An Observation on the *Padé* Table for $\sqrt{1+x}$ and the simple continued fraction expansions for $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$.

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Abstract. The main staircase sequence *Padé* approximants for $\sqrt{1+x}$ are shown to yield the convergents of the simple continued fraction expansions of $\sqrt{2}$ and $\sqrt{3}$ whilst another ordered sequence of *Padé* approximants are shown to provide the convergents of the simple continued fraction expansions of $\sqrt{5}$.

1. The Padé Table. The *Padé table* [see figure 1] of the formal power series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \quad c_0 \neq 0, \tag{1}$$

is an infinite two dimensional two dimensional array of irreducible rational functions

$$P_{n,m}(x) = \frac{A_{n,m}(x)}{B_{n,m}(x)} = \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_m x^m}{\beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_n x^n}, \quad m, n \geq 0, \tag{2}$$

in each of which the coefficients are such that the expansion of $P_{m,n}(x)$ in powers of x matches that of $f(x)$ as far as possible. The power series and its associated *Padé table* are said to be *normal* if

$$P_{n,m}(x) = \sum_{k=0}^{m+n} c_k x^k + \text{higher order terms},$$

in which case every element of the table exists and is different from any other element.

$$\begin{array}{llllll}
P_{0,0} & P_{1,0} & P_{2,0} & P_{3,0} & P_{4,0} & \ldots\ldots P_{n,0} \;\; \ldots\ldots \\
P_{0,1} & P_{1,1} & P_{2,1} & P_{3,1} & P_{4,1} & \ldots\ldots P_{n,1} \;\; \ldots\ldots \\
P_{0,2} & P_{1,2} & P_{2,2} & P_{3,2} & P_{4,2} & \ldots\ldots P_{n,2} \;\; \ldots\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
P_{0,m} & P_{1,m} & P_{2,m} & P_{3,m} & P_{4,m} & \ldots\ldots P_{n,m} \;\; \ldots\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

Figure 1. *The Padé table*

There are several methods for transforming a normal series into its *Padé table*, including variations of the quotient difference algorithm, techniques which exploit the close connection between the *Padé table* and various continued fraction expansions that correspond to the series (1). One such algorithm is the following, developed for transforming two series expansions into two point *Padé* approximations but applicable to the standard Padé table.

Set $d_1^j = {}^{-c_j}\!/_{c_{j-1}}$ for $j = 1, 2, \ldots$ and use these ratios as starting values for the *rhombus rules*

$$\left.\begin{aligned} n_{i+1}^{j} &= n_i^{j+1} + d_i^{j+1} - d_i^{j} \\ d_{i+1}^{j+1} &= n_{i+1}^{j+1} \times d_i^{j} \div n_{i+1}^{j} \end{aligned}\right\}, \tag{3}$$

for $i, j = 1, 2, 3, \ldots$ with $n_1^j = 0$ for all j and $d_i^1 = -n_i^1$ for $i = 2, 3, \ldots$.

The elements generated column by column form the $n - d$ array shown in figure 2 and it can be seen that each recurrence connects four elements that form a rhombus.

$$\begin{array}{ccccccccc} d_1^1 & n_2^1 & d_2^1 & n_3^1 & d_3^1 & n_4^1 & d_4^1 & \ldots\ldots \\ d_1^2 & n_2^2 & d_2^2 & n_3^2 & d_3^2 & n_4^2 & d_4^2 & \ldots\ldots \\ d_1^3 & n_2^3 & d_2^3 & n_3^3 & d_3^3 & n_4^3 & d_4^3 & \ldots\ldots \\ d_1^4 & n_2^4 & d_2^4 & n_3^4 & d_3^4 & n_4^4 & d_4^4 & \ldots\ldots \\ .. & .. & .. & .. & .. & .. & .. & \end{array}$$

Figure 2. *The n – d array*

The convergents of the continued fraction

$$c_0 + c_1 x + \ldots + c_{k-1} x^{k-1} + \frac{c_k x^k}{1 + d_1^k x +} \frac{n_2^k x}{1 + d_2^k x +} \frac{n_3^k x}{1 + d_3^k x +} \ldots\ldots \tag{4}$$

for $k = 1, 2, 3, \ldots$ are the *Padé* approximants $P_{1,k-1}$, $P_{2,k-1}$, $P_{3,k-1}$, $P_{4,k-1}$,....., that is those on the $(k - 1)th$ row of the Padé table. In fact the $n - d$ array contains all that is required to construct the continued fraction whose successive convergents form any chosen sequence of *Padé* approximants, provided that each member of the sequence is a neighbour of the previous member in the table. These include row sequences, staircase sequences, saw tooth sequences and battlement sequences. See McCabe [1] for details.

