A Note: Playing With Elementary Functional Integrals & Contours + Imagery

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Abstract: Starting from a rudimentary definition of functional integral and basic complex variable theory several examples of functional integrals and their values are derived. In addition, images of the complex plane in which each point is associated with a functional integral are presented. This is not the functional integral used in physics, nor are these examples related to applications.

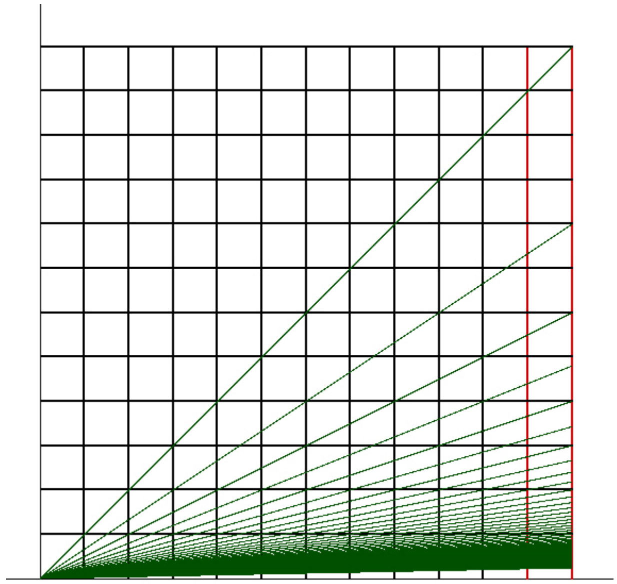
From Wikipedia: "In an ordinary integral there is a function to be integrated (the integrand) and a region of space over which to integrate the function (the domain of integration). The process of integration consists of adding up the values of the integrand for each point of the domain of integration . . . where the domain of integration is divided into smaller and smaller regions. For each small region, the value of the integrand cannot vary much, so it may be replaced by a single value. In a functional integral the domain of integration is a space of functions. For each function, the integrand returns a value to add up."

The common integral $\int_a^b f(x)dx = \underset{n\to\infty}{Lim}\sum_{k=1}^{n} f(c_{k,n})\Delta x_{k,n}$, where x is the (real) variable of integration and $f(x)$ is the integrand. $\Pi_n[a,b]=\left\{x_{k,n}\right\}_{k=1}^{n}$ is a partition of $[a,b]$ and $\Delta x_{k,n} = x_{k+1,n} - x_{k,n}$ with $x_{k,n} \le c_{k,n} \le x_{k+1,n}$. All the subintervals shrink to 0 as $n \to \infty$.

On the other hand, the integral $\int_S F[f]\, d\mu T$ has as its variable of integration "points" in a *function space* S , i.e., functions $f \in S$. And the "subintervals" are sets ($\sigma - \text{algebras}$) of functions requiring a non-negative definition of "size" or "measure", μT , paralleling the Riemann concept. $F[f]$ is a *functional* operating on a function $f \in S$ and returning a real or complex number for each such function.

I start from this very rudimentary and simple definition of functional integral and not from the usual interpretation seen in path integrals. Also, when *measures* are discussed I include *pre-measures* and *cylinder measures* under this rubric. The space S will be a space of defined *contours* in the complex plane.

In order to establish a functional integral in the context of sets of contours, S , we need a way of quantifying or measuring sets, T , within S , and a designation of a collection of such sets dependent upon n; also, a reasonable way of extracting an individual contour, $z(t)$, within each set and assigning a numerical value to it. We want the "size" of each of these sets to diminish to 0 as $n \to \infty$. Thus the "functional integral" here is a hybrid concept.



