A Primer on the Elementary Theory of Infinite Compositions of Complex Functions

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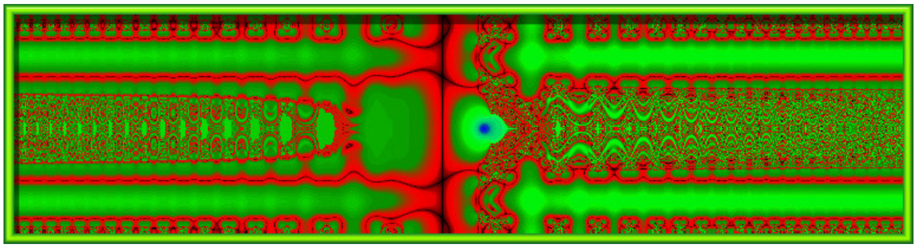

Abstract: *Elementary* means not requiring the complex functions be holomorphic. Theorem proofs are fairly simple and are outlined. This is a brief compilation of such basic results, with examples, prefaced by three fundamental theorems about holomorphic functions.

Beyond simple iteration of a single function, or even a finite sequence of functions, results appearing here refer to the following:

Inner compositions: $F_n(z) = \mathop{\mathcal{R}}_{k=1}^{n} f_k(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$, $F(z) = \lim_{n\to\infty} \mathop{\mathcal{R}}_{k=1}^{n} f_k(z)$, <u>or</u>

$$F_n(z) = \mathop{\mathcal{R}}_{k=1}^{n} f_{k,n}(z) = f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{n,n}(z), F(z) = \lim_{n\to\infty} \mathop{\mathcal{R}}_{k=1}^{n} f_{k,n}(z).$$

Outer compositions: $G_n(z) = \mathop{\mathcal{L}}_{k=1}^{n} g_k(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$, $G(z) = \lim_{n\to\infty} \mathop{\mathcal{L}}_{k=1}^{n} g_k(z)$, <u>or</u>

$$G_n(z) = \mathop{\mathcal{L}}_{k=1}^{n} g_{k,n}(z) = g_{n,n} \circ g_{n-1,n} \circ \cdots \circ g_{1,n}(z), \; G(z) = \lim_{n\to\infty} \mathop{\mathcal{L}}_{k=1}^{n} g_{k,n}(z).$$

As background, we begin with three fundamental theorems for *analytic* functions. By requiring analyticity, *Lipshitz contraction*, a frequent condition for iteration convergence, can be replaced with simple *domain contraction*. *Schwarz's lemma* is key to these results.

Theorem 1. (Henrici [1][1974]) Let f be analytic in a simply-connected region S and continuou on the closure $\overline{S}$ of S. Suppose $f(\overline{S})$ is a bounded set contained in S. Then $F_n(z) = f \circ f \circ \cdots \circ f(z) = \underset{k=1}{\overset{n}{\mathcal{R}}} f(z) \to \alpha$, the *attractive fixed point* of f in S.

Example: $f(z) = \frac{1}{4}\left(2z^2 + 1\right)$, $S = \left(|z| < \frac{3}{2}\right)$. Then $F_n(z) \to \alpha = 1 - \sqrt{\frac{1}{2}}$.

For Inner compositions, there is:

Theorem 2 : (Lorentzen [2][1990]) Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain S and continuous on the closure of S. Suppose there exists a compact set $\Omega \subset S$ such that for each k, $f_k(S) \subset \Omega$. Then $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z) = \underset{k=1}{\overset{n}{\mathcal{R}}} f_k(z) \to \alpha$, a constant, uniformly on S.

Example: The simple continued fraction $\frac{a_1\zeta}{1\ +}\ \frac{a_2\zeta}{1\ +} \cdots$ is generated by

$f_k(z) = \frac{a_k\zeta}{1+z}$, $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$. If $|\zeta| < 1$, $|z| < R < 1$, then

$|a_n| < \rho R(1-R) \Rightarrow |f_n(z)| < \rho R$, $\rho < 1$. For example, $R = \frac{1}{2}$ implies

$F_n(z) \to \alpha(\zeta)$ for $|\zeta| < 1$.

