A class of non-symmetric Laguerre-Hahn Polynomials

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Abstract

We show that if v is a regular Laguerre-Hahn form (linear functional), then the form u defined by $(x-\beta_0^2)\sigma u = -\lambda v$ and $\sigma(x-\beta_0)u = 0$ where σu is the even part of u, is also regular and Laguerre-Hahn form for every complex λ except for a discrete set of numbers depending on v. We give explicitly the recurrence coefficients and the structure relation coefficients of the orthogonal polynomials sequence associated with u and the class of the form u knowing that of v. An example related to the associated form of the first of Jacobi is worked out.

Keywords: Orthogonal polynomials; Laguerre-Hahn forms; structure relation.
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1. Introduction

In many recent papers, different construction processes of Laguerre-Hahn orthogonal polynomials (O.P) grow from well known ones, particularly the associated of classical ones. For instance, we can mention the adjunction of a finite number of Dirac's masses to Laguerre-Hahn forms [1, 6, 8], the product and the division of a form by a polynomial [2, 3, 8, 10, 12].
The whole idea of the following work is to build a new construction process of a Laguerre-Hahn form, which has not yet been treated in the literature of Laguerre-Hahn polynomials. The problem we tackle is as follows:
We study the form u, fulfilling $(x-\beta_0^2)\sigma u = -\lambda v$, $\lambda \neq 0$, $(u)_{2n+1} = \beta_0 (u)_{2n}$, where σu is the even part of u, $\beta_0 \in \mathbb{C}$ and v is a given Laguerre-Hahn form.

This paper is arranged in sections : The first provides a focus on the preliminary results and notations used in the sequel. We will also give the regularity condition and the coefficients of the three-term recurrence relation satisfied by the new family of O.P. In the second , we compute the exact class of the Laguerre-Hahn form obtained by the above modification and the structure relation of the O.P. sequence relatively to the form u will follow. In the final section, we apply our results to an example. The regular form found in the example is Laguerre-Hahn of class $\tilde{s} \leq 3$.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(v)_n := \langle v, x^n\rangle$, $n \geq 0$, the moments of v. For any form v and

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any polynomial h let $Dv = v'$, hv, δ_c, and $(x-c)^{-1}v$ be the forms defined by: $\langle v', f\rangle := -\langle v, f'\rangle$, $\langle hv, f\rangle := \langle v, hf\rangle$, $\langle \delta_c, f\rangle := f(c)$,
and $\big\langle (x-c)^{-1}v, f\big\rangle := \langle v, \theta_c f\rangle$ where $\big(\theta_c f\big)(x) = \dfrac{f(x)-f(c)}{x-c}$, $c \in \mathbb{C}$, $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $f \in \mathcal{P}$ and $v \in \mathcal{P}'$, we have

$$(x-c)^{-1}((x-c)v) = v - (v)_0\delta_c \; , \tag{1}$$

$$(x-c)((x-c)^{-1}u) = u \; . \tag{2}$$

Let us define the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ by $(\sigma f)(x) := f(x^2)$. Then, we define the even part σv of v by $\langle \sigma v, f\rangle := \langle v, \sigma f\rangle$.
Therefore, we have [7, 9]

$$f(x)(\sigma v) = \sigma(f(x^2)v) \; , \tag{3}$$

$$(\sigma v)_n = (v)_{2n} \; , n \geq 0 \; , \tag{4}$$

The form v will be called regular if we can associate with it a sequence $\{S_n\}_{n\geq 0}$ $\big(\deg(S_n) \leq n\;\big)$ such that

$$\langle v, S_n S_m\rangle = r_n \delta_{n,m} \; , \quad n, m \geq 0 \; , \quad r_n \neq 0 \; , \quad n \geq 0 \; .$$