Example: $S:\{z(t):z(t)=t+i\rho t,\ \ 0\le\rho\le 1,\ \ t\in[0,1]\}$. Think of these contours as vectors in the first quadrant. Assume the unit square is divided into $n\times n$ squares. Focus on the right-most vertical line, divided into equal subintervals. Let $T_{k,n}=\{z(t):\frac{k}{n}\le\rho<\frac{k+1}{n}\}$. The simplest measure is $\mu T_{k,n}=\dfrac{1}{n}$. Now, define a functional as $F[z_{k,n}(t)]=\left|1+i\rho_{k,n}\right|^2,\ \ \rho_{k,n}=\dfrac{k}{n}$, where the contour terminates at lowest point of the subinterval $[\frac{k}{n},\frac{k+1}{n})$. Then

$$\int_S F[z]\mu T=\underset{n\to\infty}{Lim}\sum_{k=1}^{n}\left(1+\frac{k^2}{n^2}\right)\cdot\frac{1}{n}\ \approx\ .5417,\quad n=10{,}000$$

This particular contour from $T_{k,n}$ is representative of all contours in this set since

$$\left|\tfrac{1}{n}\sum_{k=1}^{n}\left(1+\rho_{k,n}^2\right)-\tfrac{1}{n}\sum_{k=1}^{n}\left(1+\tfrac{k^2}{n^2}\right)\right|\ \le\ \sum_{k=1}^{n}\frac{2k-1}{n^3}\to 0,\ \ n\to\infty$$

Of course, this is a Riemann integral in disguise.

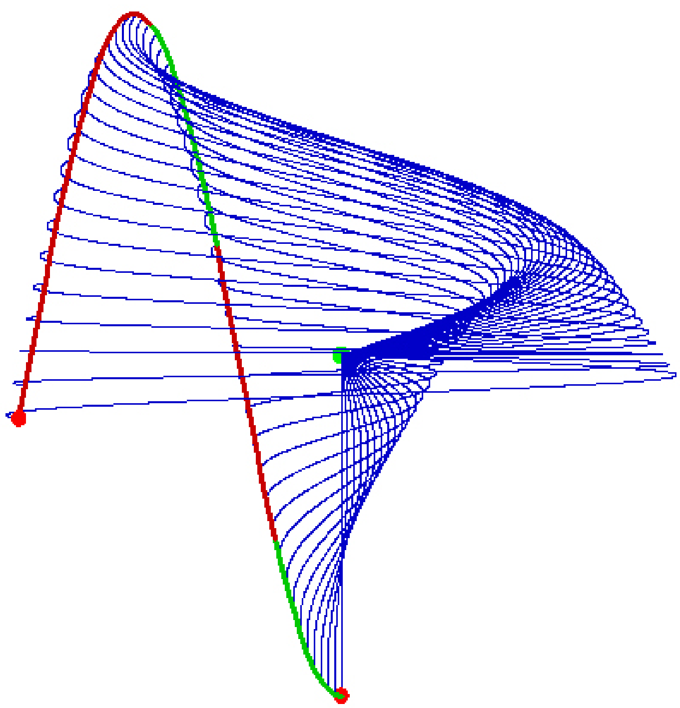
Let the set S_z be a space of contours $\zeta_s(t)$ with $t\in[0,1],\ s\in[0,1]$. s denotes a family of individual contours originating at the point z, and t ranging from 0 to 1 describes a contour. The first choice of a contour space is

$$S_z:\{\zeta_s(t):\zeta_s(t)=z+sSin(5t)-itCos(5s),\quad z=1+i\}$$

A collection of sets in this space is given by $T(a,b)=\{\zeta_s(t): 0 \le a \le s < b < 1\}$. These *cylinder sets* generate a $\sigma-\text{algebra}$. The measure we apply is the distance traversed by the points $\zeta_s(1)$ over the s-intervals (the red and green contour in the image below). The next step is to designate a particular collection of sets

$$T_{k,n}=\{\zeta_s(t): \tfrac{k}{n} \le s < \tfrac{k+1}{n}\},\ k:0 \to n,\ n \to \infty.$$

Then $\mu T_{k,n}=\int_{\frac{k}{n}}^{\frac{k+1}{n}} |d\zeta_s(1)|$. For a functional on S_z, set $F\left[\zeta_{\frac{k}{n}}\right]=\left|z-\zeta_{\frac{k}{n}}(1)\right|^2$ for that particular contour from each subset. Then $\lambda(z)=\int_{S_z} F[\zeta_s] d\mu T \ \Rightarrow\ \lambda(1+i)=\int_{S_{1+i}} F[\zeta_s] d\mu T \approx 2.2078$



An accurate representation would not show individual contours as s is a continuous variable.

