A Note: Exotic Explorations & Images in Analytic Continued Fraction Theory

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Abstract: Informal discussions of unusual continued fraction examples & theory. The images – apart from vector fields - are simple topographical (contour) maps with dark=small moduli and light=large moduli, and do not necessarily correspond to theoretical hypotheses.

1. Continued fraction fixed-point algorithm

Begin with: $CF=\frac{\alpha_1(z)}{\beta_1(z)+}\ \frac{\alpha_2(z)}{\beta_2(z)+}\ \frac{\alpha_3(z)}{\beta_3(z)+}\ \cdots$ From which one obtains

$$G_n(z)=\frac{\alpha_1(G_{n-1}(z))}{\beta_1(G_{n-1}(z))\ +}\ \frac{\alpha_2(G_{n-1}(z))}{\beta_2(G_{n-1}(z))\ +}\ \frac{\alpha_3(G_{n-1}(z))}{\beta_3(G_{n-1}(z))\ +}\ \cdots\ \frac{+\ \alpha_n(G_{n-1}(z))}{\beta_n(G_{n-1}(z))}\ ,\ G_0(z)\equiv z$$

Or

(1) $$G_n(z)=\frac{a_{1,n}(z)}{b_{1,n}(z)\ +}\ \frac{a_{2,n}(z)}{b_{2,n}(z)\ +}\ \cdots\ \frac{+\ a_{n,n}(z)}{b_{n,n}(z)}.$$

Where $a_{k,n}(z)=\alpha_k(G_{n-1}(z)),\ b_{k,n}(z)=\beta_k(G_{n-1}(z))$,

Suppose, e.g., $|z|<R,\ |\alpha_k(z)|<R,\ |\beta_k(z)|>1+\varepsilon+R$.

Thus $\left|\frac{\alpha_k(z)}{\beta_k(z)}\right|<\frac{R}{1+R}<R$, $\left|\frac{\alpha_{k-1}(z)}{\beta_{k-1}(z)+\frac{\alpha_k(z)}{\beta_k(z)}}\right|\le r<R$, etc.

When there is a contraction into a compact set involving $g_k(z)=f_1\circ f_2\circ\cdots\circ f_k(0)$, where $f_k(\zeta)=\frac{\alpha_k(z)}{\beta_k(z)+\zeta}$, and all functions are *analytic* in the region being investigated,

$G_n(z)=\mathop{\mathcal{L}}_{k=1}^{n} g_k(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z)$ and (1) converge to the fixed point, ς, of the CF if and only if the fixed points of $g_k(z)$ converge to ς [1]: $\varsigma=\frac{\alpha_1(\varsigma)}{\beta_1(\varsigma)+}\ \frac{\alpha_2(\varsigma)}{\beta_2(\varsigma)+}\ \frac{\alpha_3(\varsigma)}{\beta_3(\varsigma)+}\ \cdots$

Analyticity may be replaced by a simpler condition:

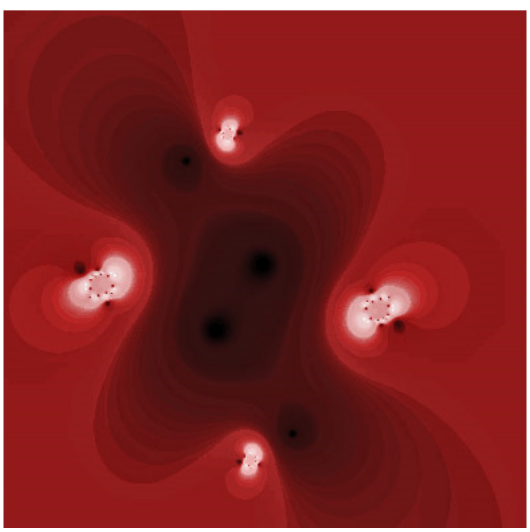
Suppose $|g_k(z)-\alpha_k| < \rho|z-\alpha_k|,\ 0 \le \rho < 1,\ \forall k,\ z \in S,\ g_k(z) \in S$ and $\alpha_n \to \alpha$.

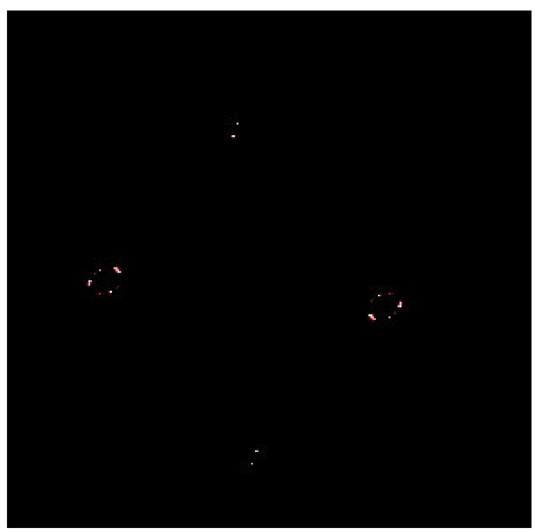
Set $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$. Then it is easily seen that

$$|G_n(z)-\alpha| < \rho^n |z-\alpha_1| + |\alpha_n - \alpha| + \sum_{k=1}^{n-1} a_k \rho^{n-k},\ a_k = |\alpha_k - \alpha_{k+1}| .$$

To show that $\sum_{k=1}^{n-1} a_k \rho^{n-k} \to 0$ is a pleasant advanced calculus exercise. Thus $G_n(z) \to \alpha$.

Example : $F(z) = \cfrac{\frac{1}{10}(z+1)^2}{1 - \cfrac{\frac{1}{10}(z+\frac{1}{2})^2}{1 + \cfrac{\frac{1}{10}(z+\frac{1}{3})^2}{1 - \cdots}}}$, $F(z)=z$?, $[-10,10]$, $n=5$. Here we use the algorithm at each point of the region to find a fixed-point ς, then we calculate $D(\varsigma)=F(\varsigma)-\varsigma$ and graph the moduli at that point in the plane. Very dark suggests $D(\varsigma)\approx 0$.

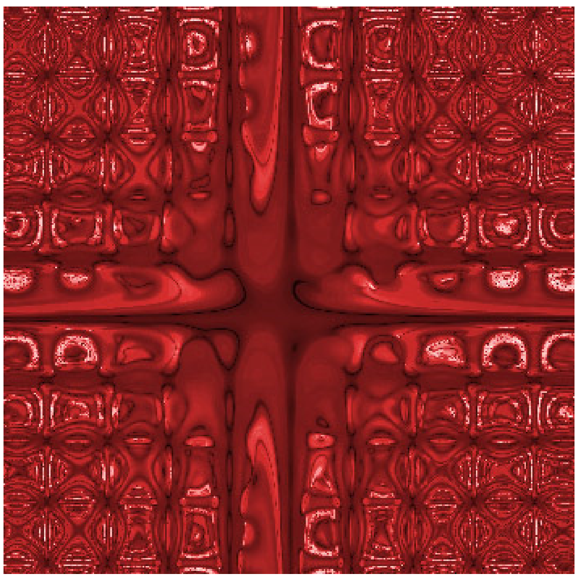




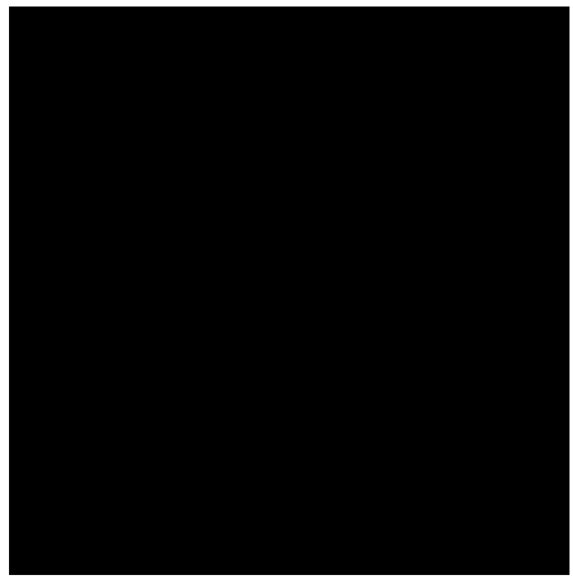
N=10 (right) shows a nearly continuous dark image, suggesting that the algorithm leads to the fixed point more or less uniformly on the region. $\varsigma \approx -.0726+.2138i$

Example : [-15,15]

$$\cfrac{\frac{1}{10}\left((x+1)Cos(y\cdot 1)+i(y+1)Sin(x\cdot 1)\right)^2}{1-\cfrac{\frac{1}{10}\left((x+1)Cos(y\cdot \frac{1}{2})+i(y+1)Sin(x\cdot \frac{1}{2})\right)^2}{1+\cfrac{\frac{1}{10}\left((x+1)Cos(y\cdot \frac{1}{3})+i(y+1)Sin(x\cdot \frac{1}{3})\right)^2}{1-\cdots}}}$$



N=3



N=10 $\varsigma \approx .14436+.04085i$

Consider the continued fraction with analytic partial numerators/denominators (holding ζ constant):

$$CF(z,\zeta(t)) = \frac{a_1(z,\zeta)}{b_1(z,\zeta)\ +}\ \frac{a_2(z,\zeta)}{b_2(z,\zeta)\ +}\ \frac{a_3(z,\zeta)}{b_3(z,\zeta)\ +}\cdots\frac{+\ a_n(z,\zeta)}{b_n(z,\zeta)\ +}\cdots,\quad t:0\to 1$$