Then $\deg(S_n) = n$, $n \geq 0$, and we can always suppose each S_n monic (i.e. $S_n(x) = x^n + \cdots$). The sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal with respect to v. It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (see, for instance, the monograph by Chihara [7])

$$\begin{aligned} &S_{n+2}(x) = (x-\xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x) \; , \quad n \geq 0 \; , \\ &S_1(x) = x - \xi_0 \; , \qquad S_0(x) = 1 \; . \end{aligned} \tag{5}$$

with $\big(\xi_n, \rho_{n+1}\big) \in \mathbb{C} \times \mathbb{C} - \{0\}$, $\quad n \geq 0$, by convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the associated sequence of first kind for the sequence $\{S_n\}_{n\geq 0}$ satisfying the three-term recurrence relation [7]

$$\begin{aligned} &S_{n+2}^{(1)}(x) = (x-\xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x) \; , \quad n \geq 0, \\ &S_1^{(1)}(x) = x - \xi_1, \qquad S_0^{(1)}(x) = 1 \; , \quad \big(S_{-1}^{(1)}(x) = 0\big) \; . \end{aligned} \tag{6}$$

Another important representation of $S_n^{(1)}$ is, (see [7])

$$S_n^{(1)}(x) := \Big\langle v, \frac{S_{n+1}(x) - S_{n+1}(\zeta)}{x-\zeta} \Big\rangle \; , n \geq 0 \; . \tag{7}$$

Also, let $\{S_n(., \mu)\}_{n\geq 0}$ be co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [7]

$$S_n(x, \mu) = S_n(x) - \mu S_{n-1}^{(1)} \; , \quad n \geq 0. \tag{8}$$

We recall that a form v is called symmetric if $(v)_{2n+1} = 0, n \geq 0$. The conditions $(v)_{2n+1} = 0$, $n \geq 0$ are equivalent to the fact the corresponding monic orthogonal polynomials sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (5) with $\xi_n = 0$, $n \geq 0$ [7].

Now let v be a regular, normalized form (i.e. $(v)_0 = 1$) and $\{S_n\}_{n\geq 0}$ be its corresponding sequence of polynomials. For a $\beta_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$, we can define a new form u as following:

$$(u)_{2n+2} - \beta_0^2(u)_{2n} = -\lambda(v)_n \ , \ (u)_{2n+1} = \beta_0(u)_{2n} \ , \ (u)_0 = 1 \ , \quad n \geq 0. \tag{9}$$

Equivalently,

$$(x - \beta_0^2)\sigma u = -\lambda v \ , \qquad \sigma\big((x - \beta_0)u\big) = 0 \ . \tag{10}$$

From (1) and (10), we have

$$\sigma u = -\lambda(x - \beta_0^2)^{-1}v + \delta_{\beta_0^2} \ . \tag{11}$$

Remarks. (i) (10) is equivalent to

$$(x^2 - \beta_0^2)u = -\lambda w \ , \tag{12}$$

where the form w defined by

$$\sigma w = v \ , \quad \sigma(x - \beta_0)w = 0 \ .$$

Notice that w is not necessarily a regular form in the problem understudy. In [2], the authors have solved where w is regular and $\beta_0 = 0$ and in [4], the problem (12) is solved when $\beta_0 \neq 0$ and w is regular.

(ii) The case $\beta_0 = 0$ is treated in [11], so henceforth we assume $\beta_0 \neq 0$.

Proposition 1. [9] *The form u defined by* (10) *is regular if and only if σu and $(x - \beta_0^2)\sigma u$ are regular.*

Proposition 2. *The form u is regular if and only if $\lambda \neq \lambda_n$, $n \geq 0$ where*

$$\lambda_0 = 0 \ , \quad \lambda_{n+1} = \frac{S_{n+1}(\beta_0^2)}{S_n^{(1)}(\beta_0^2)} \ , \ n \geq 0 \, . \tag{13}$$

Proof. We have u is defined by (10). Then, according to Proposition 1. u is regular if and only if $(x - \beta_0^2)\sigma u$ and σu are regular. But $(x - \beta_0^2)\sigma u = -\lambda v$ is regular because $\lambda \neq 0$ and v is regular. So u is regular if and only if $\sigma u = -\lambda(x - \beta_0^2)^{-1}\sigma v + \delta_{\beta_0^2}$ is regular. Or, $\{S_n\}_{n\geq 0}$ is the corresponding orthogonal sequence to v, and it was shown in [10] that $\sigma u = -\lambda(x-\beta_0^2)^{-1}v+\delta_{\beta_0^2}$ is regular if and only if $\lambda \neq 0$, and $S_n(\beta_0^2, \lambda) \neq 0$, $n \geq 0$. Then we deduce the desired result.■

