A Note: An Elementary Variation of the Banach Fixed Point Theorem

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Abstract: In the Banach theorem simple iteration of a single function converges to a unique fixed point. A variation is described in which infinite sequences of functions are composed, uniformly converging to unique points in the metric space analogous to the Banach fixed point.

We start with the

Banach Fixed Point Theorem: Let (X,d) be a non-empty complete metric space with a contraction mapping $t:X\to X$. Then t admits a unique fixed point $\alpha=t(\alpha)$. Furthermore, α can be found as follows: start with an arbitrary element $x_0\in X$ and define a sequence $\{x_n\}$ by $x_n=t(x_{n-1})$. Then $x_n\to\alpha$.

And proceed to the following

Theorem : Given a complete and bounded metric space (X,d) Let $\{t_k\}_{k=1}^{\infty}$ be a family of functions $t_k:X\to X$ such that $d(t_k(x),t_k(y))<\rho\cdot d(x,y)$, $\rho<1$, for all k, and all $x,y\in X$
Set

$$G_n(x)=t_n\circ t_{n-1}\circ\cdots\circ t_1(x) \quad \text{and} \quad F_n(x)=t_1\circ t_2\circ\cdots\circ t_n(x)$$

Then $F_n(x)\to\beta\in X$ uniformly on X. If $t_n(\alpha_n)=\alpha_n$, the unique fixed points of t_n, $G_n(x)\to\alpha$ uniformly on X if and only if $d(\alpha_k,\alpha)=\varepsilon_k\to 0$.

Proof: **W**rite $F_{k,n}(x)=t_k\circ t_{k+1}\circ\cdots\circ t_n(x)$. Then

$$d(F_{n+m}(x_0),F_n(x_0))<\rho^n d(x_0,F_{n+1,n+m}(x_0))<\rho^n Diam(X)\to 0$$

Hence $F_n(x_0)\to\beta$. Next $d(F_n(x),F_n(x_0))<\rho^n d(x,x_0)\to 0$.

Thus $d(F_n(x),\beta)\le d(F_n(x),F_n(x_0))+d(F_n(x_0),\beta)<\rho^n Diam(X)+d(F_n(x_0),\beta)\to 0,\ n\to\infty$

For the second part set $\eta_{n,k} = d(\alpha,\alpha_{n-k}) + d(\alpha,\alpha_{n-k+1}) = \eta_{n+1,k+1}$

Write $d(G_1(x),\alpha) \le d(G_1(x),\alpha_1) + d(\alpha,\alpha_1) < \rho d(x,\alpha_1) + d(\alpha,\alpha_1)$. And

$$\begin{aligned} d(G_2(x),\alpha) &\le d(G_2(x),\alpha_2) + d(\alpha,\alpha_2) < \rho d(G_1(x),\alpha_2) + d(\alpha,\alpha_2) \\ &\le \rho d(G_1(x),\alpha) + \rho d(\alpha,\alpha_2) + d(\alpha,\alpha_2) \\ &< \rho^2 d(x,\alpha_1) + \rho\eta_{2,1} + d(\alpha,\alpha_2) \end{aligned}$$

Similarly $$d(G_3(x),\alpha) < \rho^3 d(x,\alpha_1) + \sum_{k=1}^{2}\left(\rho^k \eta_{3,k}\right) + d(\alpha,\alpha_3)$$

Therefore, assume

$$d(G_n(x),\alpha) < \rho^n d(x,\alpha_1) + \sum_{k=1}^{n-1}\left(\rho^k \eta_{n,k}\right) + d(\alpha,\alpha_n)$$

By induction

$$\begin{aligned} d(G_{n+1}(x),\alpha) &< \rho d(G_n(x),\alpha) + \rho d(\alpha,\alpha_{n+1}) + d(\alpha,\alpha_{n+1}) \\ &< \rho^{n+1} d(x,\alpha_1) + \sum_{k=1}^{n-1}\left(\rho^{k+1}\eta_{n,k}\right) + \rho\left(d(\alpha,\alpha_n) + d(\alpha,\alpha_{n+1})\right) + d(\alpha,\alpha_{n+1}) \\ &= \rho^{n+1} d(x,\alpha_1) + \sum_{k=1}^{n}\left(\rho^k \eta_{n+1,k}\right) + d(\alpha,\alpha_{n+1}) \end{aligned}$$

The middle term is a null series, since $S_n = \sum_{k=1}^{n} a_k b_{n-k+1}$, $\sum_1^{\infty} a_k < \infty$, $b_j \to 0 \Rightarrow S_n \to 0$.

To show $G_n(x) \to \alpha$ uniformly implies $\alpha_n \to \alpha$, assume there exists $\left\{\alpha_{n_k}\right\}_{k=1}^{\infty}$ such that

$d\left(\alpha_{n_k},\alpha\right) > r > 0$. Now suppose n is sufficiently large that $d(G_n(x),\alpha) < \varepsilon = \frac{1-\rho}{1+\rho} r$.