The starting points for constructing the continued fractions are three term recurrence relations linking the numerator and denominator polynomials of trios of adjacent approximants, such as the following.

$P_{i-1,j}\ \ P_{i,j}\ \ P_{i+1,j}$: $P_{i+1,j}(x) = (1 + d_{i+1}^{j+1}x)\ P_{i,j}(x) + n_{i+1}^{j+1}xP_{i-1,j}(x)$ (5i)

$P_{i,j-1}$

$P_{i,j}\ \ P_{i+1,j}$: $P_{i+1,j}(x) = P_{i,j}(x) + (n_{i+1}^{j+1} + d_{i+1}^{j+1})xP_{i,j-1}(x)$ (5ii)

$P_{i-1,j}\ \ P_{i,j}$

$P_{i,j+1}$: $P_{i,j+1}(x) = P_{i,j}(x) + n_{i+1}^{j+1}xP_{i-1,j}(x)$ (5iii)

The relations (ii) and (iii) used in turn repeatedly provide the continued fraction

$$c_0 + \frac{c_1x}{1+}\frac{d_1^2x}{1+}\frac{n_2^2x}{1+}\frac{(n_2^3+d_2^3)x}{1+}\frac{n_3^3x}{1+}\frac{(n_3^4+d_3^4)x}{1+}\frac{n_4^4x}{1+}\ldots \tag{6}$$

whose convergents are the staircase sequence $P_{0,0}, P_{0,1}, P_{1,1}, P_{1,2}, P_{2,2}, P_{2,3}, P_{3,3}, \ldots\ldots$

(Note, these relations must be applied separately to the numerator polynomials and the denominator polynomials of the *Padé* approximants.)

2. A continued fraction for $\sqrt{1+x}$.

The *Maclaurin* series for the function $\sqrt{1+x}$ is

$$\sqrt{1+x} = 1 + \frac{x}{2} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^2}{2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{x^4}{4!} + \ldots\ldots = \sum_{r=0}^{\infty} c_r x^r$$

where $c_0 = 1$, $c_1 = \frac{1}{2}$ and $c_r = (-1)^{r+1}\frac{1\times 3\times 5\times \ldots\times(2r-3)}{2^r r!}$ for $r > 1$.

The $n - d$ array for this series begins

j	d_1^j	n_2^j	d_2^j	n_3^j	d_3^j	n_4^j	d_4^j
1	−1/2	3/4	−3/4	5/6	−5/6	7/8	−7/8
2	1/4	3/12	−1/6	5/12	−3/8	21/40	−1/2
3	3/6	3/24	1/8	10/40	−1/10	21/60	−1/4
4	5/8	3/40	3/10	10/60	1/12	21/84	−1/14
5	7/10	3/60	5/12	10/84	3/14	21/112	1/16
6	9/12	3/84	7/14	10/112	−5/16	21/144	1/6

It is easily established that

$$d_i^j = \frac{2j - 2i - 1}{2(i + j - 1)} \qquad \text{and} \qquad n_i^j = \frac{(i-1)(2i-1)}{2(i+j-2)(i+j-1)}, \tag{7}$$

for $i = 1, 2, 3, \ldots$ and $j = 1, 2, 3, \ldots$

Clearly when $j = i$

$$n_i^i = \frac{(i-1)(2i-1)}{2(2i-2)(2i-1)} = \frac{1}{4}$$

and when $j = i + 1$

$$n_i^{i+1} + d_i^{i+1} = \frac{(i-1)(2i-1)}{2(2i-1)(2i)} + \frac{1}{2(2i)} = \frac{i-1}{4i} + \frac{1}{4i} = \frac{1}{4}.$$