Example: Let $\zeta_s(t)=\dfrac{z\left(1+t(e^s-1)\right)}{1-t^2\left(xs+iy\right)}$, $z=x+iy$. In the image below z = 1+i. The measure and functional are the same.

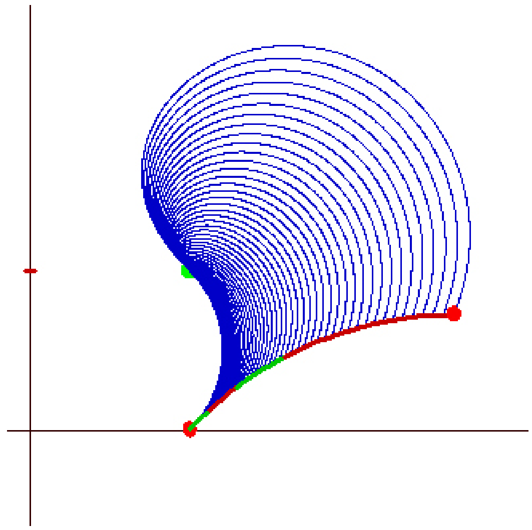
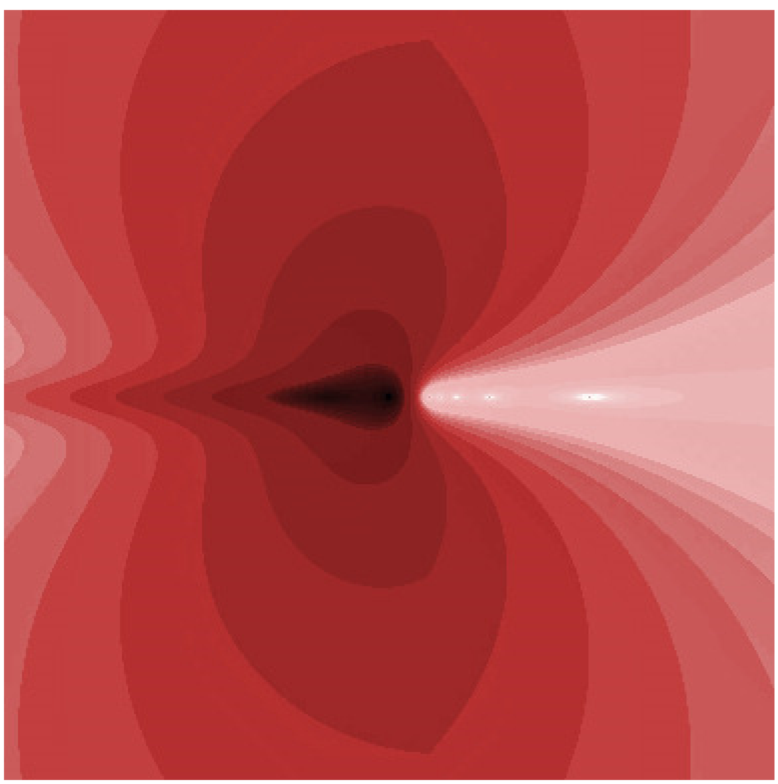
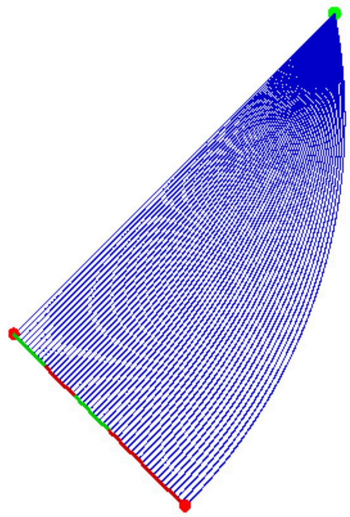