For Outer compositions, there is:

Theorem 3 : (Gill [3][1991]) Let $\{g_n\}$ be a sequence of functions *analytic* on a simply-connected domain D and continuous on the closure of D. Suppose there exists a compact set $\Omega \subset \mathrm{D}$ such that $g_n(D) \subset \Omega$ for all n. Define $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$. Then $\mathrm{G_n(z)} \to \alpha$ uniformly on the closure of D *if and only if* the sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in Ω converge to the number α.

Example: let $G(z)=\cfrac{\frac{e^z}{4}}{3+z+}\cfrac{\frac{e^z}{8}}{3+z+}\cfrac{\frac{e^z}{12}}{3+z+}\cdots$, where $|z|\le 1$. We solve the *continued fraction equation* $G(\alpha) = \alpha$ in the following way:

Set $t_n(\xi)=\dfrac{e^z/4n}{3+z+\xi}$; let $g_n(z)=t_1\circ t_2\circ\cdots\circ t_n(0)$. Now calculate $G_n(z) = g_n\circ g_{n-1}\circ\cdots\circ g_1(z)$ starting with z = 1. One obtains $\alpha = .087118118\cdots$ to ten decimal places after ten iterations.

Additional theorems on entire functions appear in the addendum. The results that follow do not require analyticity. Some of these appear in [4].

Theorem 4: Let $\{t_n\}$ be a family of complex functions defined on S, a simply-connected compact set in $\mathbb{C}$, having the properties $t_n(S)\subseteq S$ and $|t_n(z_1)-t_n(z_2)|<\rho|z_1-z_2|$, $\rho<1$, for all n, uniformly on S. Set

$$G_n(z)=t_n\circ t_{n-1}\circ\cdots\circ t_1(z) \quad\text{and}\quad F_n(z)=t_1\circ t_2\circ\cdots\circ t_n(z)$$

Then $F_n(z)\to\beta\in S$ uniformly on S. If $t_n(\alpha_n)=\alpha_n$, the unique fixed points of t_n, $G_n(z)\to\alpha$ uniformly on S if and only if $|\alpha_n-\alpha|=\varepsilon_n\to 0$.

Sketch of proof: $|F_{n+m}(z)-F_n(z)|<\rho^n|z-t_{n+1}\circ\cdots\circ t_{n+m}(z)|\le\rho^n\cdot Diam(S)\to 0$ implies $F_n(z_0)\to\beta$. Also $|F_n(z)-F_n(z_0)|\to 0$ uniformly on S. Next, Set $\sigma_n=\varepsilon_n+\varepsilon_{n+1}$. Then

$$|G_n(z)-\alpha|<\rho^n|z-\alpha_1|+\varepsilon_n+\sum_{k=1}^{n-1}\left(\rho^{n-k}\sigma_k\right)\to 0,\ n\to\infty.$$

To show $G_n(z)\to\alpha$ uniformly on S implies $\alpha_n\to\alpha$, assume there exists $\{\alpha_{n_k}\}_{k=1}^{\infty}$ such that $|\alpha_{n_k}-\alpha|>r>0$. Now suppose n is sufficiently large that $|G_n(z)-\alpha|<\varepsilon$ for $\varepsilon<\dfrac{1-\rho}{1+\rho}\cdot r$. For $n_k>n+1$

$$|G_{n_k}(z)-\alpha_{n_k}|<\rho|G_{n_k-1}(z)-\alpha|+\rho|\alpha-\alpha_{n_k}|<\rho\varepsilon+\rho|\alpha-\alpha_{n_k}| \quad\text{and}$$

$|G_{n_k}(z)-\alpha_{n_k}|>|\alpha-\alpha_{n_k}|-|G_{n_k}(z)-\alpha|$. Therefore $|G_{n_k}(z)-\alpha|>(1-\rho)r-\rho\varepsilon>\varepsilon$ $(\rightarrow\leftarrow)$#

Example : $t_k(z)=\frac{1}{2}x+i\left(\frac{k}{4(k+1)}y-\frac{1}{8}\right)$, $S=\{z:|x|<1,|y|<1\}$. Thus $\rho=\frac{1}{2}$, $|t_k(z)|<\frac{7}{8}$.

