delta_2}}$, $G_3(z) = \frac{\rho(z)}{\delta_1 + \frac{\rho(G_1(z))}{\delta_2 + \frac{\rho(G_2(z))}{\delta_3}}}$, etc. ,

With $\rho(z)$ analytic in a region described below.

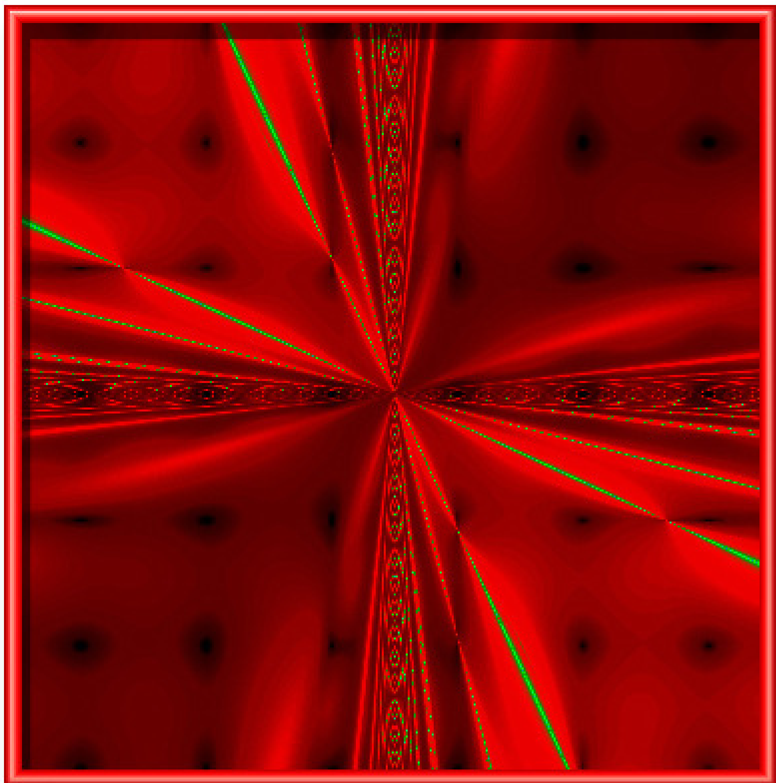
Write $T_k(\omega) = \frac{\rho(G_k(z))}{\delta_{k+1} + \omega}$. Then $CF_n(z) = T_0 \circ T_1 \circ \cdots \circ T_n(0)$.

Suppose now that $|\omega| < R$ and for $|z| < R$, $|\rho(z)| < M$; and in addition $|\delta_k| > R + \frac{M(1+\varepsilon)}{R}$.

Then for each k and relevant z, $|T_k(\omega)| < \frac{R}{1+\varepsilon} < R$

Thus the conditions of theorem 3 are met, and for each z we have $CF_n(z) \to \lambda(z)$:

Example: $\rho(z)=\dfrac{Cos(x)}{1+Cos(\frac{y}{x})+\sin(\frac{x}{y})}+i\dfrac{Sin(y)}{1+Cos(\frac{x}{y})+Sin(\frac{y}{x})}$, $\delta=10$. [-10,10], n=30