The continued fraction (6) then becomes

$$\sqrt{1+x} = 1 + \frac{x/2}{1\,+}\,\frac{x/4}{1\,+}\,\frac{x/4}{1\,+}\,\frac{x/4}{1\,+}\,\frac{x/4}{1\,+}\,\frac{x/4}{1\,+}\,\frac{x/4}{1\,+\ldots} \tag{8}$$

or, equivalently,

$$\sqrt{1+x} = 1 + \frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+\ldots}$$

Setting $x = 1$ in the right hand side yields

$$1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\frac{1}{2+\ldots}$$

which is of course the well known simple continued fraction for $\sqrt{2}$. Hence the *Padé* approximant values $P_{0,0}(1), P_{0,1}(1), P_{1,1}(1), P_{1,2}(1), P_{2,2}(1), P_{2,3}(1), P_{3,3}(1), \ldots\ldots$ are the successive convergents of the simple continued fraction for $\sqrt{2}$.

If we set $x = 2$ in the expansion (8) we obtain

$$1+\frac{2/2}{1}\,\frac{2/4}{+\;1}\,\frac{2/4}{+\;1}\,\frac{2/4}{+\;1}\,\frac{2/4}{+\;1}\,\frac{2/4}{+\;1}\,\frac{2/4}{+\;1}\,+\ldots$$

or, equivalently,

$$1+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\ldots$$

which is the simple continued fraction for the left hand side, namely $\sqrt{3}$. Hence the *Padé* approximant values $P_{0,0}(2), P_{0,1}(2), P_{1,1}(2), P_{1,2}(2), P_{2,2}(2), P_{2,3}(2), P_{3,3}(2), \ldots\ldots$ are the successive convergents of the simple continued fraction for $\sqrt{3}$.

Clearly there are no other integer values of x for which (8) will yield a simple continued fraction directly , though of course setting $x = 4$ yields the expansion

$$\sqrt{5} = 1+\frac{2}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\ldots$$

from which the simple continued fraction for the reciprocal of the *golden ratio*, namely

$$\frac{\sqrt{5}-1}{2} = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\ldots,$$

is easily obtained. The convergents of this expansion are not values of a sequence of *Padé* approximants.

However, we can obtain the simple continued fractions for $\sqrt{1+\frac{1}{N}}$ and $\sqrt{1+\frac{2}{N}}$, where N is a positive integer, from (8). They are, respectively,

$$\sqrt{1+\frac{1}{N}} = 1+\frac{1}{2N+}\frac{1}{2+}\frac{1}{2N+}\frac{1}{2+}\frac{1}{2N+}\frac{1}{2+}\frac{1}{2N+}\ldots \tag{9}$$

and

$$\sqrt{1+\frac{2}{N}} = 1+\frac{1}{N+}\frac{1}{2+}\frac{1}{N+}\frac{1}{2+}\frac{1}{N+}\frac{1}{2+}\frac{1}{N+}\frac{1}{2+}\frac{1}{N+}\ldots. \tag{10}$$

We can use these expansions to obtain simple continued fractions for further integers by setting $N = m^{2k}$ for m an integer greater than unity and $k = 2, 3, 4, \ldots$. We obtain

$$\sqrt{1+m^{2k}} = m^k+\frac{1}{2m^k+}\frac{1}{2m^k+}\frac{1}{2m^k+}\frac{1}{2m^k+}\frac{1}{2m^k+}\frac{1}{2m^k+}\frac{1}{2m^k+}\frac{1}{2m^k+}\frac{1}{2m^k+}\ldots$$

and

$$\sqrt{2+m^{2k}} = m^k+\frac{1}{m^k+}\frac{1}{2m^k+}\frac{1}{m^k+}\frac{1}{2m^k+}\frac{1}{m^k+}\frac{1}{2m^k+}\frac{1}{m^k+}\frac{1}{2m^k+}\frac{1}{m^k+}\ldots.$$

A particular case of the first of these is when $m = 2$ and $k = 1$, yielding

$$\sqrt{5} = 2 + \frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+\ldots} \quad . \tag{11}$$