Define by recursion, for each value of t:

$$g_n(z,\zeta) = \frac{a_1(g_{n-1}(z,\zeta),\zeta)}{b_1(g_{n-1}(z,\zeta),\zeta)\ +}\ \frac{a_2(g_{n-1}(z,\zeta),\zeta)}{b_2(g_{n-1}(z,\zeta),\zeta)\ +}\ \cdots\ \frac{+\ a_n(g_{n-1}(z,\zeta),\zeta)}{b_n(g_{n-1}(z,\zeta),\zeta)}$$

$$= \frac{a_{1,n}}{b_{1,n}\ +}\ \frac{a_{2,n}}{b_{2,n}\ +}\ \cdots\ \frac{+\ a_{n,n}}{b_{n,n}}$$

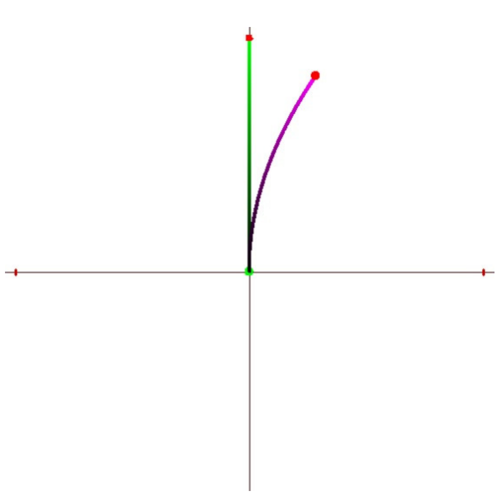
With $t\in[0,1] \Rightarrow \zeta(t)\in H$, continuous, and

$g_n(S,H)\subset\overline{\Omega}\subset S,\ g_n(\alpha_n(\zeta(t)),\zeta(t)) = \alpha_n(\zeta(t)),\ \alpha_n(\zeta(t))\to\alpha(\zeta(t))$

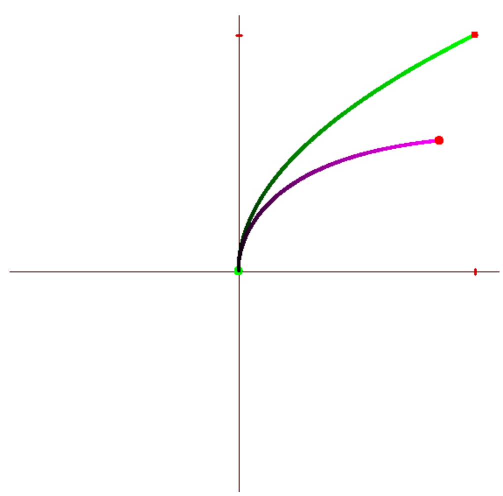
$$\Rightarrow\quad CF(\alpha(\zeta(t)),\zeta(t)) = \alpha(\zeta(t))\ ,\ [1]$$

For a given function $\zeta(t)$, the ζ - *Fixed Point Spectrum* is the contour whose points are the *fixed points* of the original CF, corresponding to the points of the contour $\zeta(t)$.

Example: $CF(z,\zeta(t)) = \frac{z+\zeta}{2\ +}\ \frac{z+\zeta}{5\ +}\ \frac{z+\zeta}{10\ +}\ \cdots\ \frac{+\ z+\zeta}{n^2+1\ +}\cdots$, [-1,1]. $\zeta(t)$ contour green.

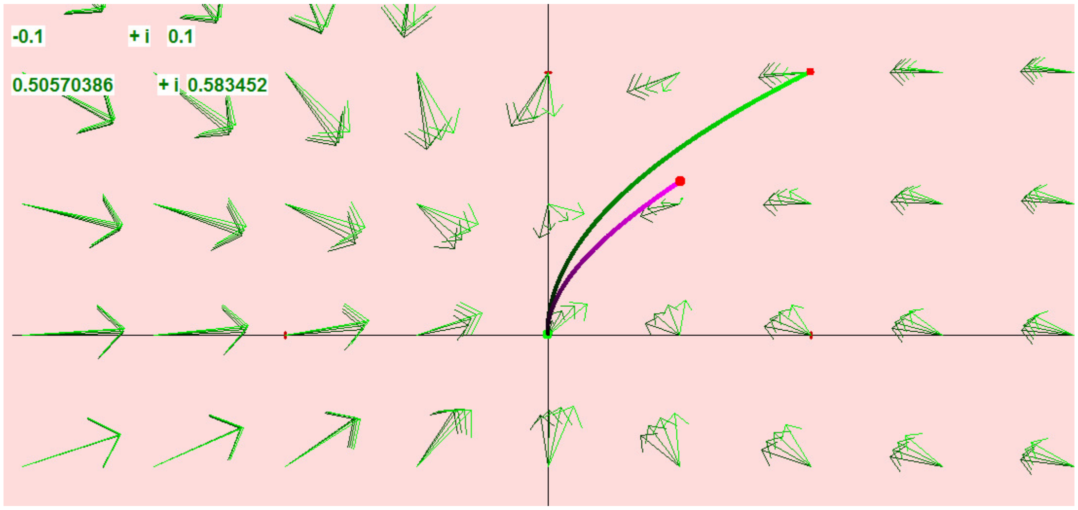


$\zeta(t) = it$



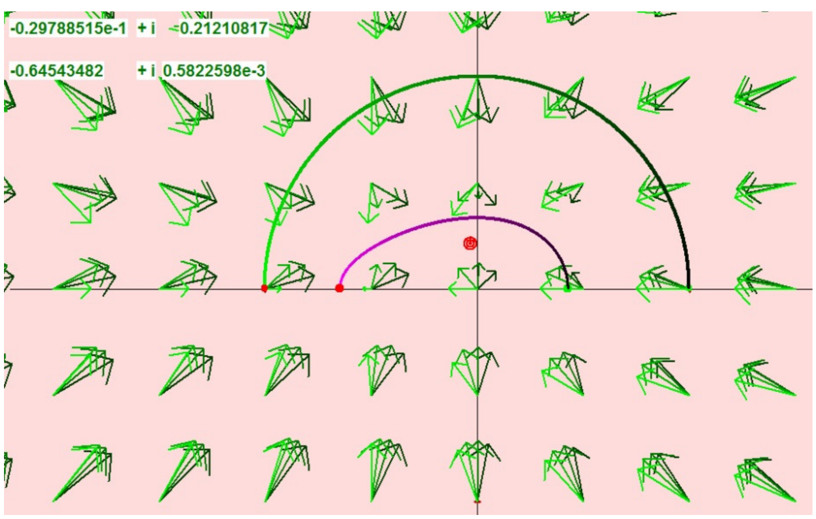
$\zeta(t) = t^2 + it$

Example: $CF(z,\zeta(t)) = \frac{z^2+\zeta}{2\ +}\ \frac{z^2+\zeta}{3\ +}\ \frac{z^2+\zeta}{4\ +}\ \ldots\ \frac{+\ z^2+\zeta}{n+1\ +} \ldots,\ \zeta(t) = t^2 + it$



The vector field is an approximation of the T-D vector field defined by the CF.

Example: $CF = \frac{xCos(y)+iySin(x)+Sin(\pi t)}{3\quad +}\ \frac{xCos(y)+iySin(x)+Cos(\pi t)}{4\quad +}\ \ldots,\quad \zeta(t) = Sin(\pi t) + iCos(\pi t)$



The progression of fixed points can be followed by observing the vectors. The top number in the upper corner is the mean of the fixed points (red dot) corresponding to the green contour.

2. Continued fractions as functions of the tail-end variable

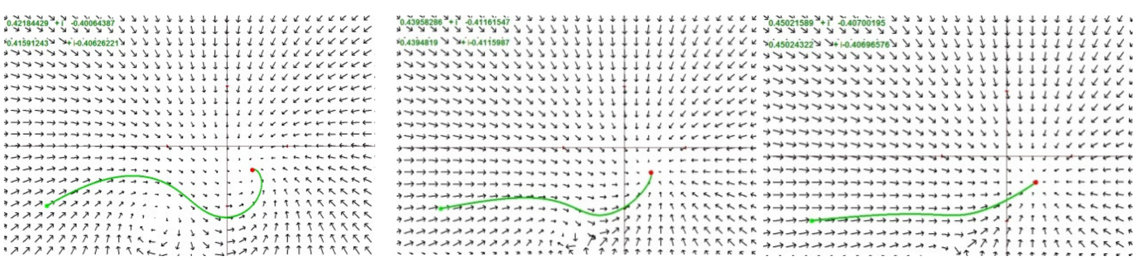
A different interpretation of fixed points of continued fractions occurs when one considers a CF to be a function of the "tail end" variable. A CF can be interpreted as a forward or inner composition of *linear fractional transformations* of the general form

$$t_n(z)=\frac{a_n(\zeta)z+b_n(\zeta)}{c_n(\zeta)z+d_n(\zeta)}\ ,\ a_n(\zeta)\equiv 0,\quad c_n(\zeta)\equiv 1\,,\qquad T_n(z)=t_1\circ t_2\circ\cdots\circ t_n(z)\ ,\ z=0\,.$$

In applications the CF is considered a function of ζ , but fixing ζ and allowing the compositional variable z to assume other complex values gives a CF as a function of z . Lorentzen's theorem [2] requires $t_n(S)\subset\bar{\Omega}\subset S$ and yields $T_n(z)\rightarrow\alpha$ for $z\in S$, where $\alpha=\alpha(\zeta),\ \alpha_n=T_n(\alpha_n)$ and

$$\alpha_n\rightarrow\alpha=T(\alpha)=\lim_{n\to\infty}T_n(\alpha_n)\,.$$

Example: $t_n(z)=\dfrac{\left(1+\frac{1}{n}i\right)z+\left(1-2i\right)}{\left(\frac{1}{n}-2i\right)z+\left(5-\frac{1}{n}i\right)}$, $T_n(-6-3i)$: n=1 , n=2 , n=3



This example shows how rapidly the composition of LFTs evolves into a simple vector field structure with a single attracting fixed point. For values of n>3 the vector field shows Zeno contours converging to the attractor along approximately straight lines.

