A Short Note on Expanding Functions as Outer Infinite Compositions

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Suppose a complex function can be expanded as an infinite composition in the following way:

(1) $F(z)=\lim_{n\to\infty}\left(f_1\circ f_2\circ\cdots\circ f_n(z)\right)$. (See [1])

And suppose the inverses of the $f_k(z)$ exist: $g_k(z)=f_k^{-1}(z)$.

Then, under suitable restrictions

(2) $G(z)=F^{-1}(z)=\lim_{n\to\infty}\left(g_n\circ g_{n-1}\circ\cdots g_1(z)\right)$, outer or left compositions.

Conditions for convergence of (1) include the following variation on theorems in [2]

***Theorem* 1** (*Gill* 2015) Suppose functions $\{f_n(z)\}$ are analytic in $S_0=\left(|z|<R_0\right)$ and $|f_n(z)-z|\le C\rho_n$, $0<\sigma=C\sum_{k=1}^{\infty}\rho_k<R_0<\infty$ in S_0 . Then there exists R such that $|z|<R<R+\sigma=R^*<R_0$ and $\lim_{n\to\infty}\left(f_1\circ f_2\circ\cdots\circ f_n(z)\right)=F(z)$ for $|z|<R$. Convergence is uniform on compact subsets.

Sketch of Proof: Set $F_{n,n+m}(z)=f_n\circ f_{n+1}\circ\cdots\circ f_{n+m}(z)$. Then it is easily determined that $|F_{n,n+m}(z)|<R+\sigma=R^*$. Now write $f_n(z)=z+\nu_n(z)$, $|\nu_n(z)|<C\rho_n$. Set $r=R_0-R^*$ and use the Cauchy Integral Formula to estimate the magnitude of $|\nu_n'(z)|$: $|\nu_n'(z)|\le\frac{C}{r}\rho_n$. Therefore $|f_n(z)-f_n(\zeta)|\le|z-\zeta|+|\nu_n(z)-\nu_n(\zeta)|<|z-\zeta|+\frac{C}{r}\rho_n|z-\zeta|=|z-\zeta|\left(1+\frac{C}{r}\rho_n\right)$ by the Fundamental Theorem. Then $|F_{1,n}(z)-F_{1,n}(\zeta)|<|z-\zeta|\cdot\prod_{k=1}^{n}\left(1+\frac{C}{r}\rho_k\right)$ and

$$|F_{1,n}(z)-F_{1,n+m}(z)|=|F_{1,n}(z)-F_{1,n}(f_{n+1}\circ\cdots\circ f_{n+m}(z))|<|z-f_{n+1}\circ\cdots\circ f_{n+m}(z)|\cdot\prod_{k=1}^{n}\left(1+\frac{C}{r}\rho_k\right)$$

Where $|f_{n+1}\circ\cdots\circ f_{n+m}(z)-z|\le\sum_{k=1}^{m-1}|f_{n+k}(F_{n+k+1,n+m}(z))-F_{n+k+1,n+m}(z)|+|f_{n+m}(z)-z|<\sum_{k=1}^{\infty}C\rho_{n+k}$

Thus $|F_{1,n}(z)-F_{1,n+m}(z)|<C\cdot\sum_{k=1}^{\infty}\rho_{n+k}\cdot\prod_{k=1}^{n}\left(1+\frac{C}{r}\rho_k\right)\ \to\ 0$ as $n\to\infty$. ||

We have, in an inexact sense,

$$|f_n(z)-z|\le C\rho_n \quad \Rightarrow \quad f_n(z)=z+\nu_n(z)\ ,\ \ |\nu_n(z)|\le C\rho_n$$, and