The sequence of convergents of this continued fraction begins

$$2, \frac{9}{4}, \frac{38}{17}, \frac{161}{72}, \frac{682}{305}, \frac{2889}{1292}, \ldots\ldots\ldots \tag{12}$$

and it was observed that these early convergents are the values of a sequence of *Padé* approximants for $\sqrt{1+x}$ evaluated at $x = 4$, namely

$$P_{1,1}(4), P_{2,3}(4), P_{4,4}(4), P_{5,6}(4), P_{7,7}(4), P_{8,9}(4), P_{10,10}(4), \ldots\ldots \tag{13}$$

It is now proved that the infinite sequence of convergents of the continued fraction (11) are this sequence of *Padé* approximants continued indefinitely. The above *Padé* approximants, are alternate members of the sequence

$$P_{1,1}(4), P_{1,2}(4), P_{2,3}(4), P_{3,3}(4), P_{4,4}(4), P_{4,5}(4), P_{5,6}(4), P_{6,6}(4), P_{7,7}(4), P_{7,8}(4), P_{8,9}(4), P_{9,9}(4), P_{10,10}(4), \ldots\ldots$$

whose values are

$$2, \frac{7}{3}, \frac{18}{8}, \frac{29}{13}, \frac{76}{34}, \frac{123}{55}, \frac{322}{144}, \frac{521}{233}, \frac{1364}{610}, \frac{2207}{987}, \frac{4578}{2584}, \ldots\ldots\ldots \, . \tag{14}$$

The reason for multiplying and dividing the odd even members by the two will be seen in what follows. These approximants can be seen in parentheses in the table below

$$\begin{array}{llllllllllll}
P_{0,0} & P_{1,0} & P_{2,0} & P_{3,0} & P_{4,0} & P_{5,0} & P_{6,0} & P_{7,0} & P_{8,0} & P_{9,0} & \ldots\ldots\ldots P_{n,0} & \ldots\ldots \\
P_{0,1} & (P_{1,1}) & P_{2,1} & P_{3,1} & P_{4,1} & P_{5,1} & P_{6,1} & P_{7,1} & P_{8,1} & P_{9,1} & \ldots\ldots\ldots\ldots P_{n,1} & \ldots\ldots \\
P_{0,2} & (P_{1,2}) & P_{2,2} & P_{3,2} & P_{4,2} & P_{5,2} & P_{6,2} & P_{7,2} & P_{8,2} & P_{9,2} & \ldots\ldots P_{n,2} & \ldots\ldots \\
P_{0,3} & P_{1,3} & (P_{2,3}) & (P_{3,3}) & P_{4,3} & P_{5,3} & P_{6,3} & P_{7,3} & P_{8,3} & P_{9,3} & \ldots\ldots P_{n,3} & \ldots\ldots \\
P_{0,4} & P_{1,4} & P_{2,4} & P_{3,4} & (P_{4,4}) & P_{5,4} & P_{6,4} & P_{7,4} & P_{8,4} & P_{9,4} & \ldots\ldots P_{n,4} & \ldots\ldots \\
P_{0,5} & P_{1,5} & P_{2,5} & P_{3,5} & (P_{4,5}) & P_{5,5} & P_{6,5} & P_{7,5} & P_{8,5} & P_{9,5} & \ldots\ldots P_{n,5} & \ldots\ldots \\
P_{0,6} & P_{1,6} & P_{2,6} & P_{3,6} & P_{4,6} & (P_{5,6}) & (P_{6,6}) & P_{7,6} & P_{8,6} & P_{9,6} & \ldots\ldots P_{n,6} & \ldots\ldots \\
P_{0,7} & P_{1,7} & P_{2,7} & P_{3,7} & P_{4,7} & P_{5,7} & P_{6,7} & (P_{7,7}) & P_{8,7} & P_{9,7} & \ldots\ldots P_{n,7} & \ldots\ldots \\
P_{0,8} & P_{1,8} & P_{2,8} & P_{3,8} & P_{4,8} & P_{5,8} & P_{6,8} & (P_{7,8}) & P_{8,8} & P_{9,8} & \ldots\ldots P_{n,8} & \ldots\ldots \\
P_{0,9} & P_{1,9} & P_{2,9} & P_{3,9} & P_{4,9} & P_{5,9} & P_{6,9} & P_{7,9} & (P_{8,9}) & P_{9,9} & \ldots\ldots P_{n,9} & \ldots\ldots
\end{array}$$