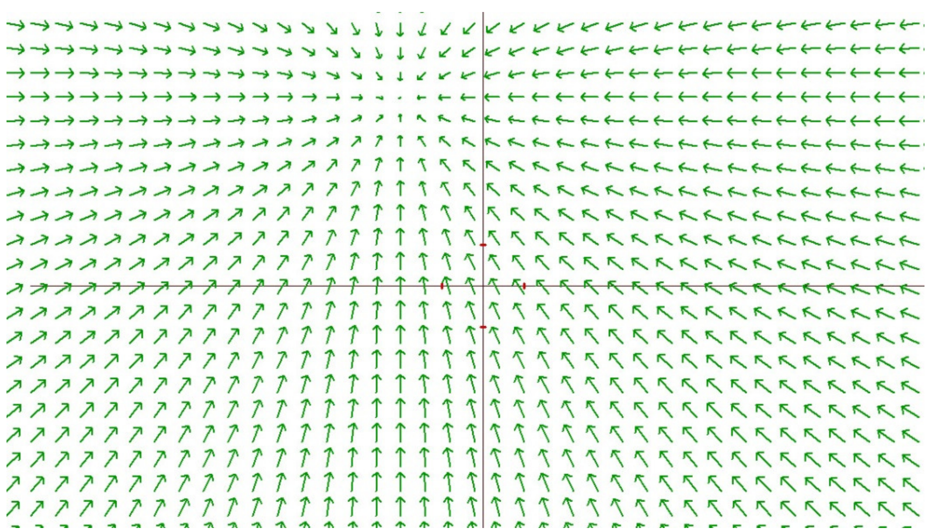
Using the following schema of *forward recursion* for continued fractions, one can locate fixed points of these continued fractions – as functions of the tail-end variable z. Or the fixed point can be found by resorting to the algorithm described earlier:

$$\frac{A_n(z)}{B_n(z)} = \frac{a_1}{b_1 \;\; +} \; \frac{a_2}{b_2 \;\; +} \; \frac{a_3}{b_3 \;\; +} \cdots \frac{a_n}{b_n + z} \;\; ,$$

$$\begin{cases} A_k = b_k A_{k-1} + a_k A_{k-2} \\ B_k = b_k B_{k-1} + a_k B_{k-2} \end{cases} , \quad \begin{cases} A_n(z) = (b_n + z) A_{n-1} + a_n A_{n-2} \\ B_n(z) = (b_n + z) B_{n-1} + a_n B_{n-2} \end{cases} \quad \Rightarrow$$

$$\frac{A_{n-1} z + A_n}{B_{n-1} z + B_n} = z \quad \Rightarrow \quad z = \frac{A_{n-1} - B_n \pm \sqrt{(A_{n-1} - B_n)^2 + 4 A_n B_{n-1}}}{2 B_{n-1}} = \varsigma$$

Example : $T_n(z) = \dfrac{4+(3+1)i}{8 \quad -} \; \dfrac{4+(3+\frac{1}{2})i}{8 \quad +} \; \dfrac{4+(3+\frac{1}{3})i}{8 \quad -} \; \cdots \; \dfrac{4+(3+\frac{1}{n})i}{8 \quad +(-1)^n \cdot z}$



$n = 15 \;\; \Rightarrow \;\; \varsigma \approx -2.047 + 4.476i$ using the algorithm in 1. rather than the schema.

3. Continuized linear fractional transformations and contour integration

A *linear fractional transformation* (LFT) , $f(\zeta)=\dfrac{a\zeta+b}{c\zeta+d}$, can be written in terms of its (one or) two *fixed points (* α and β *)* and its *multiplier or indicator (* K *)*, as

$$\frac{f(\zeta)-\alpha}{f(\zeta)-\beta}=K\cdot\frac{z-\alpha}{z-\beta} \quad\Rightarrow\quad \frac{f^{(n)}(\zeta)-\alpha}{f^{(n)}(\zeta)-\beta}=K^{n}\cdot\frac{z-\alpha}{z-\beta}$$, implying

$$f^{(n)}(\zeta)=\frac{\left(\alpha-K^{n}\beta\right)\zeta+\alpha\beta\left(K^{n}-1\right)}{\left(1-K^{n}\right)\zeta+\left(K^{n}\alpha-\beta\right)}$$, leading to a natural continuization

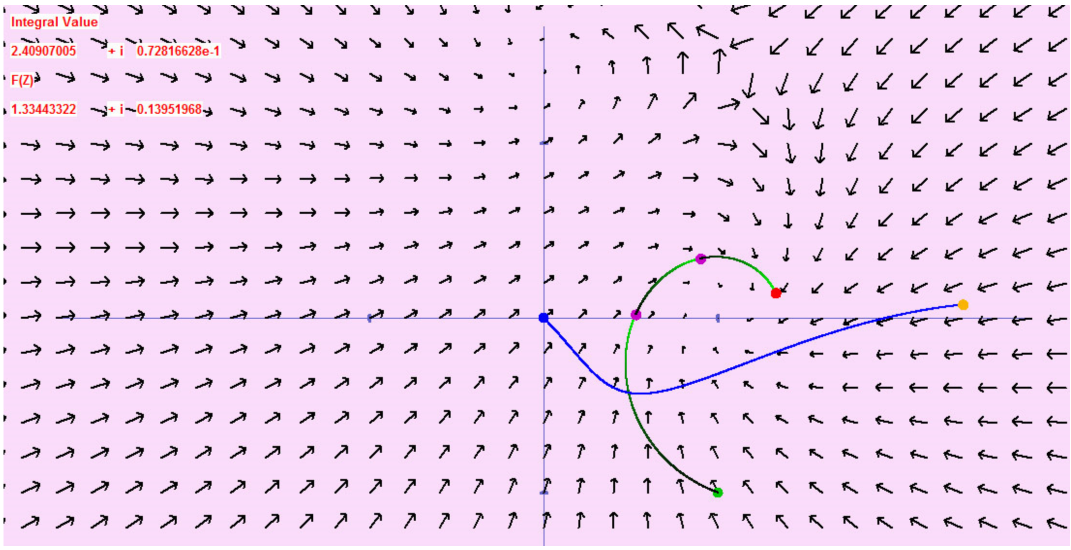
$$\mathcal{V}_{\zeta}(t)=f^{(t)}(\zeta)=\frac{\left(\alpha-K^{t}\beta\right)\zeta+\alpha\beta\left(K^{t}-1\right)}{\left(1-K^{t}\right)\zeta+\left(K^{t}\alpha-\beta\right)} \quad , \quad 0\le t\le T .$$ Then

$$f^{(0)}(\zeta)=\zeta \quad \text{and} \quad f^{(1)}(\zeta) = \frac{(\alpha-K\beta)\zeta+\alpha\beta(K-1)}{(1-K)\zeta+(K\alpha-\beta)} = f(\zeta).$$

A simple algorithm then allows the computation of a contour integral $\lambda(\zeta,\mathrm{T})=\int_0^T \mathcal{V}_{\zeta}(t)dt$.

If one interprets $\mathcal{V}_{\zeta}(t)$ as the instantaneous velocity of a particle departing from the origin, then $\lambda(\zeta,\mathrm{T})=\int_0^T \mathcal{V}_{\zeta}(t)dt$ is the displacement of the particle from the origin at the end of T time units. The distance traversed by the particle is given by: $\mathrm{L}(\zeta,\mathrm{T})=\int_0^T \left|\mathcal{V}_{\zeta}(t)\right|\cdot dt$

Example: $\alpha = 1$, $\beta = 1.5i$, $K = -.8i$, $\zeta = 1 - i$, $T = 3$, $\lambda(\zeta,3) = \int_0^3 \mathcal{V}_\zeta(t)dt \approx 2.409 + .0728i$



The velocity contour is green and the integral contour is blue.

Suppose now that a CF is generated that provides the instantaneous velocity of the particle as each discrete approximate. If, for instance, this is each minute, how can one reasonably compute the displacement of the particle at, say, 3 minutes and 23 seconds? One answer is to *continuize* the CF [8] and interpolate for the 23 seconds.

A general composition of LFTs**:**

$$f_n(\zeta) = \frac{a_n\zeta + b_n}{c_n\zeta + d_n}\ ,\quad F_n(\zeta) = f_1 \circ f_2 \circ \cdots \circ f_n(\zeta) \ \rightarrow\ F(\zeta).$$

We have $F_n(\zeta) = \dfrac{A_n\zeta + B_n}{C_n\zeta + D_n}$ with the forward recursion scheme