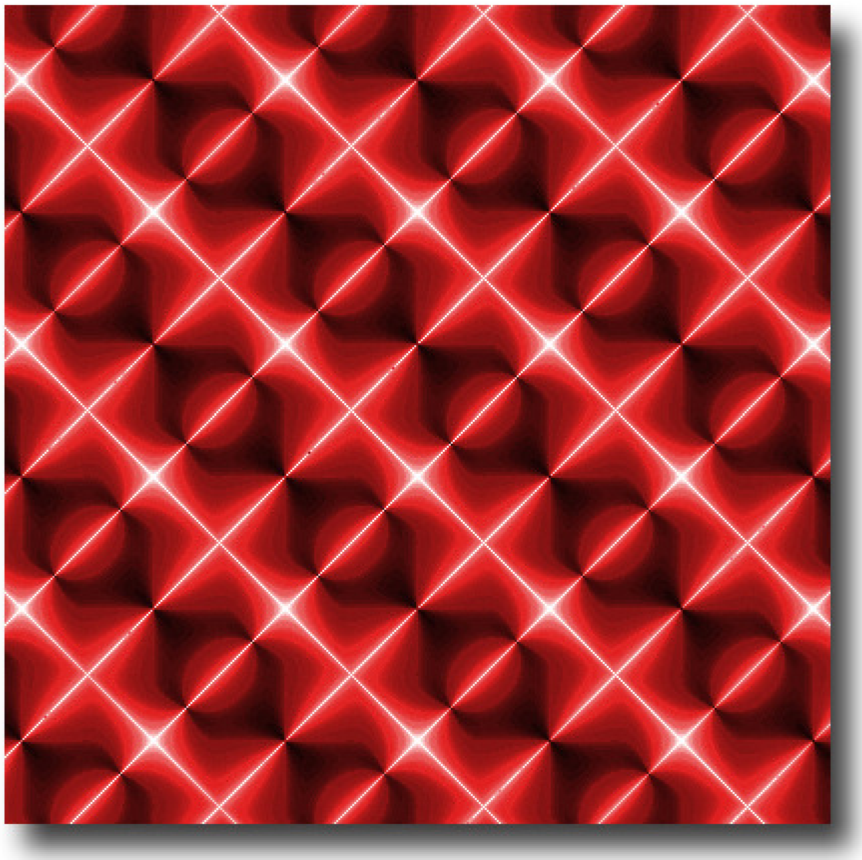


Quantum code (1)

Example : $\rho(z) = \dfrac{Cos(x+y)}{Cos(\frac{y}{x}) + Sin(\frac{x}{y})} + i \cdot \dfrac{Sin(x+y)}{Cos(\frac{x}{y}) + +Sin(\frac{y}{x})}$, $\delta = 10$, [-10,10], n=50

Quantum code (2)

Example : $\rho(z) = \dfrac{Cos(x+y)}{Cos(y)+Sin(x)} + i \cdot \dfrac{Sin(x+y)}{Cos(x)+Sin(y)}$, [-10,10]



Symmetry in red

Codependent contours: "Weave" two contours as follows:

$$z_{k,n} = z_{k-1,n} + \eta_n \varphi_1(\zeta_{k-1,n}) \text{ and } \zeta_{k,n} = \zeta_{k-1,n} + \eta_n \varphi_2(z_{k-1,n}) \quad ,$$

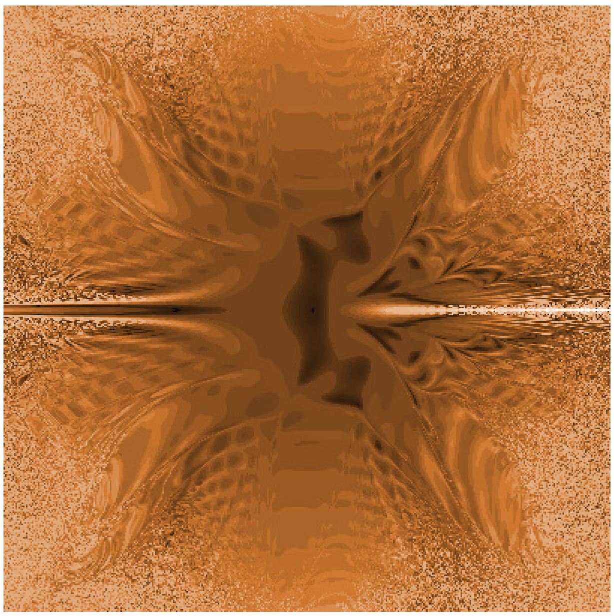
analogous to the system

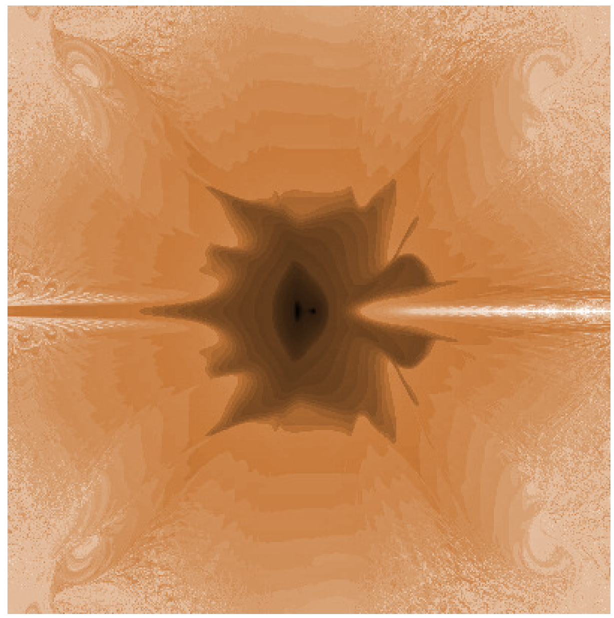
$$\frac{dz}{dt} = \varphi_1(\zeta) \text{ and } \frac{d\zeta}{dt} = \varphi_2(z) \ .$$

The idea can be extended to larger systems.