It follows that if we can derive the continued fraction whose convergents are the sequence (14), and then show that the even contraction of this fraction is (11), then we will have established that the convergents of the simple continued fraction for $\sqrt{5}$ form the ordered sequence of *Padé* approximants for $\sqrt{1+x}$, evaluated at $x = 4$, suggested above.

Clearly the first three convergents are those of $2+\frac{1}{3+}\frac{2}{2}$. We now prove that the continued fraction continues

$$2+\frac{1}{3+}\frac{2}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}..........$$

of which the even contraction is

$$2+\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}...$$

which is (11).

By making use of the three relations (5), each time using two of them and introducing and then eliminating a neighbouring *Padé* approximant, we obtain the four further relations below.

$$P_{i,j+1}(x) = (1+n_{i+1}^{j+1}x)\;\; P_{i-1,j}(x) + \left(n_i^{j+1} + d_i^{j+1}\right)xP_{i-1,j-1}(x) \qquad (15i)$$

$$P_{i,j}(x) = (1+\left(n_i^{j+1} + d_i^{j+1}\right)x)\;\; P_{i-1,j}(x) - n_i^j\left(n_i^{j+1} + d_i^{j+1}\right)x^2P_{i-2,j-1}(x) \qquad (15ii)$$

$$P_{i+1,j+1}(x) = \left\{1+\left(n_{i+2}^{j+1} + d_{i+1}^{j+1}\right)x\right\}\;\; P_{i,j}(x) + n_{i+1}^{j+1}xP_{i-1,j}(x) \qquad (15iii)$$

$$P_{i,j+1}(x) = (1+n_{i+1}^{j+1}x)\;\; P_{i,j}(x) - n_{i+1}^{j+1}\left(n_i^{j+1} + d_i^{j+1}\right)x^2P_{i-1,j-1}(x) \qquad (15iv).$$

It is easily seen that we only require these relations in the cases when $i = j$, and also verified that using the values of the $n-d$ array given earlier and setting $x = 4$ these reduce to

$$P_{i,i+1}(4) = 2P_{i-1,i}(4) + P_{i-1,i-1}(4) \qquad (16i)$$

$$P_{i,i}(4) = 2P_{i-1,i}(4) - P_{i-2,i-1}(4) \qquad (16ii)$$

$$P_{i+1,i+1}(4) = 2P_{i,i}(4) + P_{i-1,i}(4) \qquad (16iii)$$

$$P_{i,i+1}(4) = 2P_{i,i}(4) - P_{i-1,i-1}(4) \qquad (16iv)$$

Now, starting with $P_{1,1}(4) = 2$ and $P_{1,2}(4) = \frac{7}{3}$ and then, in turn, (i) for $i = 2, 3, 5, 8...$,

(ii) and (iii) for $i = 3, 6, 9, 12, ...$ and (iv) for $i = 4, 7, 10, ...$ we obtain the sequence

$$\frac{18}{8}, \frac{29}{13}, \frac{76}{34}, \frac{123}{55}, \frac{322}{144}, \frac{521}{233}, \frac{1364}{610}, \frac{2207}{987}, \frac{4578}{2584}, ...$$

which together are the initial convergents of the continued fraction

$$2 + \frac{1}{3+}\frac{2}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+}\frac{1}{2-}\frac{1}{2+} ....$$