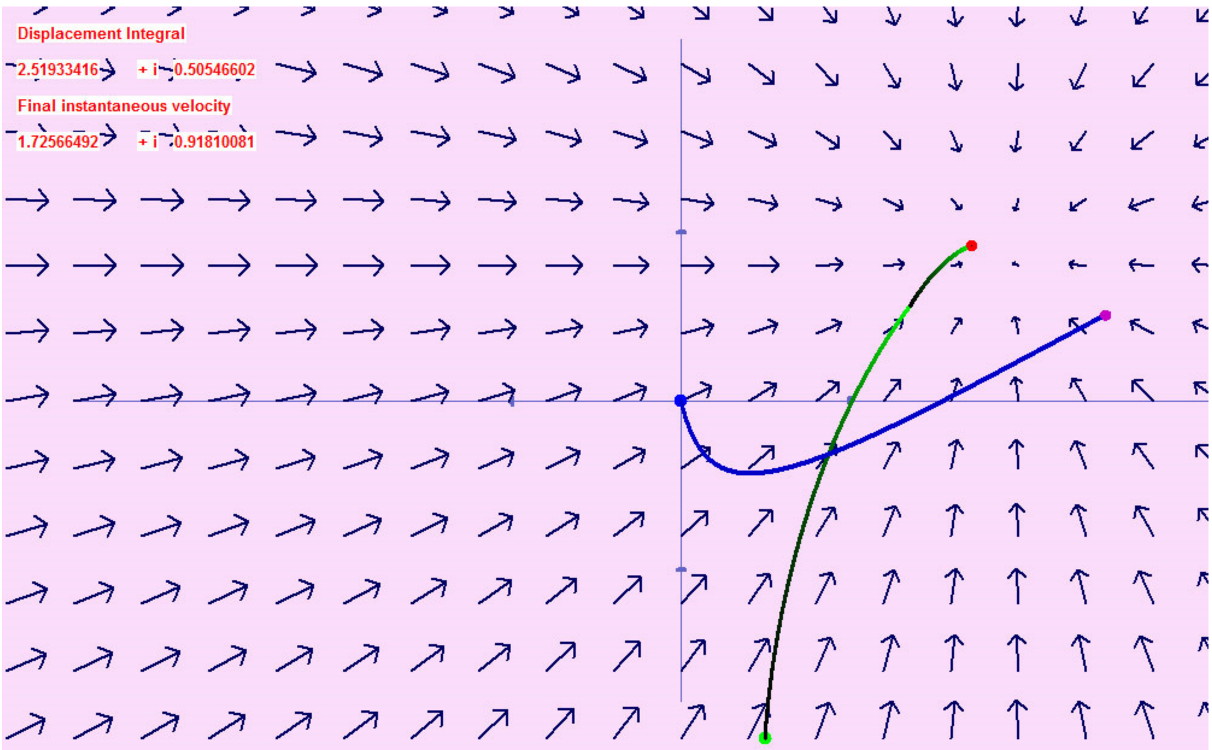
$$A_n = a_n A_{n-1} + c_n B_{n-1}, B_n = b_n A_{n-1} + d_n B_{n-1}, C_n = a_n C_{n-1} + c_n D_{n-1}, D_n = b_n C_{n-1} + d_n D_{n-1}.$$

Defining $f_n^{(t)}(\zeta) = \dfrac{\left(\alpha_n - K_n^t \beta_n\right)\zeta + \alpha_n \beta_n \left(K_n^t - 1\right)}{\left(1 - K_n^t\right)\zeta + \left(K_n^t \alpha_n - \beta_n\right)} = \dfrac{a_n^{(t)}\zeta + b_n^{(t)}}{c_n^{(t)}\zeta + d_n^{(t)}}$ generates an algorithm for continuizing the composition.

Example: $\zeta = .5 - 2i$, $\lambda(\zeta,\mathrm{T}) = \int_0^T \mathcal{V}_\zeta(t)dt$, $\alpha_k = 2 + \frac{1}{k}i$, $\beta_k = \frac{4}{k} + 5i$, $K_k = \dfrac{\alpha_k}{\beta_k}$

$$\mathcal{V}_\zeta(0) = .5 - 2i,\ \mathcal{V}_\zeta(1) = \frac{\alpha_1\beta_1}{\alpha_1+\beta_1},\ \mathcal{V}_\zeta(2) = \frac{\alpha_1\beta_1}{\alpha_1+\beta_1-\dfrac{\alpha_2\beta_2}{\alpha_2+\beta_2}},\ \mathcal{V}_\zeta(3) = \frac{\alpha_1\beta_1}{\alpha_1+\beta_1-\dfrac{\alpha_2\beta_2}{\alpha_2+\beta_2-\dfrac{\alpha_3\beta_3}{\alpha_3+\beta_3}}}$$

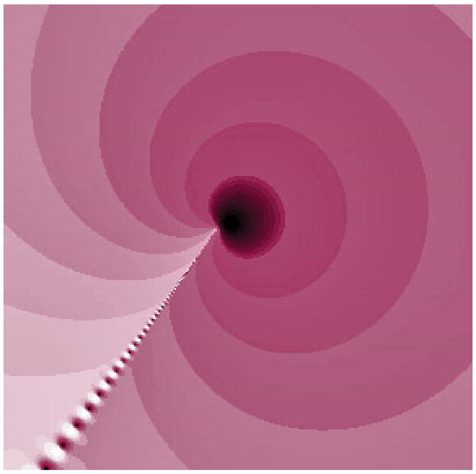
$$\lambda(\zeta,\mathrm{T}) = \int_0^{3.3833} \mathcal{V}_\zeta(t)dt \approx 2.5193 + .5055i:$$



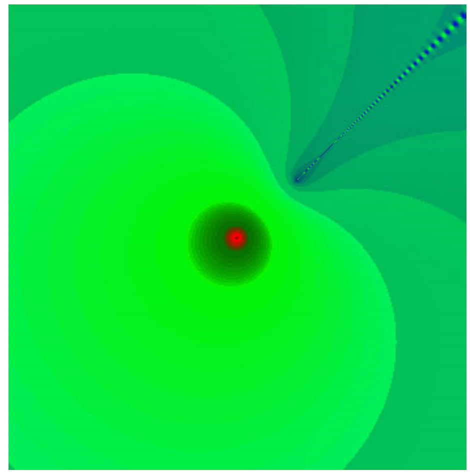
$$\mathrm{L}(\zeta,\mathrm{T}) = \int_0^T \left|\mathcal{V}_\zeta(t)\right| \cdot dt \approx 2.978$$

Examples: (Topo contours) (1) $\alpha=1+2i,\ \beta=-1+2i,\ K=.4(1+i),\ n=100,\ [-15,15],$

(2) $\alpha=-3+3i,\ \beta=5-2i,\ K=.8+.4i,\ [-60,60],\ n=100$ $\qquad \lambda(\zeta) = \int_0^1 \mathcal{V}_\zeta(t)dt\,,\ \left|\lambda(\zeta)\right|$:



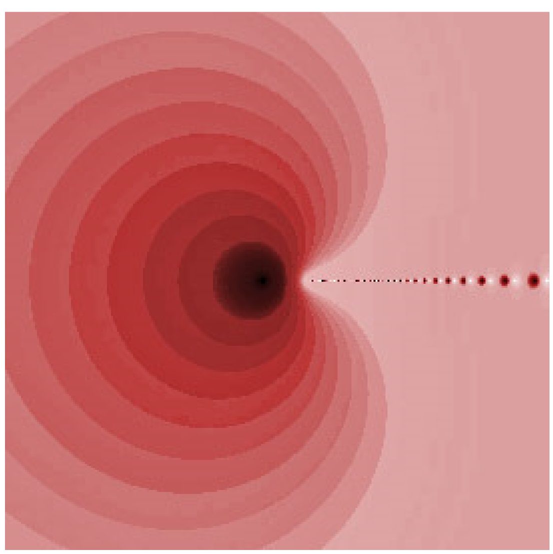
(1)



(2)

When $K=\dfrac{\alpha}{\beta}$ the LFT has the appearance $f(\zeta)=\dfrac{\alpha\beta}{\alpha+\beta-\zeta}$ and a composition of such LFTs generates a continued fraction.

Example: $\alpha=1+i,\ \beta=1-i,\ K=i,\ [-20,20],\ n=100$ $\qquad \lambda(\zeta) = \int_0^1 \mathcal{V}_\zeta(t)dt\,,\ \left|\lambda(\zeta)\right|$:



4. Fixed-point continued fractions

A *fixed-point CF* is generated by $t_n(z)=\frac{\alpha_n\beta_n}{\alpha_n+\beta_n-z}$, $T_n(0)=t_1\circ t_2\circ\cdots\circ t_n(0)$. If $|\alpha_n|<|\beta_n|$ then α_n is an *attractor* and β_n is a *repellor* of t_n. If $t_n\equiv t$ then the (periodic) FPCF will converge to $\alpha_n\equiv\alpha$. The indicator (or multiplier) $K_n=\frac{\alpha_n}{\beta_n}$ provides a means of classification; in the present case the CF is *Hyperbolic/Loxodromic* (from similar classifications of $t_n(z)=\frac{\alpha_n\beta_n}{\alpha_n+\beta_n-z}$).

Assume in what follows these conditions hold.

We have

$$FPCF=\frac{\alpha_1\beta_1}{\alpha_1+\beta_1-}\;\frac{\alpha_2\beta_2}{\alpha_2+\beta_2-}\;\frac{\alpha_3\beta_3}{\alpha_3+\beta_3-}\;\cdots$$

The *equivalent CF* of Euler, $CF_n=\frac{\alpha_1}{1-}\;\frac{\alpha_2}{1+\alpha_2-}\;\frac{\alpha_3}{1+\alpha_3-}\;\cdots\;\frac{\alpha_n}{1+\alpha_n}\;=\;\sum_{k=1}^{n}\left(\prod_{j=1}^{k}\alpha_j\right)$,

yields a specific fixed-point CF: $\frac{\alpha_2}{1+\alpha_2-}\;\frac{\alpha_3}{1+\alpha_3-}\;\cdots\;\frac{\alpha_n}{1+\alpha_n}\;=\;\frac{\sum_{k=1}^{n}\left(\prod_{j=1}^{k}\alpha_j\right)-\alpha_1}{\sum_{k=1}^{n}\left(\prod_{j=1}^{k}\alpha_j\right)}$

The general FPCF can be rewritten as

$FPCF=\beta_0\cdot\frac{\frac{\alpha_1}{\beta_0}}{\frac{\alpha_1}{\beta_1}+1-}\;\frac{\frac{\alpha_2}{\beta_1}}{\frac{\alpha_2}{\beta_2}+1-}\;\frac{\frac{\alpha_3}{\beta_2}}{\frac{\alpha_3}{\beta_3}+1-}\cdots$ which implies , under the assumption $\beta_n\equiv\beta$,

$$FPCF\;=\;\beta\cdot\left(1\;-\;\frac{1}{\sum_{k=1}^{n}\left(\prod_{j=1}^{k}\frac{\alpha_j}{\beta}\right)}\right),\quad(\alpha_1=1)$$

A $FPCF = \frac{\alpha_1\beta_1}{\alpha_1+\beta_1-}\ \frac{\alpha_2\beta_2}{\alpha_2+\beta_2-}\ \frac{\alpha_3\beta_3}{\alpha_3+\beta_3-}\cdots$ converges to α provided $\alpha_n \equiv \alpha,\ \beta_n \equiv \beta$.