Example : $\dfrac{dz}{dt} = \varphi_1(\zeta, t) = \nu Cos(\tau) + i\tau Sin(\nu)$, $\zeta = \nu + \mathrm{i}\,\tau$, $\dfrac{d\zeta}{dt} = \varphi_2(z,t) = z^2 = (x^2 - y^2) + i2xy$

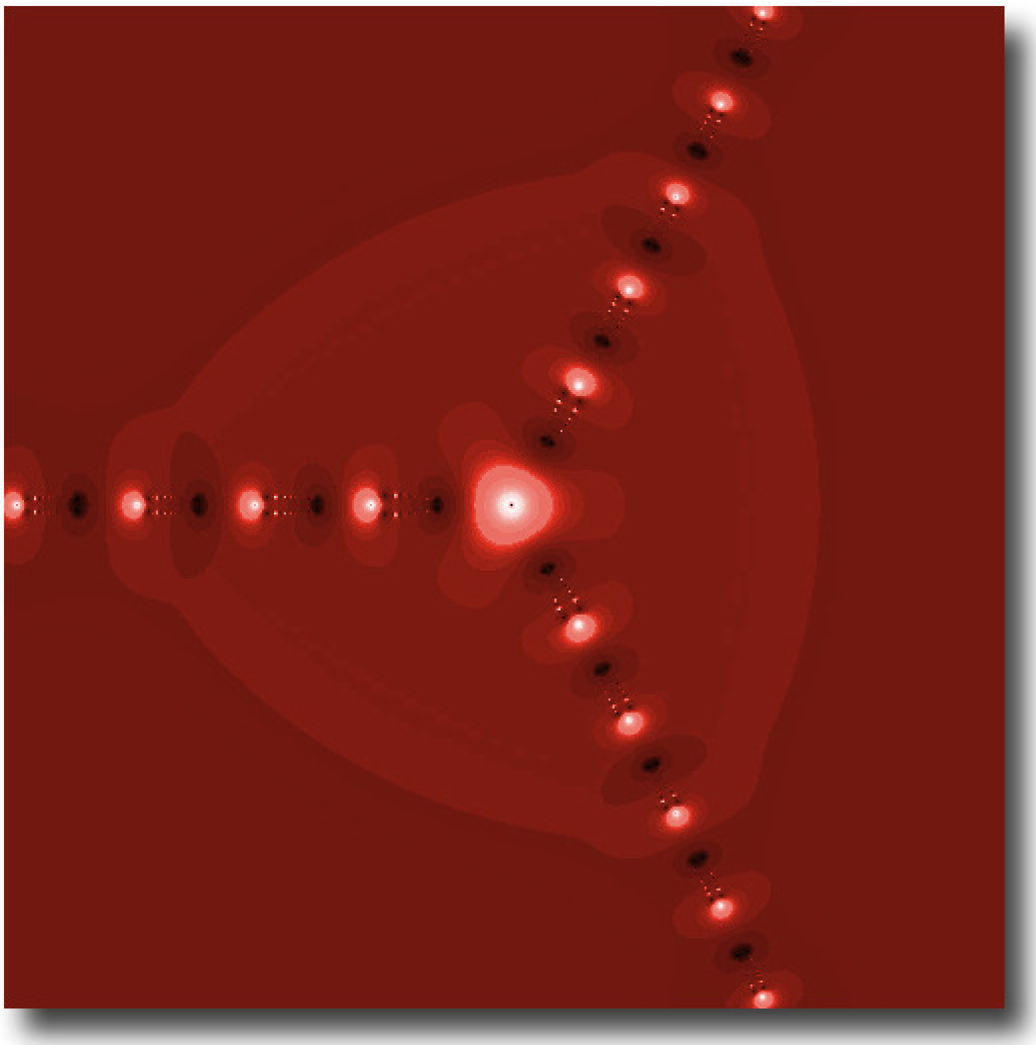
$\lambda(\alpha) = \int_0^1 \psi_1(\alpha, t)dt$: $\lambda(\alpha) = \int_0^1 \psi_2(\alpha, t)dt$: [-10,10] , n=30