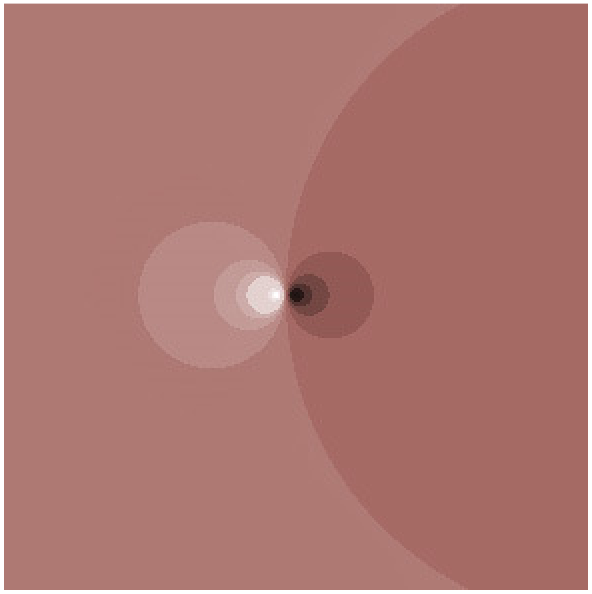
$$z=g_n(f_n(z))=g_n(z+\nu_n(z))=g_n(\omega)=\omega-\nu_n(z) \ \Rightarrow\ |g_n(\omega)-\omega|<C\rho_n$$

And the following

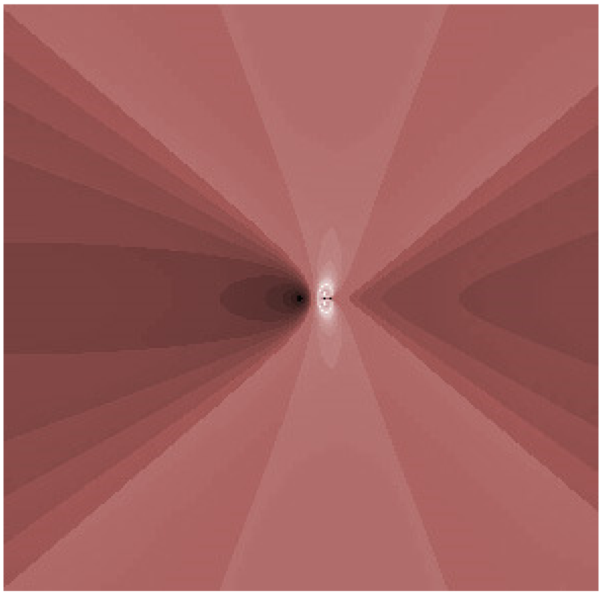
***Theorem* 3** (Gill, 2011) Let $\{g_n\}$ be a sequence of complex functions defined on $S=(|z|<M)$. Suppose there exists a sequence $\{\rho_n\}$ such that $\sum_{k=1}^{\infty}\rho_n<\infty$ and $|g_n(z)-z|<C\rho_n$ if $|z|<M$. Set $\sigma=C\sum_1^{\infty}\rho_k$ and $R_0=M-\sigma>0$. Then, for every $z\in S_0=\left(|z|<R_0\right)$, $G_n(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z) \ \rightarrow\ G(z)$, uniformly on compact subsets of S_0

The following are simple topographical images.

Example 1 : $f_n(z)=\dfrac{z}{1+\rho_n z}$, $g_n(z)=f_n^{-1}(z)=\dfrac{z}{1-\rho_n z}$, $\rho_n=\dfrac{1}{n^2}$ -10<x,y<10 n=40

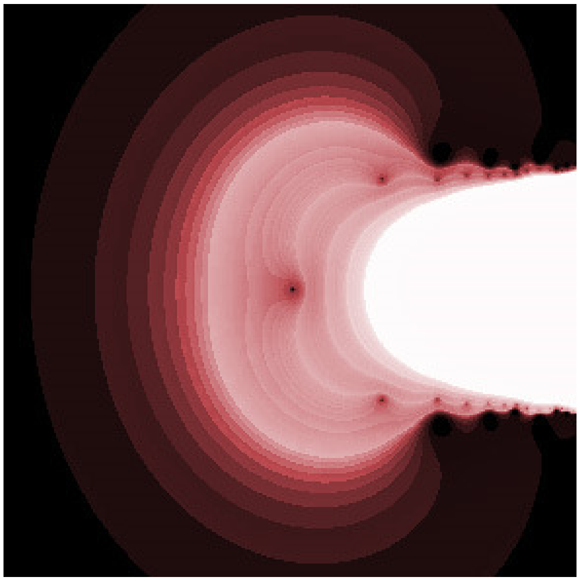


F(z)