Then $F_n(z)\to -.14384104i$, $n=20$ and $G_n(z)\to \alpha=-\frac{1}{6}i$, slowly.

Theorem 5: Let S be a bounded simply-connected domain and $t_k(S)\subseteq S$, $t_k(\alpha_k)=\alpha_k\in S$, $\alpha_k\to\alpha\in S$. Furthermore, assume

$$|t_k(z)-\alpha_k|<\rho|z-\alpha_k|,\ \rho<1,\ \forall k.$$

Set $T_n(z)=t_n\circ t_{n-1}\circ\cdots\circ t_1(z)$. Then $T_n(z)\to\alpha$ uniformly on S .

Sketch of proof: $|T_n(z)-\alpha|\le|\alpha-\alpha_n|+\rho^n|z-\alpha|+\sum_{k=1}^{n-1}\left(\rho^{n-k}\varepsilon_k\right)\ \to\ 0$, $\varepsilon_k=|\alpha_k-\alpha_{k+1}|$. #

Example : $t_k(z)=\frac{1}{2}\left(z-\frac{k}{k+1}i\right)+\frac{k}{k+1}\mathrm{i}\ \Rightarrow\ T_k(z)\to i$

Theorem 6: Suppose $g_n(z)=z+\rho_n\cdot\varphi(z)$ where there exist $R>0$ and $M>0$ such that $|z|<R\ \Rightarrow\ |\varphi(z)|<M$. Furthermore, suppose $\rho_k\ge 0$, $\sum_1^\infty\rho_k<\infty$ and $R>M\cdot\sum_{k=1}^\infty\rho_k$. Then there exists $0<R^*<R$ such that $G_n(z)\equiv g_n\circ g_{n-1}\circ\cdots\circ g_1(z)\ \to\ G(z)$ for $\{z:|z|<R^*\}$.

Sketch of proof: Assume momentarily that R^* exists. Then

$$|g_n(z)|\le\ |z|+\rho_nM\ <\ R^*+\rho_nM$$

$$|g_{n+1}\circ g_n(z)|\le\ |z|+M(\rho_n+\rho_{n+1})\ <\ R^*+M(\rho_n+\rho_{n+1})$$

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$$|g_{n+m}\circ g_{n+m-1}\circ\cdots\circ g_n(z)|\le\ |z|+M\sum_{k=n}^{n+m}\rho_k\ <\ R^*+M\sum_{k=n}^{n+m}\rho_k\le\ R^*+M\sum_{k=1}^{\infty}\rho_k$$

with $R^*<\ R-M\sum_{k=1}^\infty\rho_k$. Next,

$$\left|G_{n+1}(z)-G_n(z)\right|=\left|g_{n+1}(G_n)-G_n\right|<\rho_{n+1}M$$
$$\left|G_{n+2}(z)-G_n(z)\right|\le\left|g_{n+2}(G_{n+1})-G_{n+1}\right|+\left|g_{n+1}(G_n)-G_n\right|<(\rho_{n+2}+\rho_{n+1})M$$

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$$\left|G_{n+m}(z)-G_n(z)\right|<M\cdot\sum_{k=n}^{\infty}\rho_k\to 0 \text{ as } n\to\infty$$

There is no requirement that φ be analytic or even continuous, merely that it contract as described. #

Example : $g_k(z)=z+\frac{1}{3^k}\left(xCos(y)+iySin(x)\right)$,

Therefore $\sum_1^{\infty}\rho_k=\frac{1}{2}$, $|z|<R=1 \Rightarrow |\varphi(z)|<M=1 \Rightarrow R^*=\frac{1}{2}$.

Hence $G_n(-.1+.4i)=-.12440508+.38950871i,\ n=20$.