Mystical moths

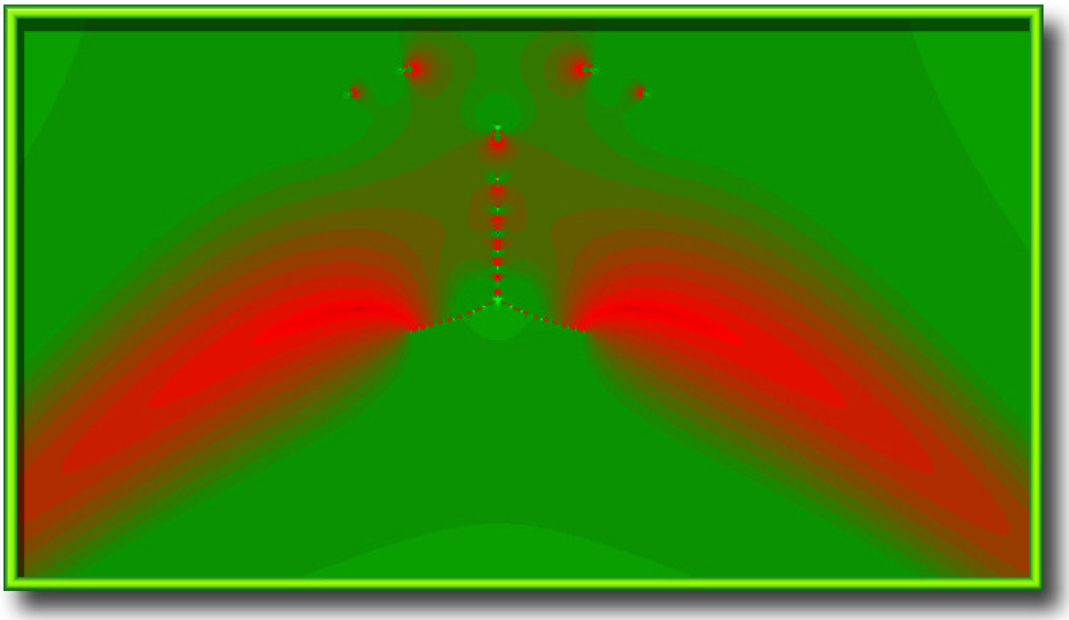
Example : $\dfrac{dz}{dt} = \varphi_1(\zeta) = \dfrac{1}{\zeta}$, $\dfrac{d\zeta}{dt} = \varphi_2(z) = \dfrac{z^2}{2}$, [-.1,.1] , n=50. $\lambda(\alpha) = \int_0^1 \psi_2(\alpha,t)dt$:



Electrostatics (magnified)

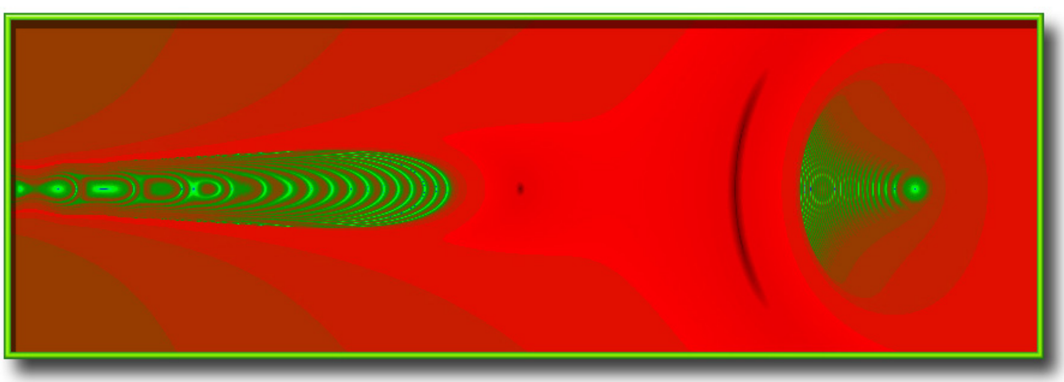
Example : $\dfrac{dz}{dt}=\varphi_1(\zeta)=\dfrac{1}{\zeta}$, $\dfrac{d\zeta}{dt}=\varphi_2(z)=\dfrac{z^2}{2}$. $\dfrac{d\omega}{dt}=\varphi_\omega(z,\zeta)=\dfrac{\varphi_1+\varphi_2}{2}$, [-1.5,1.5], n=50

$\lambda(\alpha)=\int_0^1 \psi_\omega(\alpha,t)dt$: (rotated)



Firebird

Example : $\dfrac{dz}{dt}=\varphi_1(\alpha+i\beta)=\dfrac{Cos(\alpha)-i\cdot Sin(\beta)}{\alpha^2+\beta^2}$, $\zeta=\alpha+i\beta$, $\dfrac{d\zeta}{dt}=\varphi_2(z)=\dfrac{z^2}{2}$, [-5,1]



Jet propulsion

References:

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J. Gill, *The Use of the Sequence . . . in Computing Fixed Points of . . .* , Appl. Numer. Math. 8 (1991)

The images were produced by graphics programs I have written in at least two versions of BASIC.