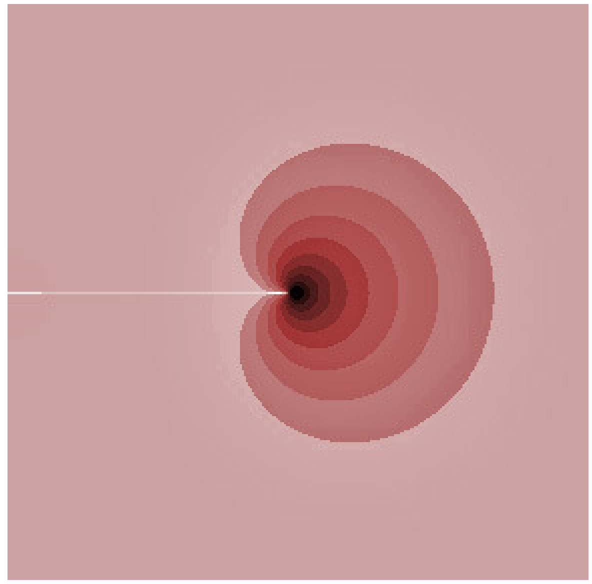


G(z)

Example 2 : $f_n(z) = z + \rho_n z^2$, $g_n(z) = \dfrac{2z}{1+\sqrt{1+4\rho_n z}}$, $\rho_n = \dfrac{1}{n^2}$ -10<x,y<10 n=40

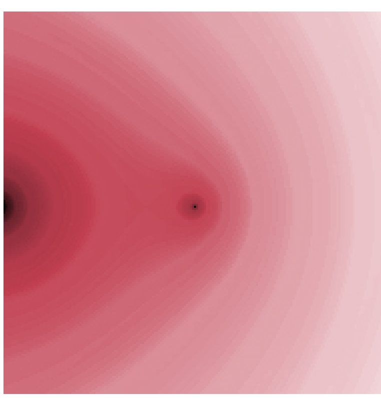


F(z)

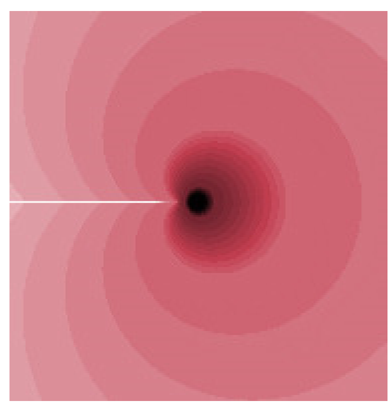


G(z)

Example 3 : $f_n(z) = z + \rho_n z^2$, $g_n(z) = \dfrac{2z}{1+\sqrt{1+4\rho_n z}}$, $\rho_n = \dfrac{1}{4^n}$ -10<x,y<10 n=40

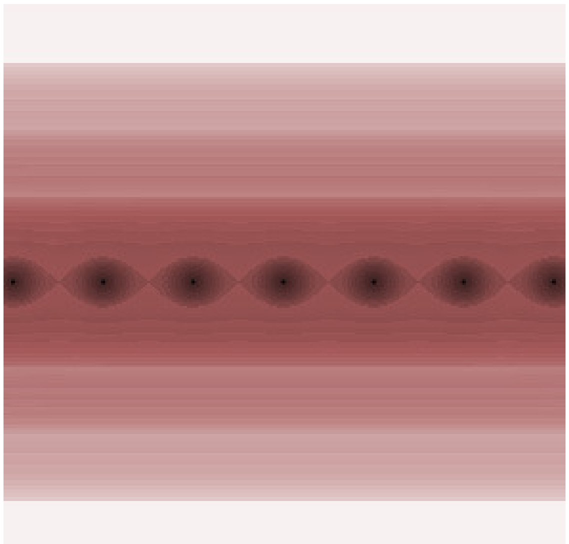


F(z)