Theorem 7: Suppose $f_n(z)=z+\rho_n\cdot\varphi(z)$ where there exist $R>0$ and $M>0$ such that $|z|<R$ and $|\zeta|<\mathrm{R} \Rightarrow |\varphi(z)|<M$ and $|\varphi(z)-\varphi(\zeta)|<r|z-\zeta|$. Furthermore, suppose $\rho_k\ge 0$, $\sum_1^{\infty}\rho_k<\infty$ and $R>M\cdot\sum_{k=1}^{\infty}\rho_k$. Then there exists $0<R^*<R$ such that

$$F_n(z)\equiv f_1\circ f_2\circ\cdots\circ f_n(z) \ \to\ \mathrm{F}(z) \text{ for } \{z:|z|<R^*\}.$$

Sketch of proof: (similar to the previous proof)

$$\left|f_{n+m}(z)-z\right|<M\rho_{n+m}\Rightarrow\left|f_{n+m}(z)\right|<R^*+M\rho_{n+m}<R$$
$$\left|f_{n+m-1}\circ f_{n+m}(z)-z\right|\le\left|f_{n+m-1}(f_{n+m}(z))-f_{n+m}(z)\right|+\left|f_{n+m}(z)-z\right|<M\rho_{n+m-1}+M\rho_{n+m}$$
$$\Rightarrow\left|f_{n+m-1}\circ f_{n+m}(z)\right|<R^*+M(\rho_{n+m-1}+\rho_{n+m})<R$$

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$$\left|f_n\circ f_{n+1}\circ\cdots\circ f_{n+m}(z)-z\right|<M\cdot\sum_{k=n}^{n+m}\rho_k$$

Next, for $R^* < R - M\sum_{k=1}^{\infty}\rho_k$,

$$\left|F_{n+m}(z)-F_n(z)\right| < \prod_{k=1}^{n}(1+r\rho_k)\cdot\left|f_{n+m}\circ f_{n+m-1}\circ\cdots\circ f_n(z)-z\right| < M\cdot\prod_{k=1}^{n}(1+r\rho_k)\cdot\sum_{k=n+1}^{n+m}\rho_k = M\cdot S_n(m)$$

With $S_n(m)\to 0$ as $n\to\infty$. #

Example : $f_k(z)=z+\frac{1}{3^k}\left(Cos(x)+iSin(y)\right),\ |z|<R=1 \ \Rightarrow\ r=1$

$\Rightarrow\ |\varphi(z)|<1\ \Rightarrow\ R^*=\frac{1}{2}.\ \ F_n(z)\to .14971038+.50302079i,\ \ n=20.$

Theorem 8: Suppose $t_k(z)=z\left(1+\rho_k\varphi_k(z)\right)$. Then set

(1) $G_n(z)=z\cdot\prod_{k=1}^{n}\left(1+\rho_k\varphi_k(G_{k-1}(z))\right),\ G_0(z)=z,\ G_k(z)=t_k\circ t_{k-1}\circ\cdots\circ t_1(z)$ <u>and</u>

(2) $\mathrm{F}_n(z)=z\cdot\prod_{k=n}^{1}\left(1+\rho_k\varphi_k(F_{k+1,n}(z))\right),\ \mathrm{F}_{n+1,n}(z)=z,\ \mathrm{F}_{k+1,n}(z)=t_{k+1}\circ t_{k+2}\circ\cdots\circ t_n(z)$ with

$|z|<R\ \Rightarrow\ |\varphi_n(z)|<M$, $\sum_{1}^{\infty}\rho_k<\infty$. Then there exists $0<R^*<R$ such that $S=\{z:|z|<R^*\}\ \Rightarrow$

$G_n(z)\to G(z)$. If, in addition, $|\varphi(z_1)-\varphi(z_2)|<r|z_1-z_2|$, then $F_n(z)\to F(z)$.
Both uniformly on S .

Sketch of proof: $|t_1(z)|<R^*(1+\rho_1 M)<R\ \Leftrightarrow\ R^*<\frac{R}{1+\rho_1 M}$ . . .