However, in the more general case where $\alpha_n \to \alpha$ and $|K_n| < \rho < 1$ it has been shown that the CF may converge more rapidly to its final value if one forms the sequence

$$T_n(\alpha) = \frac{\alpha_1\beta_1}{\alpha_1+\beta_1-}\ \frac{\alpha_2\beta_2}{\alpha_2+\beta_2-}\ \frac{\alpha_3\beta_3}{\alpha_3+\beta_3-}\ \cdots\ \frac{\alpha_n\beta_n}{\alpha_n+\beta_n-\alpha}$$

And oddly enough if α_n and β_n are analytic in ζ, say, the expression

$$T_n(\beta) = \frac{\alpha_1\beta_1}{\alpha_1+\beta_1-}\ \frac{\alpha_2\beta_2}{\alpha_2+\beta_2-}\ \frac{\alpha_3\beta_3}{\alpha_3+\beta_3-}\ \cdots\ \frac{\alpha_n\beta_n}{\alpha_n+\beta_n-\beta}$$

might *analytically continue* the normal FPCF $T_n(0) = F_n(\zeta) \to F(\zeta)$.

I introduced the concept of *attractor* fixed-point acceleration of *limit-periodic* CFs (LPCF) in [3], and Thron and Waadeland completed the theory later in [4]. I also introduced the idea of using the repellor to analytically continue the CF [5]. What follows is a rough sketch of that acceleration argument [4]:

First, observe that $\frac{a_1}{b_1-}\ \frac{a_2}{b_2-}\ \frac{a_3}{b_3-}\ \cdots\ \frac{a_n}{b_n-z} = \frac{-a_1}{-b_1+}\ \frac{-a_2}{-b_2+}\ \frac{-a_3}{-b_3+}\ \cdots\ \frac{-a_n}{-b_n+z}$

So that

$$\frac{\alpha_1\beta_1}{\alpha_1+\beta_1-}\ \frac{\alpha_2\beta_2}{\alpha_2+\beta_2-}\ \cdots\ \frac{\alpha_n\beta_n}{\alpha_n+\beta_n-z} = \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+}\ \cdots\ \frac{a_n}{b_n+z},$$

$a_k = -\alpha_k\beta_k,\ b_k = -(\alpha_k+\beta_k)$.

Assume that $\alpha_n \to \alpha,\ \beta_n \to \beta$, although the second condition is not necessary.

As before $|K_n| = \left|\frac{\alpha_n}{\beta_n}\right| < r < 1$. The LPCF converges, so call its value T.

The strategy is to find an expression for $\left|\frac{T-T_n(z)}{T-T_n(0)}\right|$ that tends to zero for $z=\alpha$. We do this by resorting to the standard forward-recursion formula from elementary CF theory:

$$T_n(z)=\frac{A_n}{B_n} = \frac{A_{n-1}(b_n+z)+A_{n-1}a_n}{B_{n-1}(b_n+z)+B_{n-2}a_n}$$

Setting the "tail end" of the CF , $v_n = t_n \circ t_{n+1} \circ \cdots$, provides $T = T_n(v_{n+1})$. Algebraic manipulation gives

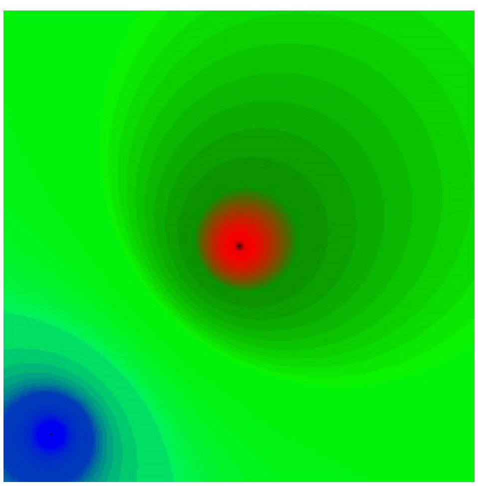
$$\left|\frac{T-T_n(z)}{T-T_n(0)}\right| = \left|\frac{1}{1+\frac{B_{n-1}}{B_n}\cdot z}\right| \cdot \left|1-\frac{z}{v_{n+1}}\right| ,$$

$\frac{B_{n-1}}{B_n} = b_n + \frac{a_n}{b_{n-1}+}\ \frac{a_{n-1}}{b_{n-2}+}\cdots\frac{a_2}{b_1+}\ \frac{a_1}{b_0}$, a reverse CF that converges [6]. It can be shown that

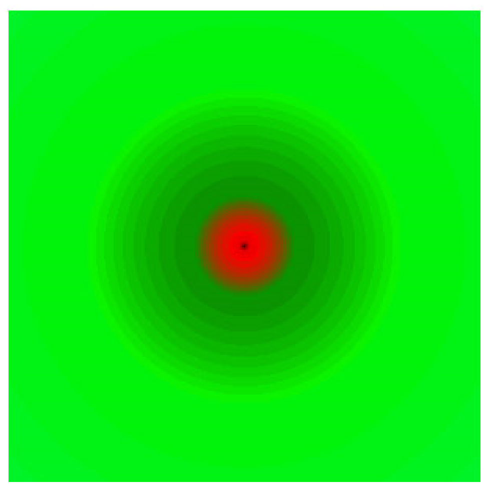
$v_n \to \alpha$, a reasonable result since $t_n \approx t$ for large values of n. Therefore $z=\alpha$ is the best accelerating factor for most LPCFs.

Example : $$FPCF = \frac{\alpha_1\beta_1}{\alpha_1+\beta_1-}\ \frac{\alpha_2\beta_2}{\alpha_2+\beta_2-}\ \frac{\alpha_3\beta_3}{\alpha_3+\beta_3-}\cdots,$$

$\alpha_1 = z,\ \alpha_k = z\cdot(1-\tfrac{1}{k}),\ \beta_k = 12(1+i),\ \ \alpha_k \to \alpha = z,\ n=5$, [-15,15]



$T_5(0)$



$T_5(\alpha) \approx T_\infty(0)$

5. A Tail-end virtual integral for continued fractions

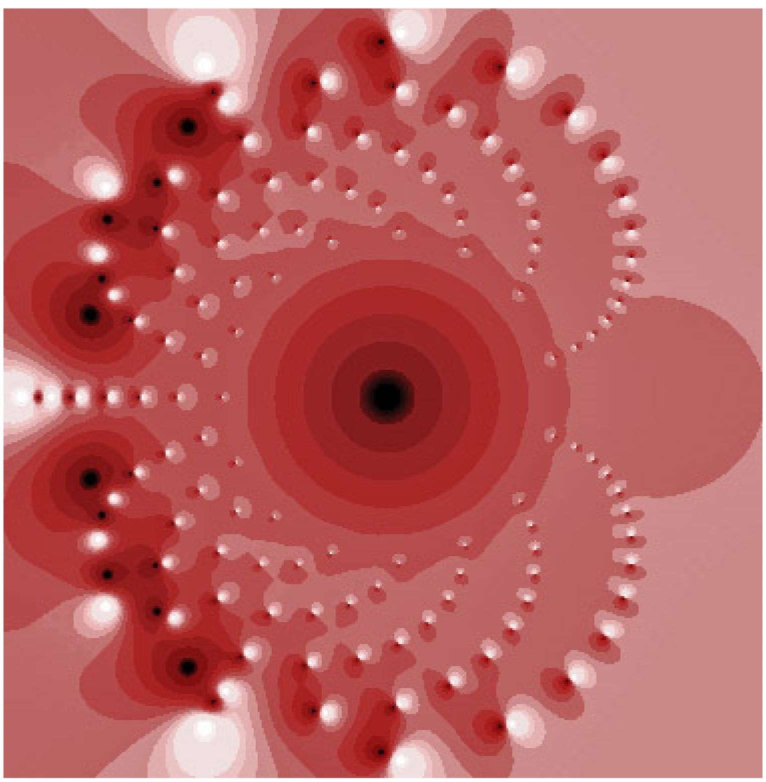
Begin with $FPCF = \frac{\alpha_1 \beta_1}{\alpha_1 + \beta_1 -} \; \frac{\alpha_2 \beta_2}{\alpha_2 + \beta_2 -} \; \frac{\alpha_3 \beta_3}{\alpha_3 + \beta_3 -} \; \ldots$ where each $\alpha_n = \alpha_n(z), \; \beta_n = \beta_n(z)$ is continuous in some $S \subset \mathbb{C}$.

Now for a given n form $\varphi(z, \frac{k}{n}) = t_k \circ t_{k+1} \circ \cdots \circ t_n(\zeta), \; \zeta = 0$ for $k = n, n-1, \cdots, 1$.