When u is regular let $\{Z_n\}_{n\geq 0}$ be its corresponding sequence of polynomials satisfying the recurrence relation

$$\begin{aligned} &Z_{n+2}(x) = (x - (-1)^{n+1}\beta_0)Z_{n+1}(x) - \gamma_{n+1}Z_n(x) \ , \quad n \geq 0 \ , \\ &Z_1(x) = x - \beta_0 \ , \quad Z_0(x) = 1 \ . \end{aligned} \tag{14}$$

Let us consider its quadratic decomposition [7, 9]:

$$Z_{2n}(x) = P_n(x^2) \ , \quad Z_{2n+1}(x) = (x - \beta_0)R_n(x^2) \ , \tag{15}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to σu and $(x - \beta_0^2)\sigma u$.

From (13), we have

$$R_n(x) = S_n(x) \ , \quad n \geq 0 \ . \tag{16}$$

Proposition 3. *We may write*

$$\gamma_1 = -\lambda \;, \quad \gamma_{2n+2} = a_n \;, \quad \gamma_{2n+3} = \frac{\rho_{n+1}}{a_n} \;, \quad n \geq 0 \;, \tag{17}$$

where

$$a_n = -\frac{S_{n+1}(\beta_0^2, \lambda)}{S_n(\beta_0^2, \lambda)} \;, \quad n \geq 0 \;. \tag{18}$$

For the proof, we need the following lemma:

Lemma 4. *We have*

$$Z_{2n}^{(1)}(x) = R_n(x^2, \lambda) \;, \quad Z_{2n+1}^{(1)}(x) = (x + \beta_0) P_n^{(1)}(x^2) \;, \quad n \geq 0 \;. \tag{19}$$

Proof. Using (7) and (15), one has

$$\begin{aligned}
Z_{2n}(\zeta) &= \langle u, \tfrac{Z_{2n+1}(x) - Z_{2n+1}(\zeta)}{x-\zeta} \rangle \quad (\, u\, acts\, on\, the\, variable\, x \,) \\
&= \langle u, \tfrac{(x-\beta_0)R_n(x^2) - (\zeta-\beta_0)R_n(\zeta^2)}{x-\zeta} \rangle \\
&= \langle u, R_n(\zeta^2) \rangle + \langle u, (x-\beta_0) \tfrac{R_n(x^2) - R_n(\zeta^2)}{x-\zeta} \rangle \\
&= R_n(\zeta^2) + \langle u, (x+\zeta)(x-\beta_0) \tfrac{R_n(x^2) - R_n(\zeta^2)}{x^2-\zeta^2} \rangle \\
&= R_n(\zeta^2) + \langle u, \big((x^2 - \beta_0^2) - (\beta_0 - \zeta)(x - \beta_0)\big) \tfrac{R_n(x^2) - R_n(\zeta^2)}{x^2-\zeta^2} \rangle \;. \\
&= R_n(\zeta^2) + \langle (x - \beta_0^2)\sigma u, \tfrac{R_n(x) - R_n(\zeta^2)}{x-\zeta^2} \rangle \qquad (from\, (10)) \\
&= R_n(\zeta^2) - \lambda \langle v, \tfrac{R_n(x) - R_n(\zeta^2)}{x-\zeta^2} \rangle \qquad (from\, (10)) \\
&= R_n(\zeta^2) - \lambda R_{n-1}^{(1)}(\zeta^2) \qquad (from\, (7)) \\
&= R_n(\zeta^2, \lambda)
\end{aligned}$$

For the proof of the second relation see the proof of the Lemma 4.2 in [5]. Hence, we get (19). ■

Proof of Proposition 3. Using (10) and the condition $\langle u, Z_2 \rangle = 0$, we obtain $\gamma_1 = -\lambda$.

From (6) and (14) where $n \longrightarrow 2n$ and taking (16) and (19) into account, we get

$$S_{n+1}\left(x^2, -\gamma_1\right) = (x - \beta_0) Z_{2n+1}^{(1)}(x) - \gamma_{2n+2} S_n\left(x^2, -\gamma_1\right) \;.$$

Substituting x by β_0 in the above equation, we obtain $\gamma_{2n+2} = a_n$.