$|G_n(z)|,|F_n(z)|<R^*\prod_{1}^{n}(1+\rho_k M)<R\ \Rightarrow\ R^*<\frac{R}{\prod_{1}^{\infty}(1+\rho_k M)}$. Thus

$\left|G_{n+m}(z)-G_n(z)\right| < MR\sum_{k=n}^{\infty}\rho_k\to 0,\ n\to\infty.$ And

$\left|F_{n+m}(z)-F_n(z)\right|<\prod_{k=1}^{m}\left(1+\rho_k\left(M+rR\right)\right)\cdot MR\sum_{k=n+1}^{\infty}\rho_k\ \to\ 0,\ n\to\infty$. #

Example : $g_k(z)=z\left(1+\frac{1}{2^k}\left(xCos(y)+iySin(x)\right)\right),\ \ |z|<R=1\ \Rightarrow\ |\varphi(z)|<M=1$

$\prod_{k=1}^{\infty}\left(1+\frac{1}{2^k}M\right)<2.4\ \Rightarrow\ R^*<.42\ ;\ \ G_{100}(.2+.3i)\approx .218362+.377394i$.

Example : Expanding functions as infinite compositions [6] frequently involves a special case of theorem 8. For instance, $F(z)=e^z-1$ can be expanded by writing $F(2z)=F(z)(F(z)+2)$.

This leads to $e^z=1+\underset{k=1}{\overset{\infty}{\mathcal{R}}}\left(z+\frac{z^2}{2^{k+1}}\right)=1+\underset{k=1}{\overset{\infty}{\mathcal{R}}}\left(z\left(1+\frac{z}{2^{k+1}}\right)\right)$.

Example : $Ln(z+1)=\underset{k=1}{\overset{\infty}{\mathcal{L}}}\left(z\left(1+\frac{1}{2^{k+1}}\cdot\left(\frac{-4z}{\left(1+\sqrt{1+\frac{4}{2^{k+1}}z}\right)^2}\right)\right)\right)$

Example : $Tan(z)=\underset{k=1}{\overset{\infty}{\mathcal{R}}}\left(z\left(1+\frac{1}{4^k}\cdot\left(\frac{z^2}{1-\frac{1}{4^k}z^2}\right)\right)\right)$

Example : $ArcTan(z)=\underset{k=1}{\overset{\infty}{\mathcal{L}}}\left(z\left(1+\frac{1}{4^k}\cdot\left(\frac{-z^2}{\left(1+\sqrt{1+\frac{z^2}{4^k}}\right)^2}\right)\right)\right)$

Example : $Sin(z)=\pm\underset{k=1}{\overset{\infty}{\mathcal{R}}}\left(z\left(1+\frac{1}{4^k}\cdot\left(\frac{-z^2}{1+\sqrt{1-\frac{z^2}{4^k}}}\right)\right)\right)$

Theorem 9: Let $t_k(z)=\frac{\rho_k(z)}{\rho_k(z)+1-z}$. Set $P_n(z)=\prod_{k=1}^{n}\rho_k(z)$.

For $G_n(z)=t_n\circ t_{n-1}\circ\cdots\circ t_1(z)$, set $S_n(z)=\sum_{k=1}^{n}\left(\prod_{j=k}^{n}\rho_j(z)\right)$.

Then $G_n(z)=\frac{S_n(z)(1-z)+zP_n(z)}{S_n(z)(1-z)+1+z(P_n(z)-1)}$.

For $F_n(z)=t_1\circ t_2\circ\cdots\circ t_n(z)$, set $T_n(z)=\sum_{k=1}^{n}\left(\prod_{j=1}^{k}\rho_j(z)\right)$.

Then $F_n(z)=\frac{T_n(z)(1-z)+zP_n(z)}{T_n(z)(1-z)+1+z(P_n(z)-1)}$

Comment: When the $\rho_k(z)=\rho_k \ \forall k$ are constants and $z=0$ this formula reduces to *Euler's equivalent (reverse and forward) continued fraction*. The proof is purely computational. As a reverse continued fraction the expansion is self-generating.