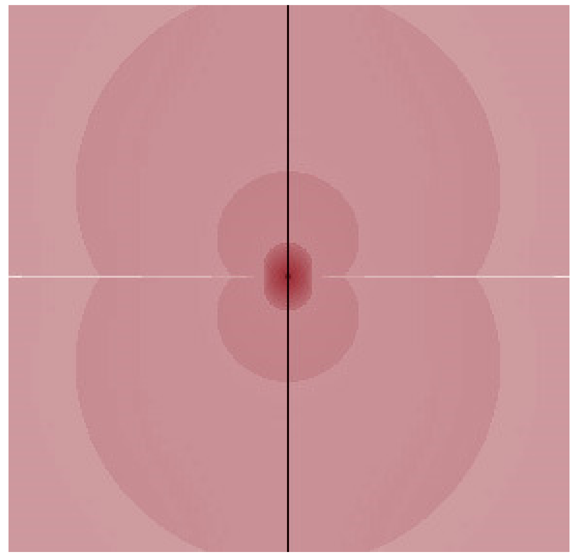


G(z)

Example 4: $Sin(z): f_n(z) = z\sqrt{1-4^{-n}z^2}$, $ArcSin(z): g_n(z) = \sqrt{\dfrac{2z^2}{1+\sqrt{1-4^{1-n}z^2}}}$ -10<x,y<10 n=40

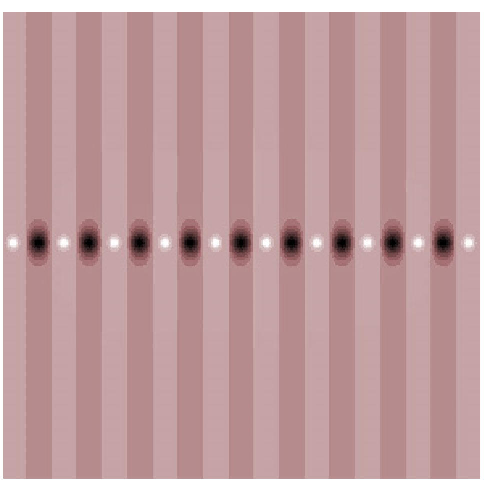


Sin(z)

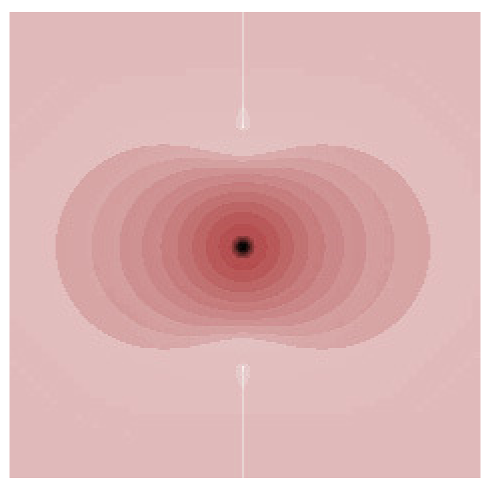


ArcSin(z)

Example 5 : $Tan(z): f_n(z) = \dfrac{z}{1-4^{-n}z^2}$, $ArcTan(z): g_n(z) = \dfrac{2z}{1+\sqrt{1+4^{-n}z^2}}$ n=40