From (14), we have

$$\gamma_{2n+2}\gamma_{2n+3} = \frac{\left\langle u, Z_{2n+2}^2 \right\rangle}{\left\langle u, Z_{2n+1}^2 \right\rangle} \frac{\left\langle u, Z_{2n+3}^2 \right\rangle}{\left\langle u, Z_{2n+2}^2 \right\rangle} = \frac{\left\langle u, Z_{2n+3}^2 \right\rangle}{\left\langle u, Z_{2n+1}^2 \right\rangle}. \tag{20}$$

Using (5), (10) and (15) – (16), equation (20) becomes

$$\gamma_{2n+2}\gamma_{2n+3} = \rho_{n+1}, \tag{21}$$

then we deduce $\gamma_{2n+3} = \dfrac{\rho_{n+1}}{a_n}$.

2. The Laguerre-Hahn case

Let us recall that a form v is called Laguerre-Hahn when it is regular and satisfies a linear non-homogeneous differential equation [2]

$$\Phi(z) S'\big(v\big)(z) = B(z) S^2\big(v\big)(z) + C_0(z) S\big(v\big)(z) + D_0(z) \;, \tag{22}$$

where Φ monic, B, C_0 and D_0 are polynomials and $S(v)(z)$ is the formal Stieltjes function of the form v, namely

$$S(v)(z) = -\sum_{n\geq 0} \frac{(v)_n}{z^{n+1}} \ . \tag{23}$$

The class of the Laguerre-Hahn form v is $s = \max\big(\deg(\Phi)-2, \deg(B)-2, \deg(C_0)-1\big)$ if and only if the following condition is satisfied [2]

$$\prod_{c\in\mathcal{Z}} (|B(c)| + |C_0(c)| + |D_0(c)|) \neq 0, \tag{24}$$

where $\mathcal{Z}$ denotes the set of zeros of Φ.

The corresponding orthogonal sequence $\{S_n\}_{n\geq 0}$ is also called Laguerre-Hahn of class s.

The Laguerre-Hahn character is invariant by shifting. Indeed, the shifted form $\hat{v} = (h_{a^{-1}} o t_{-b})v,\ a \in \mathbb{C} - \{0\},\ b \in \mathbb{C}$ satisfies

$$\hat{\Phi}(z) S'\big(\hat{v}\big)(z) = \hat{B}(z) S^2\big(\hat{v}\big)(z) + \hat{C}_0(z) S\big(\hat{v}\big)(z) + \hat{D}_0(z) \ , \tag{25}$$

with

$$\begin{cases} \hat{\Phi}(z) = a^{-k}\Phi(az+b) \ , \ \hat{B}(z) = a^{-k}B(az+b) \ , \\ \hat{C}_0(z) = a^{1-k}C_0(az+b) \ , \ \hat{D}_0(z) = a^{2-k}D_0(az+b) \ , \ k = \deg(\Phi) \ . \end{cases}$$

The forms $t_b v$ (translation of v) and $h_a v$ (dilation of v) are defined by

$$\langle t_b v, f\rangle := \langle v, f(x+b)\rangle, \quad \langle h_a v, f\rangle := \langle v, f(ax)\rangle, \quad f \in \mathcal{P}.$$

The sequence $\{\hat{S}_n(x) = a^{-n}S_n(ax+b)\}_{n\geq 0}$ is orthogonal with respect to $\hat{v}$ and fulfils (5) with

$$\hat{\xi}_n = \frac{\xi_n - b}{a}, \quad \hat{\rho}_{n+1} = \frac{\rho_{n+1}}{a^2}, \quad n \geq 0. \tag{26}$$

We can state characterizations of Laguerre-Hahn orthogonal sequences. $\{S_n\}_{n\geq 0}$ is Laguerre-Hahn of class s if and only if one of the following statements holds:
(a) The form v satisfied the functional equation [2]

$$\big(\Phi(x)v\big)' + \Psi(x)v + B(x^{-1}v^2) = 0 \tag{27}$$

with

$$\Psi(x) = -\Phi'(x) - C_0(x). \tag{28}$$

We also have the following relation:

$$D_0(x) = -\big(v\theta_0\Phi\big)'(x) - \big(v\theta_0\Psi\big)(x) - \big(v^2\theta_0^2 B\big)(x). \tag{29}$$