Example : $\rho_k(z)\equiv\rho,\ |\rho|<1 \ \Rightarrow \ \lim_{n\to\infty} F_n(z)=\lim_{n\to\infty} G_n(z)=\begin{cases}\rho, & z\neq 1\\ 1, & z=1\end{cases}$

Example : $\rho_k(z)\to\rho,\ |\rho|<1 \ \Rightarrow \ \lim_{n\to\infty} G_n(z)=\begin{cases}\rho, & z\neq 1\\ 1, & z=1\end{cases},\ \lim_{n\to\infty} F_n(z)=\begin{cases}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\prod_{j=1}^{k}\rho_j\right), & z\neq 1\\ 1, & z=1\end{cases}$

Example : $\rho_k(z)=(\frac{k}{k+1})^2 Cos(x)+i(\frac{k}{k+1})^2 Sin(y),\ z=.3-.4i$

$$G_{10,000}(z)=.73899-.003i,\quad F_{10,000}(z)=.44669-.00647i$$

Theorem 10: Let $\{f_n\}$ be a family of complex functions defined on S, a simply-connected compact set in $\mathbb{C}$, having the properties $f_n(S)\subseteq S$ and $|f_n(z_1)-f_n(z_2)|<\rho|z_1-z_2|,\ \rho<1$, for all n, uniformly on S. Set $F_n(z)=f_1\circ f_2\circ\cdots\circ f_n(z)$. Consider now a sequence $\{f_{k,n}\}_{1\leq k\leq n}$ such that $\lim_{n\to\infty} f_{k,n}(z)=f_k(z)$ uniformly on S. Write $F_{k,n}(z)=f_{1,n}\circ f_{2,n}\circ\cdots\circ f_{k,n}(z)$.
Then $\lim_{n\to\infty} F_{n,n}(z)=\lim_{n\to\infty} F_n(z)=\lambda$ for $z\in S$.

Sketch of Proof: The convergence of $F_n(z)=f_1\circ f_2\circ\cdots\circ f_n(z)$ is easily determined.

$$|F_n(z)-F_{n+p}(z)|=\left|f_1\circ\cdots\circ f_n(z)-f_1\circ\cdots\circ f_n\left(f_{n+1}\circ\cdots\circ f_{n+p}(z)\right)\right|\leq\rho^n\left|z-f_{n+1}\circ\cdots\circ f_{n+p}(z)\right|\to 0$$

And $|F_n(z_1)-F_n(z_2)|\leq\rho^n|z_1-z_2|\to 0$. Set $Z_{p,n}=f_{p+1,n}\circ f_{p+2,n}\circ\cdots\circ f_{n,n}(z)$. Then

$$|F_{n,n}(z)-\lambda|=|F_{p,n}(Z_{p,n})-\lambda|\leq|F_{p,n}(Z_{p,n})-F_p(Z_{p,n})|+|F_p(Z_{p,n})-\lambda|$$

For the second term in the inequality, choose and fix p sufficiently large that $|F_p(z)-\lambda|<\dfrac{\varepsilon}{2}$ for all $z\in S$. For the first term, choose n sufficiently large to insure $|F_{p,n}(z)-F_p(z)|<\dfrac{\varepsilon}{2}$. This is true since a finite composition of functions of the type described above, converging uniformly on S, will also converge uniformly on S. #

Example : $F_n(z) = \frac{\rho_n a_1 \zeta}{1\ +}\ \frac{\rho_n a_2 \zeta}{1\ +}\ \frac{\rho_n a_3 \zeta}{1\ +}\ \cdots\ \frac{\rho_n a_n \zeta}{1+z}$, $\rho_n = 1 - \frac{1}{n}$. $f_{k,n}(z) = \frac{\rho_n a_k \zeta}{1+z}$.

$f_{k,n}(z) \to f_k(z) = \frac{a_k \zeta}{1+z}$. Suppose $|z| < R < \frac{1}{2}$, $|a_k| < R$, $|\zeta| < R$ and. Then $|f_{k,n}(z)| < R$

and $|f_{k,n}(z_1) - f_{k,n}(z_2)| < r|z_1 - z_2|$, $r < 1$. Also $|f_{k,n}(z) - f_k(z)| < R \cdot \frac{1}{n} \to 0$.

Therefore $\lim_{n\to\infty} F_{n,n}(z) = \lim_{n\to\infty} F_n(z) = \lambda(\zeta)$.