For $n_k > n+1$, $d\left(G_{n_k}(x),\alpha_{n_k}\right) < \rho d\left(G_{n_k-1}(x),\alpha\right) + \rho d\left(\alpha,\alpha_{n_k}\right) < \rho\varepsilon + \rho d\left(\alpha,\alpha_{n_k}\right)$

Then $d\left(G_{n_k}(x),\alpha_{n_k}\right) > d\left(\alpha,\alpha_{n_k}\right) - d\left(G_{n_k}(x),\alpha\right)$, giving

$$d\left(G_{n_k}(x),\alpha\right) > (1-\rho)r - \rho\varepsilon > \varepsilon, \quad (\rightarrow\leftarrow)$$

Example : $X = S \subset \mathbb{C}$, usual metric. $t_k(z) = \frac{1}{2}x + i\left(\frac{k}{4(k+1)}y - \frac{1}{8}\right)$, $S = \{z : |x| < 1, |y| < 1\}$. Thus $\rho = \frac{1}{2}$, $|t_k(z)| < \frac{7}{8}$. Then $F_n(z) \to -.14384104i$, $n = 20$, and $G_n(z) \to \alpha = -\frac{1}{6}i$ slowly.

Example : **L**et $z(t)$ be a contour in $\mathbb{C}$ defined on $t \in [0,1]$ and let S_α be the set consisting of all such contours with $z(0) = \alpha$. Let $S_\alpha(M)$ be the subset of S_α for which $\underset{t\in[0,1]}{Sup}|z(t)| \le M$, $M > |\alpha|$. Let $M = 1$ and $\alpha = .1$. Now define a sequence of operators $\{T_k\}_{k=1}^{\infty}$ having the property $T_k z(t) \in S_\alpha(M)$ for $z(t) \in S_\alpha(M)$. Set

$$z(t) = x(t) + iy(t) = (.6tSin(10t+5)+.1) + i(.6tCos(10t+5)+.1) \quad \text{and}$$

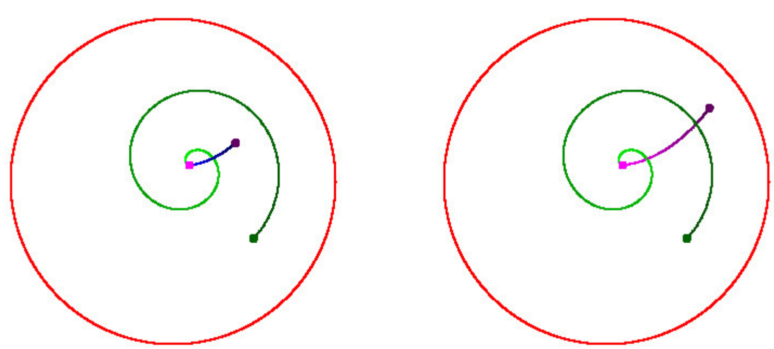
$$T_k z(t) = (\rho_k tCos\,y(t) + .1) + i(\rho_k tSin\,x(t) + .1),\ \ \rho_k = \frac{3}{5}\frac{k}{k+1}.$$ The metric here is

$$d(z_1(t), z_2(t)) = \underset{t\in[0,1]}{Sup}|z_1(t) - z_2(t)|.$$ Then

$$\begin{aligned}|T_k z_1(t) - T_k z_2(t)|^2 &= \rho_k^2 \left|(tCos\,y_1(t) - tCos\,y_2(t)) + i(tSin\,x_1(t) - tSin\,x_2(t))\right|^2 \\ &= \rho_k^2 \left|tSin\zeta_1(t)|y_1(t) - y_2(t)| + itCos\zeta_2(t)|x_1(t) - x_2(t)|\right|^2 \\ &\le \rho_k^2\left((y_1(t) - y_2(t))^2 + (x_1(t) - x_2(t))^2\right) = \rho_k^2 |z_1(t) - z_2(t)|^2\end{aligned}$$

Therefore $|T_k z_1(t) - T_k z_2(t)| \le \rho_k |z_1(t) - z_2(t)| \;\Rightarrow\; d(T_k z_1(t), T_k z_2(t)) \le \rho_k d(z_1(t), z_2(t))$.

$$T_1 \circ T_2 \circ \cdots \circ T_n z(t) \to \beta(t) \quad \text{and} \quad T_n \circ T_{n-1} \circ \cdots \circ T_1 z(t) \to \alpha(t)$$



$z(t)$ *green*, $\beta(t)$ *purple* n=1000 $z(t)$ *green*, $\alpha(t)$ *purple*

Example : **I**n this example two methods of *contour composition* are described and illustrated.