And create a Riemann-like sum:

$$\lambda_n(z) = \tfrac{1}{n}\varphi(z, \tfrac{n}{n}) + \tfrac{1}{n}\varphi(z, \tfrac{n-1}{n}) + \cdots + \tfrac{1}{n}\varphi(z, \tfrac{1}{n}) \approx \int_0^1 \psi(z,t)\,dt$$

Example : $\alpha_k(z) \equiv z, \; \beta_k(z) = k$, n=15 , [-15,15] $\int_0^1 \psi(z,t)\,dt \approx$



6. Self-generating continued fraction series

A variation on a contour generator produces a self-generating series whose terms are continued fractions:

$$g_{k,n}(z)=z+\frac{\eta_n\varphi(z,\frac{k}{n})}{z},\ \ \eta_n=\frac{1}{n}.\ \ \ G_{k,n}(z)=g_{k,n}\circ g_{k-1,n}\circ\cdots\circ g_{1,n}(z),\ \ G_n(z)=G_{n,n}(z).$$

$$G_n(z)-z=\eta_n\left(\frac{\varphi(z,\frac{1}{n})}{z}\right)+\eta_n\left(\frac{\varphi(G_{1,n}(z,\frac{2}{n}))}{G_{1,n}(z,\frac{2}{n})}\right)+\cdots+\eta_n\left(\frac{\varphi(G_{n-1,n}(z,\frac{n-1}{n}))}{G_{n-1,n}(z,\frac{n-1}{n})}\right)$$

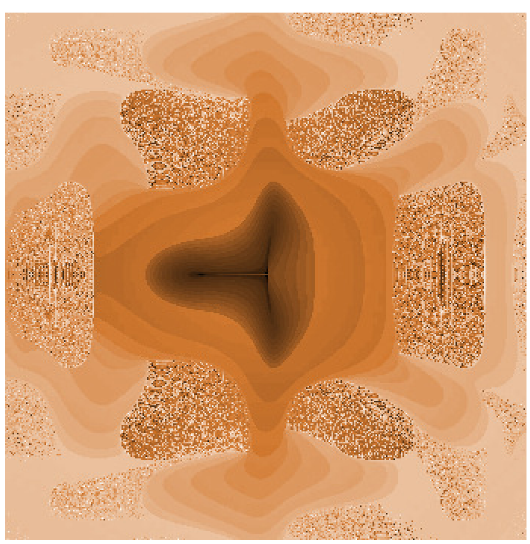
$$=\eta_n\psi(z,\tfrac{1}{n})+\eta_n\psi(z,\tfrac{2}{n})+\cdots+\eta_n\psi(z,\tfrac{n-1}{n})\sim\int_0^1\psi(z,t)dt$$

E.g., $$G_3(z)=z+\frac{\eta_3\varphi(z,\frac{1}{3})}{z}+\frac{\eta_3\varphi(G_{1,3}(z))}{z+\frac{\eta_3\varphi(z,\frac{1}{3})}{z}}+\frac{\eta_3\varphi(G_{2,3}(z))}{z+\frac{\eta_3\varphi(z,\frac{1}{3})}{z}+\frac{\eta_3\varphi(G_{1,3}(z))}{z+\frac{\eta_3\varphi(z,\frac{1}{3})}{z}}}$$

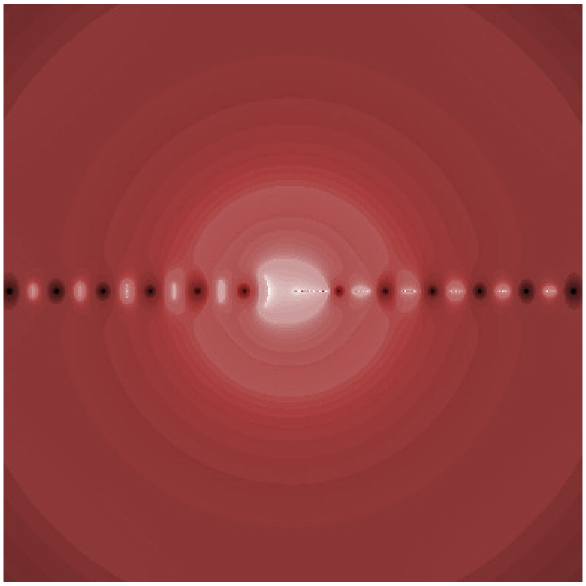
The convergence of this expansion depends upon the solvability of a DE.

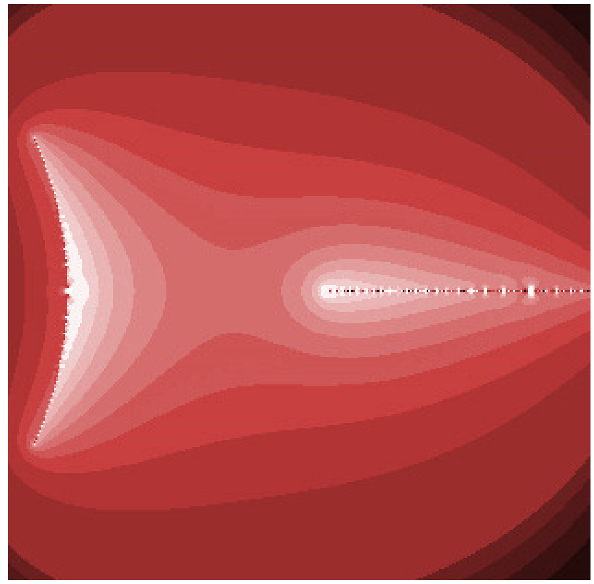
Example : $g_{k,n}(z)=z+\eta_n\cdot\frac{2\frac{k}{n}e^{-z}}{z}\ \Rightarrow\ \frac{dz}{dt}=\frac{2e^{-z}t}{z}\ \Rightarrow\ e^{z(t)}=\frac{t^2+c_0}{z(t)-1}$. In other instances it can merely be shown the DE is solvable

Example : $\varphi(z)=\frac{xCos(y)}{1+2Sin(x^2)Cos(y^2)}+i\frac{ySin(x)}{1+2Sin(x^2)Cos(y^2)}$, [-3,3], $\lambda^*(z)=z+\int_0^1\psi(z,t)dt$



Example : $\varphi(z)=Tan(z)$, [-20,20] and [-2,2], n=30 $\lambda(z)=\int_0^1 \psi(z,t)dt$:





7. Self-generating Tannery continued fractions

Consider the CF $C_n(z)=\frac{\rho(z)}{\delta_{1,n}\ +}\ \frac{\rho(G_{1,n}(z))}{\delta_{2,n}\ +}\ \frac{\rho(G_{2,n}(z))}{\delta_{3,n}\ +}\ \cdots\ \frac{\rho(G_{n-1,n}(z))}{\delta_{n,n}}$.

Where $G_{1,n}(z)=\frac{\rho(z)}{\delta_{1,n}}$, $G_{2,n}(z)=\cfrac{\rho(z)}{\delta_{1,n}+\cfrac{\rho(G_{1,n}(z))}{\delta_{2,n}}}$, $G_{3,n}(z)=\cfrac{\rho(z)}{\delta_{1,n}+\cfrac{\rho(G_{1,n}(z))}{\delta_{2,n}+\cfrac{\rho(G_{2,n}(z))}{\delta_{3,n}}}}$, $\cdots$.

Using the forward expansion formula the kth approximant is $C_{k,n}(z)=\frac{A_{k,n}(z)}{B_{k,n}(z)}$ with

$A_{k,n}=\delta_{k,n}A_{k-1,n}+\rho(G_{k-1,n}(z))A_{k-2,n}$, $B_{k,n}=\delta_{k,n}B_{k-1,n}+\rho(G_{k-1,n}(z))B_{k-2,n}$,
$A_{-1}=1, A_0=0, B_{-1}=0, B_0=1$.

Set $\delta_{k,n}=\delta_n\equiv\frac{1}{n}$ and $\lambda(z)=\lim_{n\to\infty}\sum_1^n A_{2k,n}(z)$, where

$$A_{2k,n}(z)=\sum_{l=1}^{k}\left(\delta_n\prod_{1}^{k}\rho(G_{j,n}(z))\right)+\varepsilon_{k,n}=\frac{1}{n}\cdot\sum_{l=1}^{k}\left(\prod_{1}^{k}\rho(G_{j,n}(z))\right)+\varepsilon_{k,n}\sim\frac{1}{n}\psi(z,\tfrac{k}{n}),$$

with the products difficult to describe other than $j \le k$.

The $\varepsilon_{k,n}$ consist of higher order terms (involving $\frac{1}{n^p}, p \ge 2$) that are discarded.

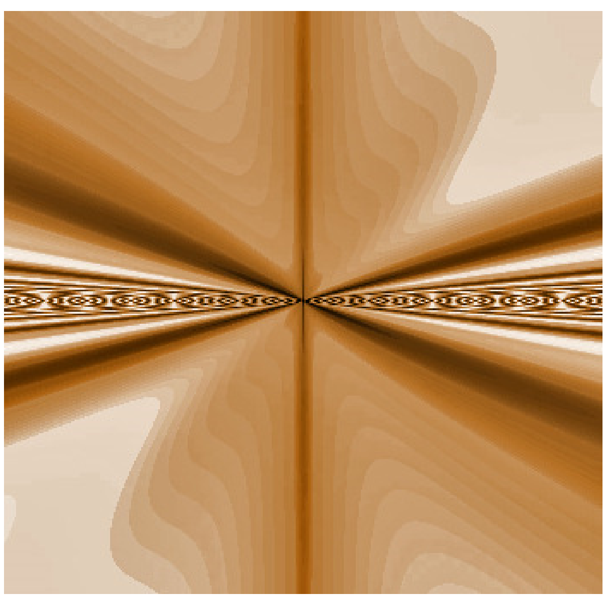
Thus $\lambda(z)=\lim_{n\to\infty}\sum_{k=1}^{n}A_{2k,n}(z)$, $\sum_{k=1}^{n}A_{2k,n}(z)\sim\frac{1}{n}\psi(z,\tfrac{1}{n})+\cdots+\frac{1}{n}\psi(z,\tfrac{n}{n})$