(b) $\{S_n\}_{n\geq 0}$ fulfills the following structure relation (written in a compact form):

$$\begin{aligned} &\Phi(x)S'_{n+1}(x) - B(x)S_n^{(1)}(x) = \\ &\frac{1}{2}\big(C_{n+1}(x) - C_0(x)\big)S_{n+1}(x) - \rho_{n+1}D_{n+1}(x)S_n(x) \ , \ n \geq 0, \end{aligned} \tag{30}$$

where

$$\left\{\begin{array}{l} C_{n+1}(x) = -C_n(x) + 2(x-\xi_n)D_n(x)\ , n \geq 0\ , \\ \rho_{n+1}D_{n+1}(x) = -\Phi(x) + \rho_n D_{n-1}(x) - (x-\xi_n)C_n(x) + \\ \qquad\qquad\qquad + (x-\xi_n)^2 D_n(x)\ , n \geq 0\ , \end{array}\right. \tag{31}$$

Φ , $C_0(x)$ and $D_0(x)$ are the same polynomials introduced in (a); ξ_n, ρ_n are the coefficients of the three term recurrence relation (5). Notice that $D_{-1}(x) = B(x), \deg C_n \leq s+1$ and $\deg D_n \leq s, n \geq 0$ [2].

In the sequel the form v will be supposed Laguerre-Hahn form of class s satisfying (22) and (27) and using a dilation in the variable β_0, we can take him equal to one.

Proposition 5. *For every $\lambda \in \mathbb{C} - \{0\}$ such that $S_n(1,\lambda) \neq 0, n \geq 0$, the form u defined by* (10) *is regular and Laguerre-Hahn. It satisfies*

$$\tilde{\Phi}(z)S'\big(u\big)(z) = \tilde{B}(z)S^2\big(u\big)(z) + \tilde{C}_0(z)S\big(u\big)(z) + \tilde{D}_0(z), \tag{32}$$

where

$$\left\{\begin{array}{l} \tilde{\Phi}(z) = (z-1)\Phi(z^2), \\ \tilde{B}(z) = -2\lambda^{-1}z(z-1)^2B(z^2), \\ \tilde{C}_0(z) = 2z(z-1)C_0(z^2) - \Phi(z^2) - 4\lambda^{-1}z(z-1)B(z^2), \\ \tilde{D}_0(z) = -2z\big(\lambda^{-1}B(z^2) + \lambda D_0(z^2) - C_0(z^2)\big), \end{array}\right. \tag{33}$$

and u is of class $\tilde{s}$ such that $\tilde{s} \leq 2s+3$.

Proof. From (10) and (23), we have

$$S\big(v\big)(z^2) = -\lambda^{-1}(z-1)S\big(u\big)(z) - \lambda^{-1}. \tag{34}$$

Make a change of variable $z \longrightarrow z^2$ in (22), multiply by $-2\lambda z$ and substitute (34) in the obtained equation, we get (32) – (33).
Then, $\deg(\tilde{\Phi}) \leq 2s+5$, $\deg(\tilde{B}) \leq 2s+7$ and $\deg(\tilde{C}_0) \leq 2s+4$.
Thus, $\tilde{s} = \max(\deg(\tilde{\Phi})-2, \deg(\tilde{B})-2, \deg(\tilde{C}_0)-1) \leq 2s+5$. ■

As an immediate consequence of (32) – (33), the form u satisfies the functional equation

$$\big(\tilde{\Phi}u\big)' + \tilde{\Psi}u + \tilde{B}(x^{-1}u^2) = 0\ , \tag{35}$$

where $\tilde{\Phi}$ is the polynomial defined by (33) and

$$\tilde{\Psi}(x) = -\tilde{\Phi}'(x) - \tilde{C}_0(x) = 2x(x-1)\Psi(x^2) + 4\lambda^{-1}x(x-1)B(x^2)\ . \tag{36}$$

Proposition 6. *The class of u depends only on the zeros $x=0$ and $x=1$ of $\tilde{\Phi}$.*