Theorem 11: Let $\{g_{k,n}\}_{1\le k\le n}$ be a family of complex functions defined on S, a simply-connected compact set in $\mathbb{C}$, having the properties $g_{k,n}(S) \subseteq S$ and $|g_{k,n}(z_1) - g_{k,n}(z_2)| < \rho|z_1 - z_2|$, $\rho < 1$, $1 \le k \le n$. Write $G_{k,n}(z) = g_{k,n} \circ g_{k-1,n} \circ \cdots \circ g_{1,n}(z)$ and let $\{\alpha_{k,n}\}$ be the unique fixed points $\alpha_{k,n} = g_{k,n}(\alpha_{k,n})$. Suppose $\alpha_{n,n} \to \alpha$ and $|\alpha_{k,n} - \alpha_{k+1,n}| < \varepsilon_n \to 0$, $1 \le k \le n-1$. Then, for $z \in S$, $G_n(z) = G_{n,n}(z) \to \alpha$.

Proof: $|G_n(z) - \alpha| \le \rho^n |z - \alpha_{1,n}| + \sum_{k=1}^{n-1} \rho^{n-k} |\alpha_{k,n} - \alpha_{k+1,n}| + |\alpha_{n,n} - \alpha|$

$$\le \rho^n |z - \alpha_{1,n}| + \frac{\rho \varepsilon_n}{1-\rho} + |\alpha_{n,n} - \alpha| \to 0 \text{ as } n \to \infty. \#$$

Observe that it is not necessary that $\alpha_{k,n} \to \alpha$:

Example : $g_{k,n}(z) = \frac{1}{2}\left(z + \frac{k}{n}\right)$. Then $|z| \le R$, $1 \le R \Rightarrow |g_{k,n}(z)| \le R$. And

$|\alpha_{k,n} - \alpha_{k+1,n}| = \frac{1}{n} = \varepsilon_n \to 0$ and $\alpha_{k,n} = \frac{k}{n} \to 0$ $\forall k$ fixed, $\alpha_{n-k,n} = \frac{n-k}{n} \to 1$ $\forall k$ fixed, whereas $\alpha_{n,n} \equiv 1$. Computer evaluation gives $G_{10}(2+3i) \approx .9 + .003i$ and $G_{100}(2+3i) \approx .99$.

Theorem 12: Let $\{f_{k,n}\}_{1\le k\le n}$ be a family of complex functions defined on S, a simply-connected compact set in $\mathbb{C}$, having the properties $f_{k,n}(S)\subseteq S$ and $|f_{k,n}(z_1)-f_{k,n}(z_2)|<\rho|z_1-z_2|$, $\rho<1$, $1\le k\le n$. Write $F_{k,n}(z)=f_{1,n}\circ f_{2,n}\circ\cdots\circ f_{k,n}(z)$ and let $\{\alpha_{k,n}\}$ be the unique fixed points $\alpha_{k,n}=f_{k,n}(\alpha_{k,n})$. Suppose $\alpha_{1,n}\to\alpha$ and $|\alpha_{k,n}-\alpha_{k+1,n}|<\varepsilon_n\to 0$, $1\le k\le n-1$. Then, for $z\in S$, $F_{n,n}(z)\to\alpha$

Proof: $|F_{n,n}(z)-\alpha|\le\rho^n|z-\alpha_{n,n}|+\sum_{k=1}^{n-1}\rho^k|\alpha_{k,n}-\alpha_{k+1,n}|+|\alpha_{1,n}-\alpha|$

$$\le\rho^n|z-\alpha_{n,n}|+\frac{\rho\varepsilon_n}{1-\rho}+|\alpha_{1,n}-\alpha| \quad\to\quad 0 \quad \text{as } n\to\infty.$$ #

Example : $f_{k,n}(z)=\frac{1}{2}\left(z+1-\frac{k}{n}\right)$. $|z|\le R$, $1\le R \ \Rightarrow\ |f_{k,n}(z)|\le R$,

$|\alpha_{k,n}-\alpha_{k+1,n}|=\frac{1}{n}=\varepsilon_n\to 0$, $\alpha_{k,n}=1-\frac{k}{n}\to 1$ $\forall k$ fixed , whereas $\alpha_{n-k,n}=1-\frac{n-k}{n}\to 0$ $\forall k$ fixed, and $\alpha_{1,n}\to\alpha=1$.