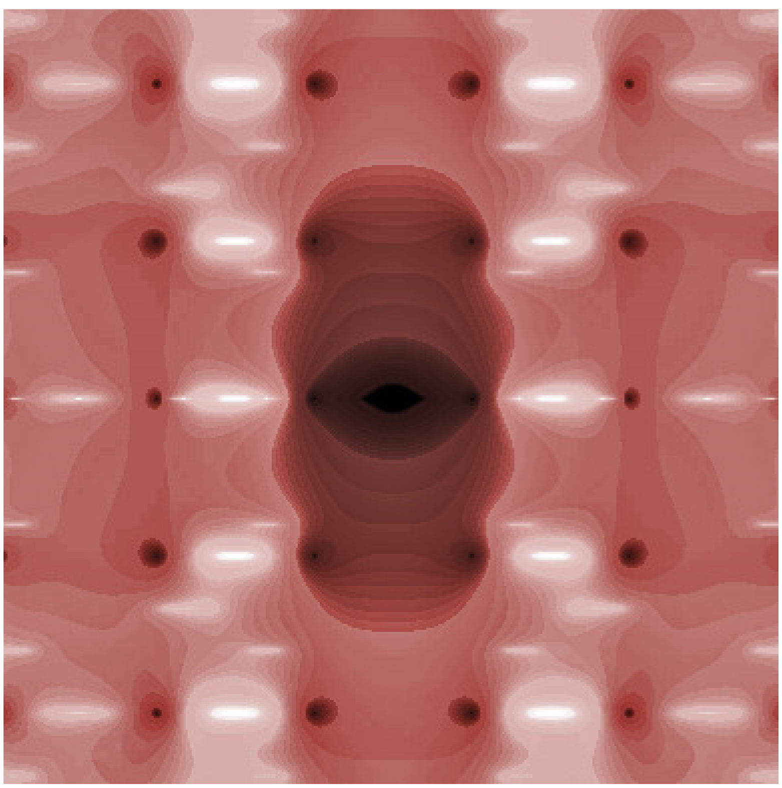
Image over a region: Dark = small moduli , light = large. $\lambda(z)=\int_{S_z} F[\zeta_s]d\mu T$:



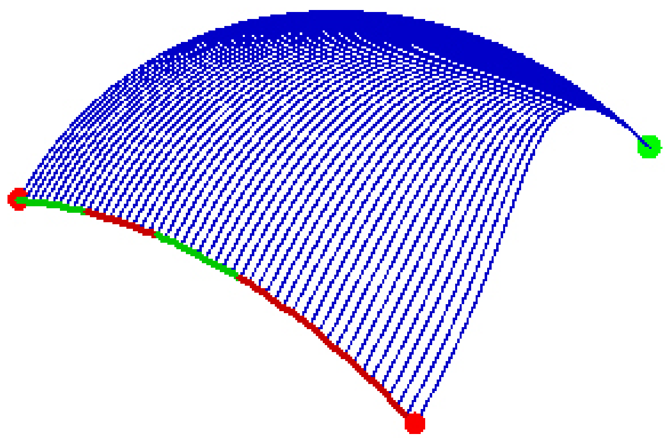
[-10,10]