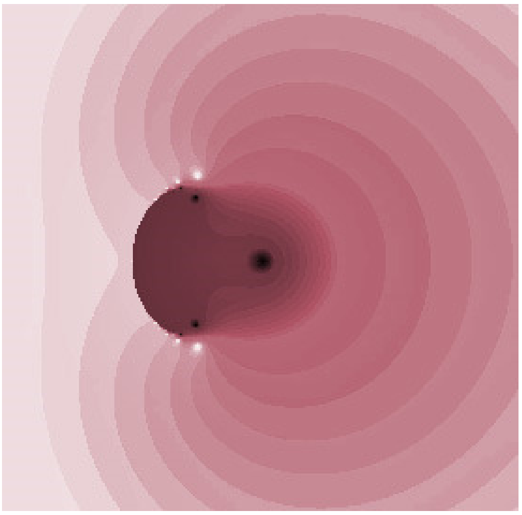


F(z) -15<x,y<15

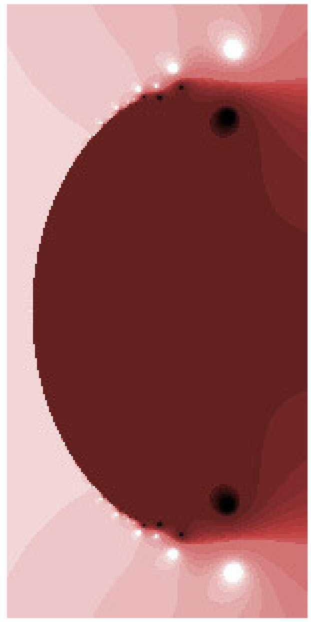


G(z) -2<x,y<2

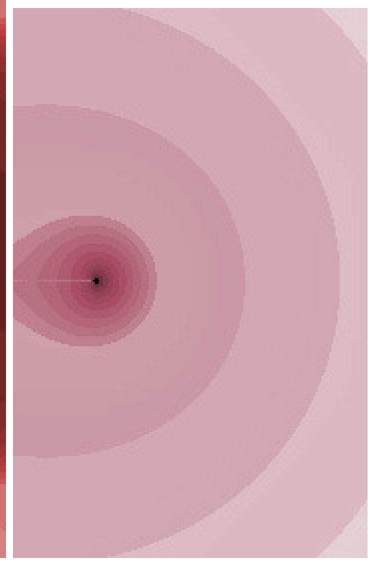
Example 6: $f_n(z) = z + \rho_n \frac{z^2}{1+z}$, $g_n(z) = \frac{2z}{\sqrt{z^2 + 2(1+2\rho_n)z + 1} \; + \; 1 - z}$, $\rho_n = \frac{1}{n^2}$ n=40