Theorem 13: Suppose $g_{k,n}(z)=z+\frac{1}{n}\varphi(z,\frac{k}{n})$, $\varphi\in C(S\times I)$, $I=[0,1]$, $z\in S\Rightarrow g_{k,n}\in S$
Set $G_{0,n}(z)=z$, $G_{k,n}(z)=g_{k,n}\circ g_{k-1,n}\circ\cdots\circ g_{1,n}(z)$, $G_n(z)=G_{n,n}(z)$. Then

$$G_n(z)-z=\frac{1}{n}\cdot\sum_{k=1}^{n}\varphi(G_{k-1,n}(z),\tfrac{k}{n})=\frac{1}{n}\sum_{k=1}^{n}\psi(z,\tfrac{k}{n})\to\int_0^1\psi(z,t)dt=\lambda(z).$$

Comment: This is simply a discrete analogue of $\frac{dz}{dt}=\varphi(z,t)$, $0\le t\le 1$, having an *exact* solution, $Z(t)$, by inspection or under certain conditions described in the *Picard–Lindelöf* theorem. Then $\lambda(z_0):=\int_0^1\psi(z_0,t)dt=\int_0^1\varphi(Z(t),t)dt=Z(1)-z_0$.

Example : Let $g_{k,n}(z)=z+\frac{1}{n}\left(2\frac{k}{n}z^2\right)$, $z\neq 0$. Then

$$\frac{dz}{dt}=\varphi(z,t)=2tz^2 \Rightarrow Z(t)=\frac{z_0}{1-z_0t^2} \Rightarrow \lambda(z_0)=\frac{z_0^{\ 2}}{1-z_0}$$

Addendum & References:

Theorem (Kojima [5][2010]) Consider *entire* functions $f_n(z)=z+\sum_{r=2}^{\infty}c_{n,r}z^r$ with complex coefficients. Set $C_n=\sup_{r=2,3,4...}\left\{\left|c_{n,r}\right|^{\frac{1}{r-1}}\right\}$. Then the convergence of the series $\sum_{n=1}^{\infty}C_n$ implies $f(z)=\mathop{\Re}_{n=1}^{\infty}f_n(z)$ exists and is entire.

Theorem (Gill, [4][2011]) Consider *entire* functions whose linear coefficients *approach* one:

Let $f_n(z)=a_nz+a_{2,n}z^2+\cdots+a_{k,n}z^k+\cdots$, where $a_n\to 1$,

and $\left|a_{k,n}\right|\le\rho_n^{k-1}$ with $\sum\rho_n<\infty$. Set $\varepsilon_n=\left|a_n-1\right|$ with $\sum\varepsilon_n<\infty$, and $\alpha_n=\left|a_n\right|$

Then $\lim_{n\to\infty}\left(f_1\circ f_2\circ\cdots\circ f_n(z)\right)=F(z)$, entire, with uniform convergence on compact sets in the complex plane.

Theorem (Gill, [4][2010]) Consider functions $g_n(z)=a_nz+c_{n,2}z^2+c_{n,3}z^3+\cdots$ with complex coefficients. Set $\rho_n=\sup_{r=2,3,4...}\left\{\left|c_{n,r}\right|^{\frac{1}{r-1}}\right\}$ and $\varepsilon_n=\left|a_n-1\right|$ with $\sum\varepsilon_n<\infty$ If

$\rho_n<\frac{\delta_n}{RM_1M_2}$, where $\sum\delta_n<\infty$, $\prod\left(1+\delta_n\right)<M_1$, $\prod\left(1+\varepsilon_n\right)<M_2$,

Then $G(z)=\lim_{n\to\infty}\left(g_n\circ g_{n-1}\circ\cdots g_1(z)\right)$ exists and is analytic for $|z|<R$. Convergence is uniform on compact subsets of $(|z|<R)$.

Image on page one: From Theorem 9, $t_k(z)=\dfrac{\rho_k(z)}{\rho_k(z)+1-z}$,

$$\rho_k(z)\equiv xCos(y^2)+iySin(x^2),\ \ G_n(z)\to G(z),\ \ [-5,5],\ \ n=50$$

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