Proof. Since v is a Laguerre-Hahn form of class s, $S(v)(z)$ satisfies (22), where the polynomials Φ, B, C_0 and D_0 are coprime. Let $\tilde{\Phi}$, $\tilde{B}$, $\tilde{C}_0$ and $\tilde{D}_0$ be as in Proposition 5. Let d be a zero of $\tilde{\Phi}$ different from 0 and 1, this implies that $\Phi(d^2) = 0$. We know that $|B(d^2)| + |C_0(d^2)| + |D_0(d^2)| \neq 0$

(i) If $B(d^2) \neq 0$, then $\tilde{B}(d) \neq 0$,

(ii) if $B(d^2) = 0$ and $C_0(d^2) \neq 0$, then $\tilde{C}_0(d) \neq 0$,

(iii) if $B(d^2) = C_0(d^2) = 0$, then $\tilde{D}_0(d) \neq 0$,
whence $|\tilde{B}(d)| + |\tilde{C}_0(d)| + |\tilde{D}_0(d)| \neq 0$. ■

Concerning the class of u, we have the following result.

Proposition 7. *Let $t = \deg(\Phi)$, $r = \deg(B)$, $p = \deg(C_0)$, $X(z) = C_0(z) - \lambda D_0(z) - \lambda^{-1}B(z)$ and $Y(z) = C_0(z) - \Phi^{'}(z) - 2\lambda^{-1}B(z)$, where the polynomials Φ, B, C_0 and D_0 are defined in (22).*
For every $\lambda \neq \lambda_n$, $n \geq 0$, the linear functional u defined by (10) is regular and Laguerre-Hahn form of class $\tilde{s}$ satisfying (32). Moreover:
1) If $\Phi(0)(|\Phi(1)| + |X(1)|) \neq 0$, then

$$\tilde{s} = \begin{cases} 2s+5 & if\, r > p,\ t < r+1\,, \\ 2s+3 & otherwise\,. \end{cases} \tag{37}$$

2) If either

$$\Phi(0) = 0 \text{ and } |\Phi(1)| + |X(1)| \neq 0$$

or

$$\Phi(1) = X(1) = 0 \text{ and } \Phi(0)(|Y(1)| + |X^{'}(1)|) \neq 0,$$

then

$$\tilde{s} = \begin{cases} 2s+4 & if\, r > p,\ t < r+1\,, \\ 2s+2 & otherwise\,. \end{cases} \tag{38}$$

3) If either

$$\Phi(1) = X(1) = \Phi(0) = 0 \text{ and } |Y(1)| + |X^{'}(1)| \neq 0$$

or

$$\Phi(1) = X(1) = X^{'}(1) = Y(1) = 0 \text{ and } \Phi(0) \neq 0,$$

then

$$\tilde{s} = \begin{cases} 2s+3 & if\, r > p,\ t < r+1\,. \\ 2s+1 & otherwise\,. \end{cases} \tag{39}$$

4) If $\Phi(1) = X(1) = X^{'}(1) = Y(1) = \Phi(0) = 0$, then

$$\tilde{s} = \begin{cases} 2s+2 & if\, r > p,\ t < r+1\,, \\ 2s & otherwise\,. \end{cases} \tag{40}$$

Proof. 1) If $\Phi(0)(|\Phi(1)| + |X(1)|) \neq 0$, then it is not possible to simplify (32) according to Proposition 6. and the standard criterion (24). From (32), we have $\deg(\tilde{\Phi}) = 2t+1$, $\deg(\tilde{B}) = 2r+3$ and $\tilde{p} := \deg(\tilde{C_0}) \leq \max(2p+2, 2t, 2r+2)$.
We will distinguish two cases:
(a) $p < \max(r, t-1)$, then $\tilde{p} \leq \max(2r+2, 2t)$ and $\tilde{s} = \max(2r+1, 2t-1)$. If $t < r+1$, then $\tilde{s} = 2r+1 = 2s+5$. If $t \geq r+1$, then $\tilde{s} = 2t-1 = 2s+3$.
(b) $p \geq \max(r, t-1)$, then $\tilde{p} = 2p+2$ and $\tilde{s} = 2p+1 = 2s+3$.
Thus, from the above situation, we deduce (37).