Then, in a <u>very</u> speculative sense $\lambda(z)\sim\int_0^1\psi(z,t)dt$

What happens when $\delta_{k,n}\to\delta_k$ for each k ? The following may apply:

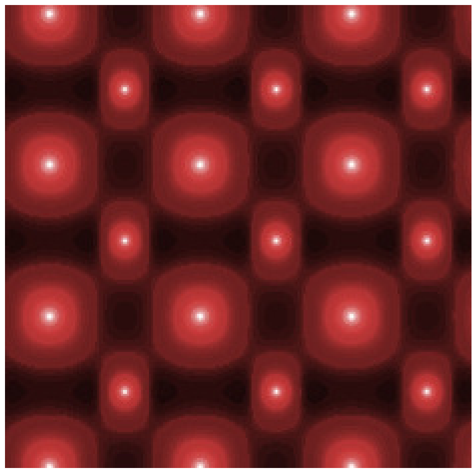
Theorem 1: (*Gill*, [7],1992) Let $\{f_{k,n}\}$, $1\le k\le n$ be a family of functions analytic on a simply-connected domain D. Suppose there exists a compact set $\Omega\subset D$ such that for each k and n, $f_{k,n}(D)\subset\Omega$ and, in addition, $\lim_{n\to\infty}f_{k,n}(z)=f_k(z)$ uniformly on D for each k. Then , with $F_{p,n}(z)=f_{1,n}\circ f_{2,n}\circ\cdots\circ f_{p,n}(z)$, $F_{n,n}(z)\to\lambda$, a constant function , as $n\to\infty$, uniformly on D.

Example : $\rho(z)=x\exp(4Cos(x/y))+iy\exp(4Sin(x/y))$, $\delta_{k,n}=\frac{1}{n}$, [-10,10] , n=40, $\lambda(z)$:



Example : $\rho(z)=\frac{2-Cos(y)}{1-Sin(x)Cos(y)}+i\frac{1-2Sin(x)}{1-Sin(x)Cos(y)}$, [-10,10], n=40,

We replace $\frac{1}{n}$ by $\delta_{k,n}=\frac{1}{n}Cos(\frac{k}{n})$. $\lambda(z)$:



7b. Self-generating continued fractions

We investigate a simplified version of the CF in **7**.

Theorem 2: (Lorentzen [2], 1990) Let $\{f_n\}$ be a sequence of functions *analytic* on a simply-connected domain D. Suppose there exists a compact set $\Omega\subset D$ such that for each n, $f_n(D)\subset\Omega$. Then $F_n(z)=f_1\circ f_2\circ\cdots\circ f_n(z)$ converges uniformly in D to a constant function F(z) = λ.

Consider

$$CF_n(z)=\frac{\rho(z)}{\delta_1\ +}\ \frac{\rho(G_1(z))}{\delta_2\ +}\ \frac{\rho(G_2(z))}{\delta_3\ +}\ \cdots\ \frac{\rho(G_n(z))}{\delta_{n+1}}\ ,$$

Where $G_1(z)=\frac{\rho(z)}{\delta_1}$, $G_2(z)=\frac{\rho(z)}{\delta_1+\frac{\rho(G_1(z))}{\delta_2}}$, $G_3(z)=\frac{\rho(z)}{\delta_1+\frac{\rho(G_1(z))}{\delta_2+\frac{\rho(G_2(z))}{\delta_3}}}$, etc. ,

With $\rho(z)$ analytic in a region described below.

Write $T_k(\omega)=\frac{\rho(G_k(z))}{\delta_{k+1}+\omega}$. Then $CF_n(z)=T_0\circ T_1\circ\cdots\circ T_n(0)$.

Suppose now that $|\omega| < R$ and for $|z| < R$, $|\rho(z)| < M$; and in addition $|\delta_k| > R + \frac{M(1+\varepsilon)}{R}$.

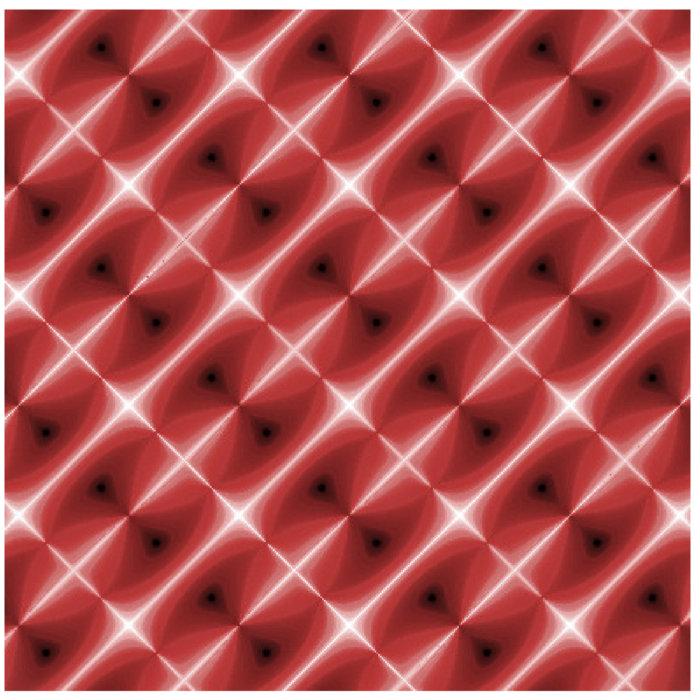
Then for each k and relevant z, $|T_k(\omega)| < \frac{R}{1+\varepsilon} < R$

Thus the conditions of theorem 2 are met, and for each z we have $CF_n(z) \to \lambda(z)$.

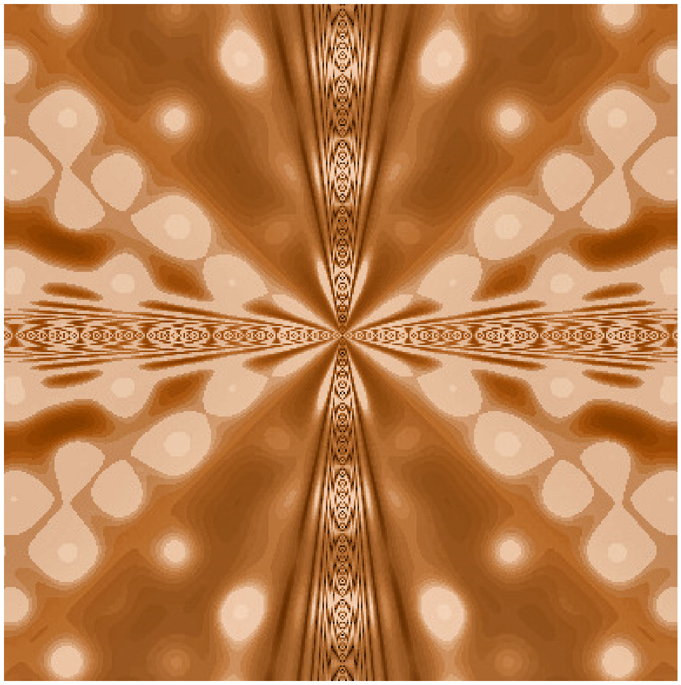
There are ways around the requirement of *analyticity* in theorem 1:

Consider $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$ with each function defined on a domain $S \subset \mathbb{C}$ that is bounded and convex, and suppose $f_n(S) \subseteq S$. Furthermore let the family of functions be *uniformly Lipschitz*. I.e., There exists a number $0 \le \rho < 1$ such that $|f_n(z) - f_n(\zeta)| < \rho|z - \zeta|$ for all n and all points in $S \subset \mathbb{C}$. Then $|F_n(z) - F_{n+m}(z)| < \rho^n \cdot Diam(S) \to 0,\ n \to \infty$. Also $|F_n(z) - F_n(\zeta)| < \rho^n \cdot Diam(S) \to 0$. Therefore $F_n(z) \to \lambda\ \forall z \in S$.

Example : $\rho(z) = \frac{Cos(x+y)}{Cos(y)+Sin(x)} + i\frac{Sin(x-y)}{Cos(x)+Sin(y)}$, $\delta = 10$, [-10,10], n=30:



Example : $\rho(z) = \frac{5Cos(y/x)}{1+Cos(y)+Sin(x)} + i\frac{5Sin(x/y)}{1+Cos(x)+Sin(y)}$, $\delta = 10$, [-10,10],n=30:



8. Fixed-point equivalent continued fractions

From Euler's *equivalent continued fraction* ,

$$CF_n = \frac{\rho_1}{1-}\ \frac{\rho_2}{1+\rho_2-}\ \frac{\rho_3}{1+\rho_3-}\ \cdots\ \frac{\rho_n}{1+\rho_n} = \sum_{k=1}^{n}\left(\prod_{j=1}^{k}\rho_j\right)$$

Consider, now, a sequence of functions $\{\alpha_{k,n}(z)\}_{k=1}^{n}$, $n \to \infty$, so that

$$\frac{\alpha_{1,n}(z)}{1-}\ \frac{\alpha_{2,n}(z)}{1+\alpha_{2,n}(z)-}\ \cdots\ \frac{\alpha_{n,n}(z)}{1+\alpha_{n,n}(z)} = \sum_{m=1}^{n}\left(\prod_{k=1}^{m}\alpha_{k,n}(z)\right) .$$

Clearly the convergence of CF_n is equivalent to the convergence of the series on the right.

If $\prod_{k=1}^{m}\alpha_{k,n}(z)\to\beta_m,\ n\to\infty,\left|\prod_{k=1}^{m}\alpha_{k,n}(z)\right|<\mathrm{M}_m\ ,\ \sum M_m<\infty$ for all $z\in S$, Tannery's theorem can be applied, showing the CF converges to $\sum\beta_m$.

