# MOMENTS VIA TRANSFORMS

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Abstract. Connections are explicated between the $n$-th moments $\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s)$ of parameter $s$ associated with the modified Laguerre distribution having weight function $w(t) := F(t)e^{-st}$ on the non-negative reals and the Laplace transform $f(s)$ of the modifying function $F(t)$. In particular, $\mu_0^{\mathcal{L}}\left\{F(t)\right\}(s) = f(s)$, $\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) = -\frac{d}{ds}\mu_{n-1}^{\mathcal{L}}\left\{F(t)\right\}(s)$, and, thus,

$$\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) = (-1)^n \frac{d^n}{ds^n} f(s)$$

if $n$ is a non-negative integer. For negative $n$, there is a corresponding iterated integral formula. Specific examples of moment and strong moment distributions are given, and transforms related to other classical systems are described.

A fine starting point for basic definitions and examples of moment sequences is Chihara's text in orthogonal polynomials [2] and the survey [7] of strong moment theory by Jones and Njåstad, which also provides an extensive bibliography for further reference. In addition, the NBS handbook [1] may be useful to the reader.

The Laplace transform $f(s)$ of a function $F(t)$,

$$f(s) = \mathcal{L}\left\{F(t)\right\}(s) := \int_0^\infty F(t)e^{-st}\, dt, \tag{0.1}$$

is reminiscent of the moments $\mu_n^{\mathcal{L}}$ of the Laguerre distribution,

$$\mu_n^{\mathcal{L}} := \int_0^\infty t^n e^{-t}\, dt,\ n = 0,\ 1,\ 2,\ \ldots. \tag{0.2}$$

By introducing a parameter $s > 0$, modifying the weight function slightly to $e^{-st}$, the corresponding moments $\mu_n^{\mathcal{L}}(s)$ are Laplace transforms.

$$\mu_n^{\mathcal{L}}(s) := \int_0^\infty t^n e^{-st}\, dt,\ n = 0,\ 1,\ 2,\ \ldots. \tag{0.3}$$

Since

$$\frac{d}{ds}\int_0^\infty t^k e^{-st}\, dt = \int_0^\infty \frac{\partial}{\partial s}\left(t^k e^{-st}\right)\, dt = \int_0^\infty \left(-t^{k+1}e^{-st}\right)\, dt,$$

$\frac{d}{ds}\mu_k^{\mathcal{L}}(s) = -\mu_{k+1}^{\mathcal{L}}(s)$. Thus, by iteration,

$$\mu_n^{\mathcal{L}}(s) = (-1)^n \frac{d^n}{ds^n}\mu_0^{\mathcal{L}}(s);$$

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that is,

$$\mu_n^{\mathcal{L}}(s) = (-1)^n \frac{d^n}{ds^n}\frac{1}{s} = \frac{n!}{s^{n+1}},\ n = 0,\ 1,\ 2,\ \ldots. \tag{0.4}$$

For example, with $s = 1$, we recover the formula for the $n$-th moment of the Laguerre distribution, $\mu_n^{\mathcal{L}} = n!$.

## 1. Laguerre-Type Moments Via Laplace Transform

In order to apply the previous ideas to strong moment bisequences such as

$$\int_0^\infty t^n t^{-1} e^{-t-1/t} dt,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots, \tag{1.1}$$

insert a real-valued modifying function $F(t)$ with domain containing the non-negative real numbers and consider the integrals

$$\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) := \int_0^\infty t^n F(t) e^{-st} dt. \tag{1.2}$$

Applying elementary methods of calculus, if $\mu_k^{\mathcal{L}}\left\{F(t)\right\}(s)$ and $\mu_{k+1}^{\mathcal{L}}\left\{F(t)\right\}(s)$ exist, then

$$\mu_{k+1}^{\mathcal{L}}\left\{F(t)\right\}(s) = -\frac{d}{ds}\mu_k^{\mathcal{L}}\left\{F(t)\right\}(s), \tag{1.3a}$$

and

$$\mu_k^{\mathcal{L}}\left\{F(t)\right\}(s) = \int_s^\infty \mu_{k+1}^{\mathcal{L}}\left\{F(t)\right\}(x)\ dx. \tag{1.3b}$$

By Definitions (0.1) and (1.2),

$$\mu_0^{\mathcal{L}}\left\{F(t)\right\}(s) := \int_0^\infty F(t) e^{-st}\ dt = f(s),$$

and, hence, considering Equations (1.3), we have

**Theorem.** *Let $s > 0$. If $\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) := \int_0^\infty t^n F(t) e^{-st} dt$ is the $n$-th moment of a moment distribution function $\psi$ and $f(s)$ is the Laplace transform of $F(t)$, then*

$$\mu_0^{\mathcal{L}}\left\{F(t)\right\}(s) = f(s), \tag{1.4a}$$

$$\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) = (-1)^n \frac{d^n}{ds^n} f(s)\ \textit{for}\ n = 1,\ 2,\ 3,\ \ldots, \tag{1.4b}$$