Suppose we have two contours: $z_1(t)=3t-it^2$ and $z_2(t)=2t-\frac{i}{2}t^2$. How can we "compose" one with the other? One method would be to simply express

$$z(t)=z_2\circ z_1(t)=3t\left(2-t^2\right)+\frac{i}{2}t^2\left(t^2-13\right)$$

But there is a more interesting compositional procedure that involves the differential equations giving rise to these contours. Let us write

$$\gamma_1:\frac{dz}{dt}=\varphi_1(z,t) \quad \text{and} \quad \gamma_2:\frac{dz}{dt}=\varphi_2(z,t)$$

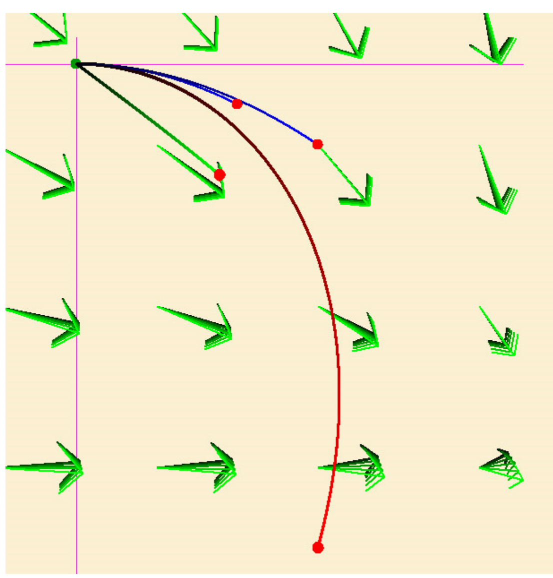
And define $$\gamma_2\circ\gamma_1:\frac{dz}{dt}=\varphi(z,t)=\varphi_2\left(\varphi_1(z,t),t\right)$$

Of course these expressions are not given at the outset, but may be derived as follows:

$$\frac{dz_1}{dt}=3-2it=3-2i\left(\frac{1}{3}(z+it^2)\right)=3+\frac{2}{3}t^2-\frac{2i}{3}z=\varphi_1(z,t)$$. Similarly

$$\frac{dz_2}{dt}=2+\frac{1}{4}t^2-\frac{i}{2}z=\varphi_2(z,t)$$. Therefore, $$\frac{dz}{dt}=\varphi(z,t)=\left(2+\frac{1}{4}t^2-\frac{1}{3}x\right)-i\left(\frac{3}{2}+\frac{1}{3}t^2+\frac{1}{3}y\right).$$

The red contour is $z(t)=z_2\circ z_1(t)$; the two blue contours are $z_1(t)$ and $z_2(t)$. The green contour is $\gamma=\gamma_2\circ\gamma_1$. The vector field is $f(z,t)=\varphi(z,t)+z$.



This peculiar composition can be expressed as $\frac{dz_2}{dt} = \varphi_2\left(\frac{dz_1}{dt}, t\right) = \varphi_2\left(\varphi_1(z_1(t),t),t\right)$ which presents a problem of interpretation, as the $z's$ are not the same. However, separating φ_1 from its differential equation opens the possibility for a new $z = z(t)$, satisfying $\frac{dz}{dt} = \varphi(z,t)$.

From the previous example let $S_\alpha(M)$ be the subset of S_α for which $\underset{t\in[0,1]}{Sup}\left|z(t)\right| \le M$, $M > \left|\alpha\right|$. Now, assume $\alpha = 0$, $M = 1$ and $\varphi_k(z,t) = \varphi_k(z(t),t)$ is an operator on this set such that

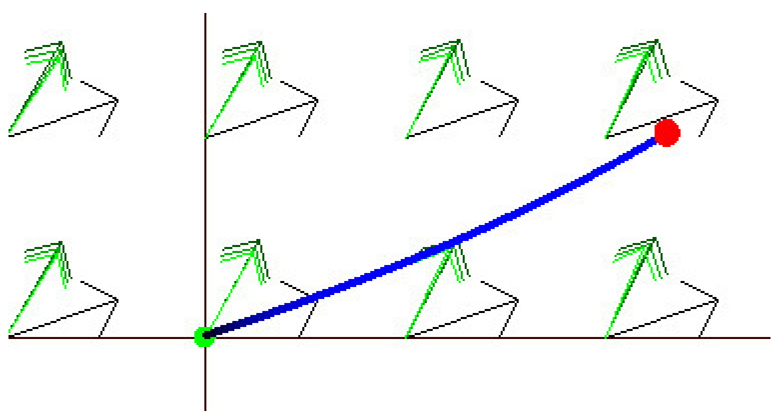
$d\left(\varphi_k(z_1,t), \varphi_k(z_2,t)\right) < \rho d\left(z_1(t), z_2(t)\right)$, which entails the inequality

$\left|\varphi_k(z_1,t) - \varphi_k(z_2,t)\right| < \rho\left|z_1(t) - z_2(t)\right|$, $\rho < 1$, $\forall k \ge 1$, $\forall t \in [0,1]$. Also, require $\varphi_k \to \varphi$.

Then $\frac{dz}{dt} = \varphi(z,t) = \mathop{\mathcal{L}}_{k=1}^{\infty} \varphi_k(z,t)$ defines a unique contour in $S_\alpha(M)$.

For example, set

$$\varphi_k(z,t) = \frac{2k}{5(k+1)}\left(Cos(tx) + \frac{1}{2}\right) \ + \ i\frac{2k}{5(k+1)}\left(Sin(ty) + \frac{1}{2}\right).$$



$z(1) \approx .578 + .254i$ n=500