The even contraction of the continued fraction

$$b_0 + \frac{a_1}{b_1 +}\frac{a_2}{b_2 +}\frac{a_3}{b_3 +}\frac{a_4}{b_4 +}\frac{a_5}{b_5 +}\frac{a_6}{b_6 +}\frac{a_7}{b_7 +}\frac{a_8}{b_8 +}\frac{a_9}{b_9 +}\frac{a_{10}}{b_{10} +} ....$$

is

$$b_0 + \frac{a_1 b_2}{b_1 b_2 + a_2 -}\frac{a_2 a_3 b_4}{(b_2 b_3 + a_3) b_4 + b_2 a_4 -}\frac{a_4 a_5 b_2 b_6}{(b_4 b_5 + a_5) b_6 + b_4 a_6 - .....-}\frac{a_{2n-2} a_{2n-1} b_{2n-4} b_{2n}}{(b_{2n-2} b_{2n-1} + a_{2n-1}) b_{2n} + b_{2n-2} a_{2n} + ...}.$$

Applying this transformation to the above continued fraction yields

$$2 + \frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+...},$$

the simple continued fraction for $\sqrt{5}$. The convergents are an ordered sequence following a series of *Knight's moves* in a chess board, indicated by those in parentheses below.

$P_{0,0}$ $P_{1,0}$ $P_{2,0}$ $P_{3,0}$ $P_{4,0}$ $P_{5,0}$ $P_{6,0}$ $P_{7,0}$ $P_{8,0}$ $P_{9,0}$ ....$P_{n,0}$ .....

$P_{0,1}$ $(P_{1,1})$ $P_{2,1}$ $P_{3,1}$ $P_{4,1}$ $P_{5,1}$ $P_{6,1}$ $P_{7,1}$ $P_{8,1}$ $P_{9,1}$ ....$P_{n,1}$ .....

$P_{0,2}$ $P_{1,2}$ $P_{2,2}$ $P_{3,2}$ $P_{4,2}$ $P_{5,2}$ $P_{6,2}$ $P_{7,2}$ $P_{8,2}$ $P_{9,2}$ ......$P_{n,2}$ .....

$P_{0,3}$ $P_{1,3}$ $(P_{2,3})$ $P_{3,3}$ $P_{4,3}$ $P_{5,3}$ $P_{6,3}$ $P_{7,3}$ $P_{8,3}$ $P_{9,3}$ ......$P_{n,3}$ .....

$P_{0,4}$ $P_{1,4}$ $P_{2,4}$ $P_{3,4}$ $(P_{4,4})$ $P_{5,4}$ $P_{6,4}$ $P_{7,4}$ $P_{8,4}$ $P_{9,4}$ ......$P_{n,4}$ .....

$P_{0,5}$ $P_{1,5}$ $P_{2,5}$ $P_{3,5}$ $P_{4,5}$ $P_{5,5}$ $P_{6,5}$ $P_{7,5}$ $P_{8,5}$ $P_{9,5}$ ......$P_{n,5}$ .....

$P_{0,6}$ $P_{1,6}$ $P_{2,6}$ $P_{3,6}$ $P_{4,6}$ $(P_{5,6})$ $P_{6,6}$ $P_{7,6}$ $P_{8,6}$ $P_{9,6}$ ......$P_{n,6}$ .....

$P_{0,7}$ $P_{1,7}$ $P_{2,7}$ $P_{3,7}$ $P_{4,7}$ $P_{5,7}$ $P_{6,7}$ $(P_{7,7})$ $P_{8,7}$ $P_{9,7}$ ......$P_{n,7}$ .....

$P_{0,8}$ $P_{1,8}$ $P_{2,8}$ $P_{3,8}$ $P_{4,8}$ $P_{5,8}$ $P_{6,8}$ $P_{7,8}$ $P_{8,8}$ $P_{9,8}$ ......$P_{n,8}$ .....

$P_{0,9}$ $P_{1,9}$ $P_{2,9}$ $P_{3,9}$ $P_{4,9}$ $P_{5,9}$ $P_{6,9}$ $P_{7,9}$ $(P_{8,9})$ $P_{9,9}$ ......$P_{n,9}$ .....

Hence there are ordered sequences of *Padé* approximants for $\sqrt{1+x}$ which yield the convergents of the simple continued fraction expansions of $\sqrt{2}$ and $\sqrt{3}$ and $\sqrt{5}$.

Are there such sequences which yield the convergents of the simple continued fraction expansions of $\sqrt{N}$ for other integers greater than 5?

REFERENCES

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