*and, furthermore, if $\psi$ is a strong moment distribution function, then*

$$\mu_n^{\mathcal{L}}\left\{F(t)\right\}(s) = I^{|n|} f(s)\ \textit{for}\ n = -1,\ -2,\ -3,\ \ldots, \tag{1.4c}$$

*where $I^1 f(s) := \int_s^\infty f(x)\ dx$ and $I^{|n|} f(s) := \int_s^\infty I^{|n|-1} f(x)\ dx$ for $|n| > 1$.*

## 2. Example

Reconsider (1.1), which can be rewritten as

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1)=\int_0^\infty t^{n-1}e^{-t-1/t}dt.$$

These are the moments of a strong moment distribution function, and the Laplace transform of $t^{-1}e^{-1/t}$ is known to be $2K_0(2\sqrt{s})$, where $K_\nu(z)$ is a modified Bessel function of the second kind. By the theorem above,

$$\mu_0^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=2K_0(2\sqrt{s}), \tag{2.1a}$$

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=(-1)^n\frac{d^n}{ds^n}2K_0(2\sqrt{s})\ for\ n=1,\ 2,\ 3,\ \ldots, \tag{2.1b}$$

and

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=I^{|n|}2K_0(2\sqrt{s})\ for\ n=-1,\ -2,\ -3,\ \ldots, \tag{2.1c}$$

where $I^1 2K_0(2\sqrt{s}) := \int_s^\infty 2K_0(2\sqrt{x})\,dx$ and $I^{|n|}2K_0(2\sqrt{s}) := \int_s^\infty I^{|n|-1}2K_0(2\sqrt{x})\,dx$ for $|n|>1$.

Now, using (2.1a), (1.3a), elementary calculus, and the known fact that $\frac{d}{dz}\left(\frac{1}{z^n}K_n(z)\right)=\frac{-1}{z^n}K_{n+1}(z)$, an induction argument shows that $\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=\frac{2}{\sqrt{s^n}}K_n(2\sqrt{s})$ for $n=0,\ 1,\ 2,\ \ldots$. Also, by a change of variables $t\to 1/(sx)$, it follows that $\mu_{-n}^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=s^n\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)$, and, subsequently,

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(s)=\int_0^\infty t^{n-1}e^{-st-1/t}dt=\frac{2}{\sqrt{s^n}}K_{|n|}(2\sqrt{s}) \tag{2.2}$$

$$for\ n=0,\ \pm1,\ \pm2,\ \ldots.$$

A greater symmetry in these moments results when $s=1$; in particular,

$$\mu_{-n}^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1)=\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1), \tag{2.3}$$

since, in this case,

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1)=\int_0^\infty t^{n-1}e^{-t-1/t}dt=2K_{|n|}(2) \tag{2.4}$$

$$for\ n=0,\ \pm1,\ \pm2,\ \ldots.$$

Table 1 gives numerical values for these moments.

TABLE 1. Numerical Values of the Moments

$$\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1) = \int_0^\infty t^{n-1}e^{-t-1/t}dt = 2K_{|n|}(2).$$

| $n$ | $\mu_n^{\mathcal{L}}\left\{t^{-1}e^{-1/t}\right\}(1)$ |
|---|---|
| 0 | 0.22779 |
| $\pm 1$ | 0.27973 |
| $\pm 2$ | 0.50752 |
| $\pm 3$ | 1.29477 |
| $\pm 4$ | 4.39183 |
| $\pm 5$ | 18.86210 |

Remark: The recursion $\mu_{n+1} = n\mu_n + \mu_{n-1}$ holds for $n > 1$.

TABLE 2. Moments $\mu_n = \int_0^\infty t^n F(t)e^{-st}dt$, Parameter $s > 0$.

| $F(t)$ | $\mu_n$ | *Comments* |
|---|---|---|
| $t^{-1}e^{-1/t}$ | $\frac{2}{\sqrt{s^n}}K_{\|n\|}(2\sqrt{s})$ | modified Bessel function $K_\nu(z)$ |
| $\frac{1}{\sqrt{\pi t}}e^{-1/t}$ | $(-1)^n \frac{d^n}{ds^n}\frac{e^{-2\sqrt{s}}}{\sqrt{s}}$ | $n \geq 0$ |
| | $I^{\|n\|}\frac{e^{-2\sqrt{s}}}{\sqrt{s}}$ | $n < 0$, see Note below |
| $\frac{k}{2\sqrt{\pi t^3}}e^{\frac{-k^2}{4t}}$ | $(-1)^n \frac{d^n}{ds^n}e^{-k\sqrt{s}}$ | $0 < k < \infty,\ n \geq 0$ |
| | $I^{\|n\|}e^{-k\sqrt{s}}$ | $n < 0$, see Note below |
| $\left[1 + \|t-k\|/(t-k)\right]/2$ | $\frac{e^{-sk}}{s}\sum_{j=0}^{n}\binom{n}{j}k^j s^{j-n}$ | $0 \leq k < \infty,\ n \geq 0$ |
| | $I^{\|n\|}\frac{e^{-sk}}{s}$ | $0 < k < \infty,\ n < 0$, see Note below |
| 1 | $\frac{n!}{s^{n+1}}$ | $n = 0, 1, 2, \ldots$ |
| $t^m$ | $\frac{\Gamma(m+n+1)}{s^{m+n+1}}$ | $m + n + 1 > 0$, gamma function $\Gamma(z)$ |