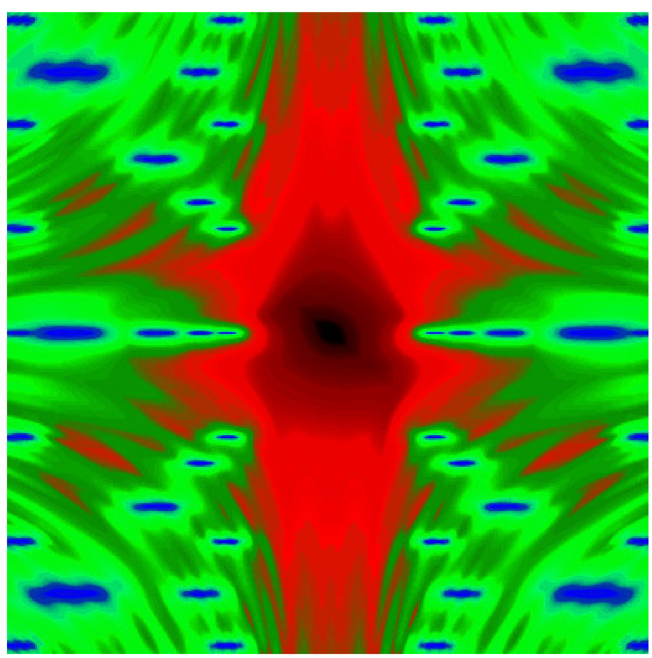
Example: $\zeta_s(t)=\dfrac{z}{1+t\left(Cos(xs)-iSin(ys)\right)}$, $z=x+iy$. The same details as in the previous example. z=1+I in the first image. $\lambda(z)=\int\limits_{S_z} F\left[\zeta_s\right]d\mu T$: The image is over [-10,10].





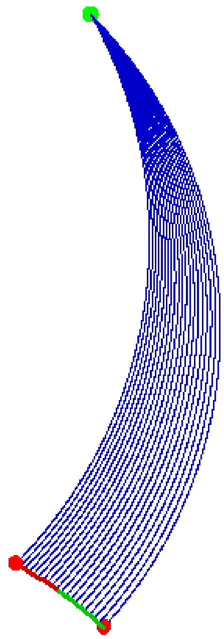
Example: $\zeta_s(t) = \dfrac{z\left(1+iSin(t)Cos(sxy)\right)}{1+t\left(Cos(xs)-iSin(ys)\right)}$, $z = x+iy$ z=1+I $\lambda(z) = \int_{S_z} F\left[\zeta_s\right] d\mu T$:

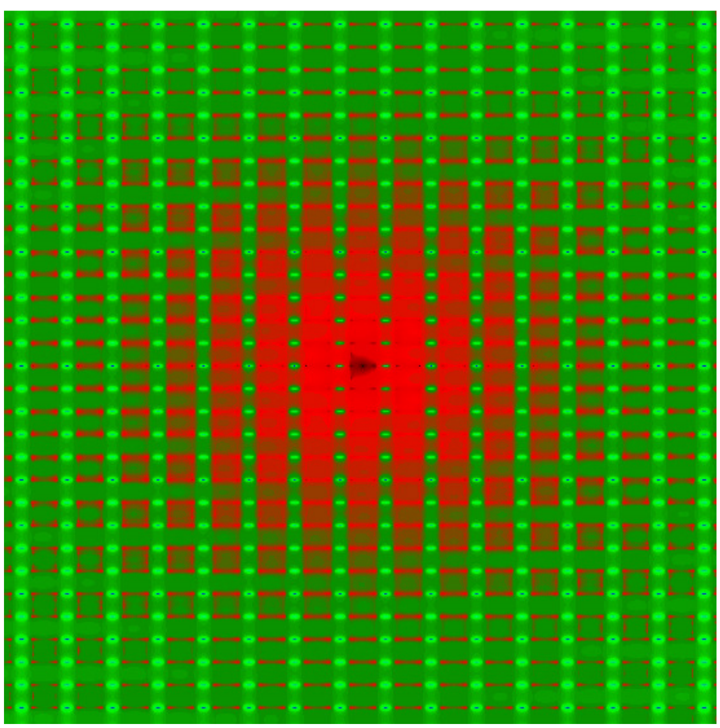




[-10,10]