F(z) -2<x,y<2

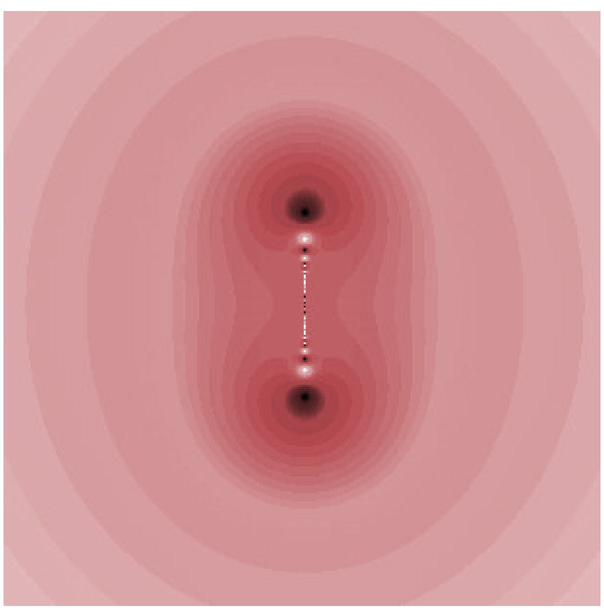


F(z) enlarged

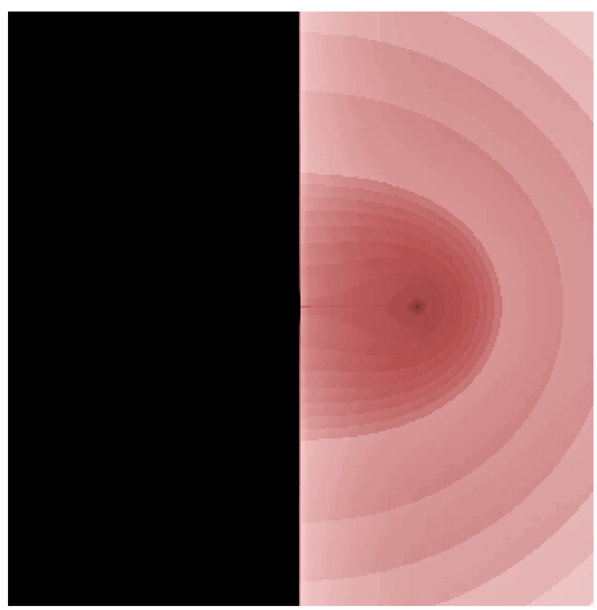


G(z) -3<x<10

Example 7 : $f_n(z) = z + \frac{\rho_n}{z}$, $g_n(z) = \frac{z + \sqrt{z^2 - 4\rho_n}}{2}$, $\rho_n = \frac{1}{2^n}$ n=40

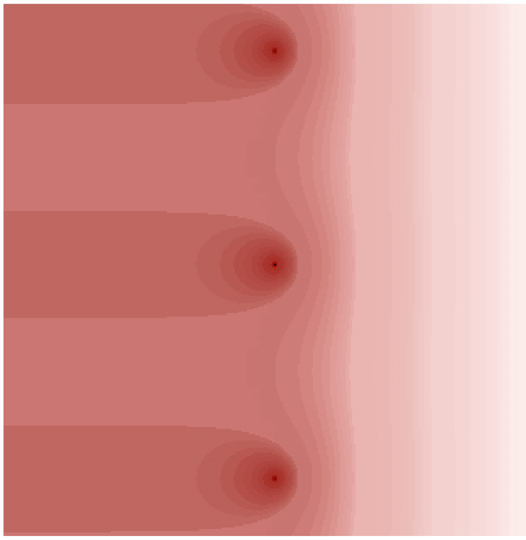


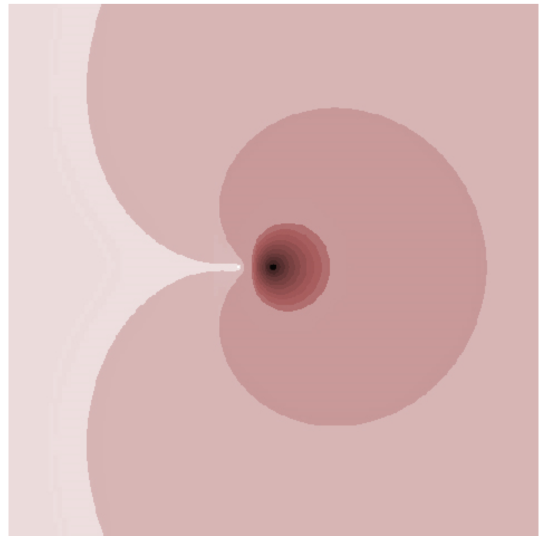
F(z) -4<x,y<4



G(z) Re(z)<0 iterates to zero.

Example 8 : $e^z - 1$: $f_n(z) = z + \frac{z^2}{2^{n+1}}$, $Ln(1+z)$: $g_n(z) = \frac{2z}{1+\sqrt{1+4\left(\frac{1}{2^{n+1}}\right)z}}$ n=40





F(z) -8<x,y<8 G(z)

[1] J. Gill, *A Note on Expanding Functions into Infinite Compositions*, Comm. Anal. Th. Cont. Frac., Vol XX (2014)

[2] J. Gill, *Convergence of infinite compositions of complex functions*, Comm. Anal. Th. Cont. Frac., Vol XIX (2012)