Note: $I^1 f(s) := \int_s^\infty f(x)\ dx$ and $I^{|n|}f(s) := \int_s^\infty I^{|n|-1}f(x)\ dx$ for $|n| > 1$.

## 3. Further Examples and Transforms

A few formulas, including (2.2), for the moments of modified Laguerre distributions which can be obtained via the above theorem are given in Table 2.

Other integral transformations can be employed to obtain formulas for moment sequences, but more than a cursory glance is beyond the goals of the present article. Several extensions of the one just explicated are possible. For distributions associated with a weight function $w(t)$ on an interval $I$, investigations lead back to parameterized moment functionals $\mathcal{M}$ similar to the Laplace transform; specifically, (3.1a) $$\mathcal{M}\left\{F(t)\right\}(s) := \int_I F(t)w(t)e^{(1-s)t}\ dt,$$

whose moments

(3.1b) $$\mu_n^{\mathcal{M}}\left\{F(t)\right\}(s) := \int_I t^n F(t)w(t)e^{(1-s)t}\ dt$$

satisfy

(3.1c) $$\mu_{n+1}^{\mathcal{M}}\left\{F(t)\right\}(s) = -\frac{d}{ds}\mu_n^{\mathcal{M}}\left\{F(t)\right\}(s).$$

As desired, when $w(t) = e^{-t}$ and $I$ is the non-negative reals, $\mathcal{M} = \mathcal{L}$ and $\mu_n^{\mathcal{M}}\left\{F(t)\right\}(s) = \mu_n^{\mathcal{L}}\left\{F(t)\right\}(s)$.

With $w(t) = (1-t)^\alpha(1+t)^\beta$ on $I = (-1, 1)$, the transform and moments from (3.1) in the classical Jacobi case are

(3.2a) $$\mathcal{P}^{(\alpha,\beta)}\left\{F(t)\right\}(s) := \int_{-1}^{1} F(t)(1-t)^\alpha(1+t)^\beta e^{(1-s)t}dt \quad (Jacobi)$$

and

(3.2b) $$\mu_n^{\mathcal{P}^{(\alpha,\beta)}}\left\{F(t)\right\}(s) := \int_{-1}^{1} t^n F(t)(1-t)^\alpha(1+t)^\beta e^{(1-s)t}dt.$$

For example, various choices of the parameters $\alpha$ and $\beta$ give transforms associated with the Legendre and Chebyshev distributions,

(3.3) $$\mathcal{P}\left\{F(t)\right\}(s) := \int_{-1}^{1} F(t)e^{(1-s)t}dt \quad (Legendre)$$

and

(3.4) $$\mathcal{T}\left\{F(t)\right\}(s) := \int_{-1}^{1} F(t)\frac{e^{(1-s)t}dt}{\sqrt{1-t^2}} \quad (Chebyshev).$$

Of course, integral transformations which may or may not reduce to the Laplace transform as discussed previously can be considered. One may wish to preserve certain aspects of the distributions and moments that may not carry over using (3.1). For example,

(3.5) $$\mathcal{H}\left\{F(t)\right\}(s) := \int_{-\infty}^{\infty} F(t)e^{-st^2}dt \quad (Hermite)$$

can be examined with one eye on the look out for the relationship to the Laguerre case involving the Laplace transform. Here, the relation

(3.6) $$\mu_{n+2}^{\mathcal{H}}\left\{F(t)\right\}(s) = -\frac{d}{ds}\mu_n^{\mathcal{H}}\left\{F(t)\right\}(s)$$

between the moments

(3.7) $$\mu_n^{\mathcal{H}}\left\{F(t)\right\}(s) := \int_{-\infty}^{\infty} t^n F(t)e^{-st^2}dt$$

exists, which in the case that $F$ is an even function will retain symmetry of the Hermite distribution.

## 4. Concluding Remarks

For the theory of orthogonal functions, there is value in adding to the representation of moments. Contributions of the methods described in the present article to the literature on research in this area include (1) differential formulas for moments and (2) a versatile, systematic and elementary way of modifying moment functionals; in particular, for the construction of positive-definite strong moment functionals. Although providing a different approach, the present article continues the theme (2) of works of Hagler as cited in the following list of references.

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