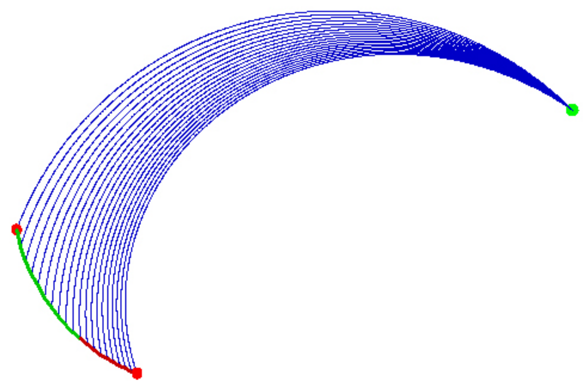
Example: $\zeta_s(t) = \dfrac{z}{1+t\left(sCos(5x)+iSin(5y)\right)}$, $z = x+iy$, z=1+I, $\lambda(z) = \int_{S_z} F\left[\zeta_s\right] d\mu T$:

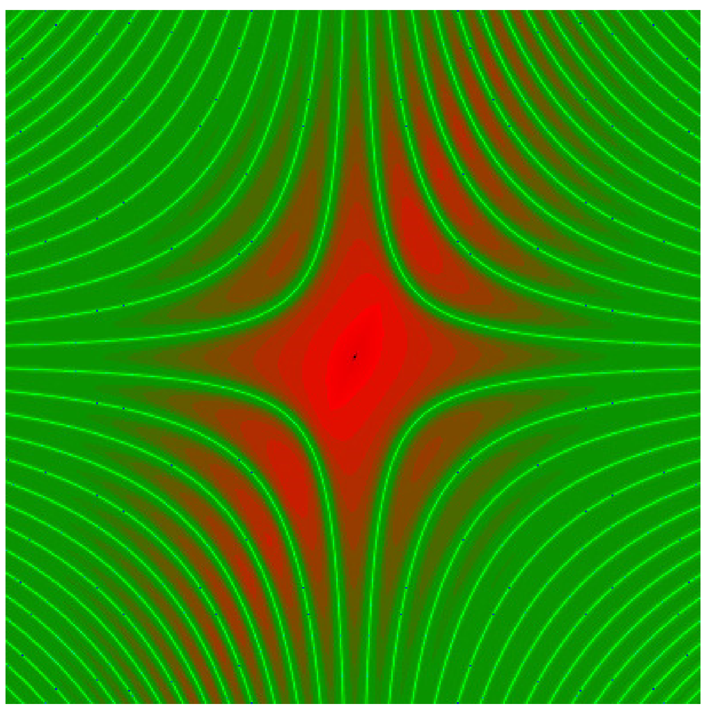




[-10,10]

Example: $\zeta_s(t) = \dfrac{z\left(Cos(2t) + iSin(2t)\right)}{1 + ts\left(Cos(xy) + iSin(xy)\right)}$, $z = x + iy$, z=1+I $\lambda(z) = \int_{S_z} d\mu T$:



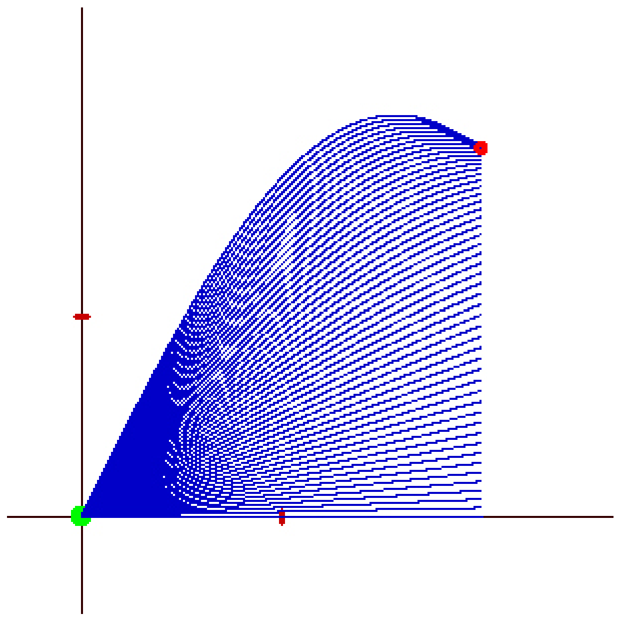


[-10,10]