In order to generate a virtual integral, set $\psi_n(z,\tfrac{m}{n})=n\cdot\prod_{k=1}^{m}\alpha_{k,n}(z)$. And we have :

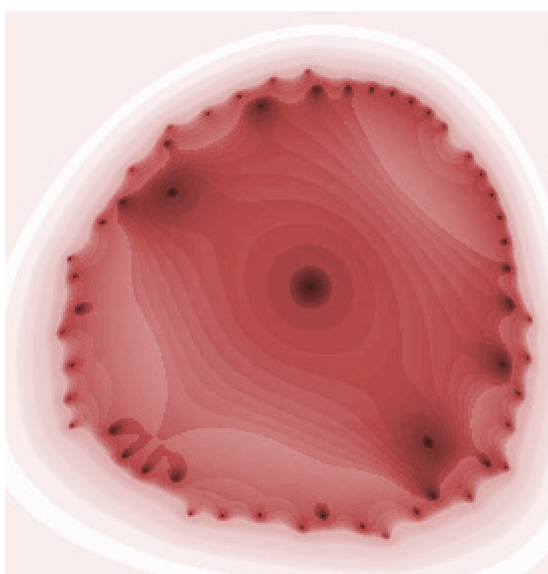
$$\sum_{m=1}^{n}\left(\prod_{k=1}^{m}\alpha_{k,n}(z)\right) = \frac{1}{n}\psi_n(z,\tfrac{1}{n}) + \frac{1}{n}\psi_n(z,\tfrac{2}{n}) + \cdots + \frac{1}{n}\psi_n(z,\tfrac{n}{n}) \sim \int_0^1\tilde{\psi}(z,t)dt$$

The $\{\alpha_{k,n}(z)\}$ and $\{1\}$ are the two *fixed points* of the *linear fractional transformations*

$$t_{k,n}(\zeta)=\frac{\alpha_{k,n}(z)}{1+\alpha_{k,n}(z)-\zeta} \quad\Rightarrow\quad C_n(z)=\frac{\alpha_{1,n}(z)}{1-T_n(z)}\ ,\ T_n(z)=t_{1,n}\circ t_{2,n}\circ\cdots\circ t_{n,n}(0)$$

And if $\alpha_{k,n}(z)\equiv\alpha(z)$, $|\alpha(z)|<1$, $z\in S$, it can be shown that $T_n(\omega)\to\alpha(z)$ as $n\to\infty$ for $\omega\neq 1$. Under these conditions $\frac{1}{n}\psi_n(z,\tfrac{m}{n})=(\alpha(z))^m$ and $\int_0^1\tilde{\psi}(z,t)dt = \frac{\alpha(z)}{1-\alpha(z)}$.

Example : $\alpha(z)=xCos(x+y)+iySin(x-y)$, n=20, 1.9<x,y<2.7, $\int_0^1\tilde{\psi}(z,t)dt = \frac{\alpha(z)}{1-\alpha(z)}$:



This is a magnification of one of the periodic uneven ovals that populate the plane. Outside the ovals the CF diverges, and the dark points on the interior are zeros or close to zeros.

9. Mixing Euler continued fractions with contours

Another *virtual integral* approach with the Euler CF is to associate the components of the fraction with a certain contour:

$$g_{k,n}(z)=z+\eta_n\left(\prod_{j=0}^{k-2}\rho(G_{j,n}(z))\right)\rho(z)\ ,\ G_{k,n}(z)=g_{k,n}\circ g_{k-1,n}\circ\cdots\circ g_{1,n}(z),\ \eta_n=\frac{1}{n},$$

which corresponds to

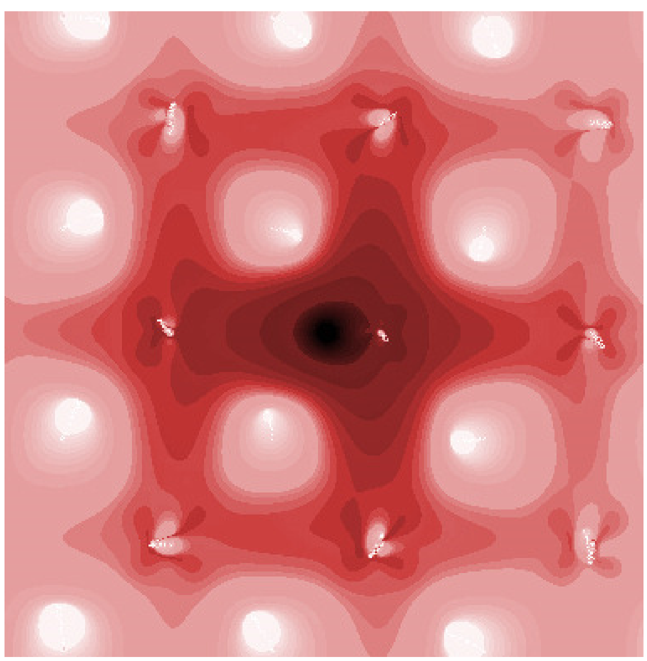
$$\frac{\rho(G_{0,n}(z))}{1-}\ \frac{\rho(G_{1,n}(z))}{1+\rho(G_{1,n}(z))-}\ \frac{\rho(G_{2,n}(z))}{1+\rho(G_{2,n}(z))-}\cdots\frac{\rho(G_{n-1,n}(z))}{1+\rho(G_{n-1,n}(z))}=\sum_{k=0}^{n}\left(\prod_{j=0}^{k}\rho(G_{j,n}(z))\right)$$

So that $G_{n,n}(z)-z=\frac{1}{n}\psi(z,\tfrac{1}{n})+\frac{1}{n}\psi(z,\tfrac{2}{n})+\cdots+\frac{1}{n}\psi(z,\tfrac{n}{n})\sim\int_0^1\psi(z,t)dt$ where

$$\psi(z,\tfrac{k}{n})=\prod_{j=0}^{k}\rho(G_{j,n}(z)).$$

Example : $\Phi(x,y;t)=(x+Sin(y))+i(y-Cos(x))-(t^2+it+c)=0\Rightarrow\ \frac{dz}{dt}=\rho(z,t)$,

$\rho(z,t)=\frac{2t-Cos(y)}{1-Sin(x)Cos(y)}+i\cdot\frac{1-2tSin(x)}{1-Sin(x)Cos(y)}$, [-15,15] , $\lambda(z)=\int_0^1\psi(z,t)dt$:



10. Reverse self-generating fixed-point continued fractions

Set $g_k(\zeta)=\dfrac{\rho(G_{k-1}(z))}{1+\rho(G_{k-1}(z))-\zeta}$, $z\,\&\,\zeta\in S$, $g_k(\zeta)\in S$ for a suitably well-behaved $\rho(z)$.

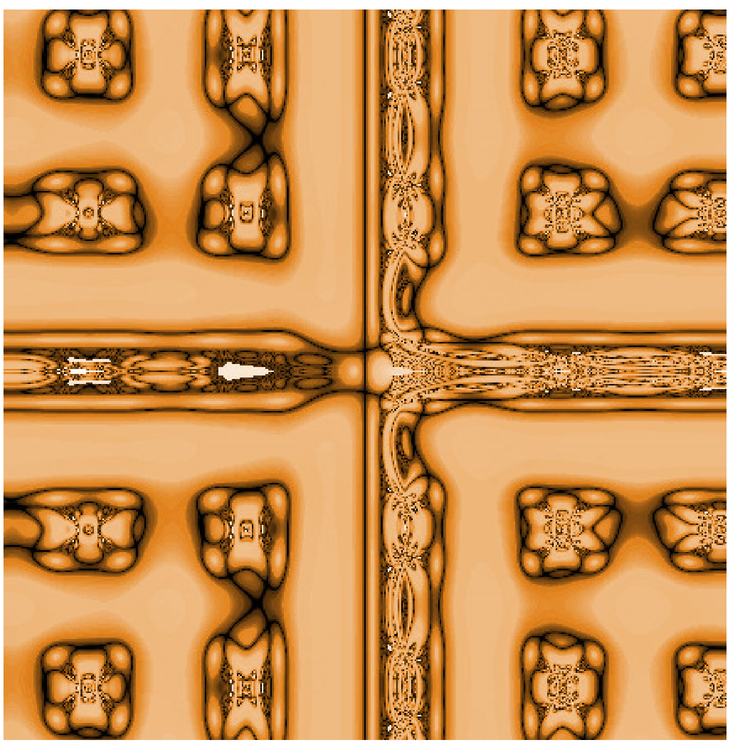
Form $G_k(z)=\mathop{\pmb{\mathcal{L}}}_{j=1}^{k} g_j(0)$, $\lim_{n\to\infty} G_n(z)=G(z)$. We have

$$G_n(z)=\frac{\rho(G_{n-1}(z))}{1+\rho(G_{n-1}(z))-}\;\frac{\rho(G_{n-2}(z))}{1+\rho(G_{n-2}(z))-}\;\cdots\;\frac{\rho(G_1(z))}{1+\rho(G_1(z))-}\;\frac{\rho(z)}{1+\rho(z)-\zeta},\quad \zeta=0$$

Abbreviate $\rho_k=\rho(G_{k-1}(z))$. Then $G_n(z)=\dfrac{S_n(z)}{1+S_n(z)}$, $S_n(z)=\sum_{k=1}^{n}\left(\prod_{j=k}^{n}\rho_j\right)=a_1(n)+a_2(n)+\cdots+a_n(n)$

If $a_k(n)\to a_k$, $|a_k(n)|\le M_k$, $\sum M_k<\infty$, Tannery's Theorem will insure convergence of $S_n(z)$.

Example : $\rho(z)=\rho(x+iy)=xCos(y)+iySin(x)$, [-15,15], n=30



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