A contour area measure:

Example: $\zeta_s(t) = 2t + i(s+1)Sin(2st)$. Let $T(a,b) = \{\zeta_s(t) : 0 \le a \le s < b < 1\}$. This cylinder set generates a sigma-algebra. For a measure, set $\mu T(a,b) = \text{Area between } \zeta_a(t) \text{ and } \zeta_b(t)$. And for a functional, set $F[\zeta_b(t)] = |\zeta_b(1)|$. Specify $T_{k,n} = \{\zeta_s(t) : \frac{k}{n} \le s < \frac{k+1}{n}\}$. Then

$$\int_{S_0} F[\zeta_s] d\mu T \;=\; \lim_{n\to\infty} \sum_{k=0}^{n} F[\zeta_{k/n}(1)]\mu T_{k,n} \approx 3.2455.$$

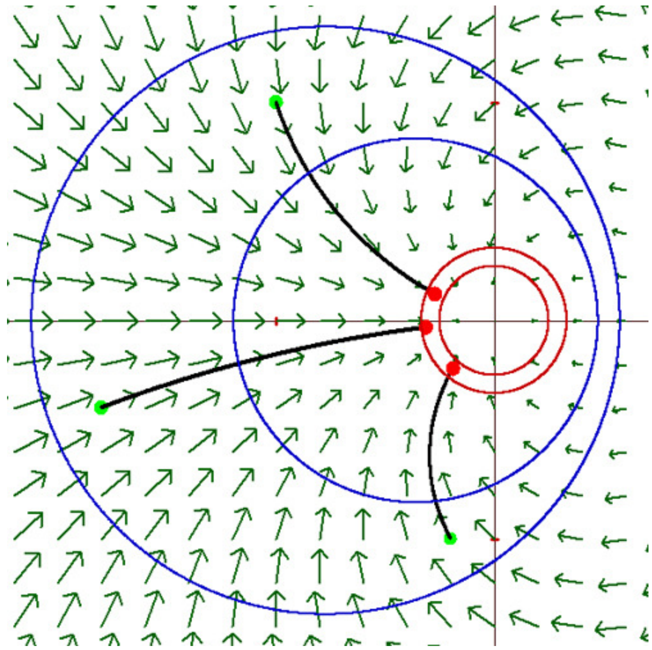


With $$\mu T_{k,n} = \left(1 + \frac{n}{k+1}\right)\left(1 - Cos(2\tfrac{k+1}{n})\right) \;-\; \left(1 + \frac{n}{k}\right)\left(1 - Cos(2\tfrac{k}{n})\right)$$

Contours are related to complex vector fields in the following way:

Given a vector field $f(z)$ in $\mathbb{C}$, to determine a formula for a contour, $z(t)$, write $\frac{dz}{dt} = \varphi(z) = f(z) - z$ and solve for $z(t)$. For instance, $\varphi(z) = z^2 \Rightarrow f(z) = z^2 + z$, leading to $z(t) = \frac{z_0}{1 - z_0 t}$.

This approach is taken in the following example where the focus is upon an attractor in a complex vector field:



Example: Consider the set of contours in the vector space $f(z)=z^2$. There are two fixed points in the space, $\alpha=0$ (attractor) and $\beta=1$ (repulsor), that govern flow. Each circle centered at $\alpha=0$ is a locus of terminating points of contours $z(t),\ t\in[0,1]$. Corresponding to each such circle of radius r_j (in red in figure) is a circle of radius r_j^* (in blue) of initial points of these contours. Let S be the space of contours of the form $\left\{z(t):|z(1)|\le\frac{1}{2}\right\}$.

Now set $\varepsilon_{k,n}=\dfrac{k}{2n},\ 1\le k\le n$. For the rings $R_{k,n}$ about the origin, formed by consecutive radii $\varepsilon_{k,n},\ \varepsilon_{k+1,n}$, set $T_{k,n}=\{z(t):z(1)\in R_{k,n}\}$. We have $\dfrac{dz}{dt}=\varphi(z)=f(z)-z \quad\Rightarrow\quad z(t)=\dfrac{z_0}{z_0(1-e^t)+e^t}$. Partition $D=\left(|z|\le\dfrac{1}{2}\right)$ into diminishing rings

$D=\underset{n\to\infty}{Lim}\bigcup_{k=1}^{n}R_{k,n},\ \ R_{k,n}=\left(|z|\le\varepsilon_{k,n}\right)\cap\left(|z|>\varepsilon_{k+1,n}\right)$. Now observe that

$T_{k,n}=\left\{z(t):z(1)\in R_{k,n}\right\}$ *corresponds* in a one-to-one manner to $Z_{k,n}=\left\{z_0:z(1)\in R_{k,n}\right\}$. The two blue circles in the figure define the boundary of $Z_{k,n}$.

An obvious choice for $\mu T_{k,n}$ would be the area between the blue circles. However, the algebra involved is unpleasant, so, having no application in mind, we default to the much simpler difference between the two circumferences:

$$\Delta C_{k,n} = 4e\pi n \frac{c^2k^2 + c^2k - 4n^2}{\left(c^2k^2 - 4n^2\right)\left(c^2(k+1)^2 - 4n^2\right)}, \quad c = e-1$$

The remaining decisions are to designate a $z_{k,n}(t) \in T_{k,n}$ as $n \to \infty$ and $F[z(t)]$ in a way that

$\sum_{k=1}^{n} F[z_{k,n}(t)] \cdot \mu T_{k,n}$ is a proper mathematical object and is computable. Toward this end observe that very large values of n produce $z_0{'s}$ virtually on the circumference of a (blue) circle $C_{k,n}$. So that <u>all</u> $z_{k,n}(t) \in T_{k,n}$ roughly correspond to points on this one circle. Hence, we choose

$$F[z_{k,n}(t)] = C_{k,n} = \frac{4e\pi nk}{\left(c^2k^2 - 4n^2\right)}$$

Define

$$\int_S F[z] d\mu T = \underset{n\to\infty}{Lim} \sum_{k=1}^{n} F[z_{k,n}(t)] \cdot \mu T_{k,n} = 16e^2\pi^2 n^2 \underset{n\to\infty}{Lim} \sum_{k=1}^{n} \left(\frac{k(c^2k^2 + c^2k - 4n^2)}{\left(c^2k^2 - 4n^2\right)^2 \left(c^2(k+1)^2 - 4n^2\right)} \right).$$

And $\int_S F[z] d\mu T \approx 44.324, \quad n = 100{,}000$.

Example: The same as the one above, except $\varepsilon_k = \frac{1}{k},\ 2 \le k \to \infty$. In this context the integrand is a step-functional and the integral collapses to a common series. $T_k = \{z(t) : z(1) \in R_k\}$ where $R_k = \left(|z| \le \varepsilon_k\right) \cap \left(|z| > \varepsilon_{k+1}\right)$. Again, T_k corresponds to $Z_k = \{z_0 : z(1) \in R_k\}$, the region between two blue circles in the figure. The choice of the area between the two circles as a measure of T_k is reasonable as the algebra is fairly simple:

$$\mu T_k = \Delta A_k = \pi e^2 \frac{(k^2(k+1)^2 - c^4)(2k+1)}{\left(k^2 - c^2\right)^2 \left((k+1)^2 - c^2\right)^2}, \quad c = e-1.$$

And $$F[z_k(t)] = C_k = \frac{2e\pi k}{\left(k^2 - c^2\right)}.$$

Hence $\int_S F[z] d\mu T = \sum_{k=1}^{\infty} F[z_k(t)] \cdot \mu T_k = \sum_{k=1}^{\infty} \left(\frac{2e^3\pi^2 k(k^2(k+1)^2 - c^4)(2k+1)}{\left(k^2 - c^2\right)^3 \left((k+1)^2 - c^2\right)^2} \right) \approx 396.47, \ n = 100$.