2) Using (24), we obtain the following cases:

(i) If $\Phi(0)=0$ and $|\Phi(1)|+|X(1)|\neq 0$, then it is possible to simplify (32) – (33) by z. Thus, u fulfills (32) with

$$\begin{cases} \tilde{\Phi}(z)=z(z-1)(\theta_0\Phi)(z^2)\ , \\ \tilde{B}(z)=-2\lambda^{-1}(z-1)^2B(z^2)\ , \\ \tilde{C}_0(z)=2(z-1)C_0(z^2)-z(\theta_0\Phi)(z^2)-4\lambda^{-1}(z-1)B(z^2)\ , \\ \tilde{D}_0(z)=2C_0(z^2)-2\lambda D_0(z^2)-2\lambda^{-1}B(z^2)\ . \end{cases} \tag{41}$$

Hence, we get $\tilde{B}(0)=-2\lambda^{-1}B(0)\ ,\ \tilde{C}_0(0)=-2C_0(0)+4\lambda^{-1}B(0)$ and $\tilde{D}_0(0)=2C_0(0)-2\lambda D_0(0)-2\lambda^{-1}B(0)$.
Or $|B(0)|+|C_0(0)|+|D_0(0)|\neq 0$, then it is not possible to simplify (32) – (41), which means that the class of u verifies (38).

(ii) If $\Phi(1)=X(1)=0$ and $\Phi(0)(|Y(1)|+|X^{'}(1)|)\neq 0$, then u fulfills (32) with

$$\begin{cases} \tilde{\Phi}(z)=\Phi(z^2)\ , \\ \tilde{B}(z)=-2\lambda^{-1}z(z-1)B(z^2)\ , \\ \tilde{C}_0(z)=2zC_0(z^2)-(z+1)(\theta_1\Phi)(z^2)-4\lambda^{-1}zB(z^2)\ , \\ \tilde{D}_0(z)=2C_0(z^2)-2\lambda D_0(z^2)-2\lambda^{-1}B(z^2)+ \\ \qquad\qquad +2(z+1)(\theta_1(C_0-\lambda D_0-\lambda^{-1}B))(z^2)\ , \end{cases} \tag{42}$$

and the class of u verifies (38).

3) Using the standard criterion (24), we obtain the following cases:
(i) If $\Phi(1)=X(1)=\Phi(0)=0$, then we can simplify (32) – (41) by $z-1$.
We obtain

$$\begin{cases} \tilde{\Phi}(z)=z(\theta_0\Phi)(z^2)\ , \\ \tilde{B}(z)=-2\lambda^{-1}(z-1)B(z^2)\ , \\ \tilde{C}_0(z)=2C_0(z^2)-(\theta_0\Phi)(z^2)-(z+1)(\theta_0\theta_1\Phi)(z^2)-4\lambda^{-1}B(z^2)\ , \\ \tilde{D}_0(z)=-2(z+1)\big(\theta_1(\lambda^{-1}B+\lambda D_0-C_0)\big)(z^2)\ . \end{cases} \tag{43}$$

Therefore the class of u verifies (39) if $|Y(1)|+|X^{'}(1)|\neq 0$.

(ii) If $\Phi(1)=X(1)=X^{'}(1)=Y(1)=0$, then we can simplify (32) – (42) by $(z-1)$.
We get

$$\begin{cases} \tilde{\Phi}(z)=(z+1)(\theta_1\Phi)(z^2)\ , \\ \tilde{B}(z)=-2\lambda^{-1}zB(z^2)\ , \\ \tilde{C}_0(z)=2C_0(z^2)-(\theta_1\Phi)(z^2)-4\lambda^{-1}B(z^2)+ \\ \qquad\qquad +2(z+1)\big(\theta_1\big(C_0-\theta_1\Phi-2\lambda^{-1}B\big)\big)(z^2)\ , \\ \tilde{D}_0(z)=-2(z+2)\big(\theta_1(\lambda^{-1}B+\lambda D_0-C_0)\big)(z^2)- \\ \qquad\qquad -4(z+1)\big(\theta_1^2(\lambda^{-1}B+\lambda D_0-C_0)\big)(z^2)\ . \end{cases} \tag{44}$$

Thus the class of u verifies (39) if $\Phi(0)\neq 0$.
Assuming that $\tilde{\Phi}(1)=2\Phi^{'}(1)=0$, then from the condition $Y(1)=0$ we obtain

$C_0(1) = 2\lambda^{-1}B(1)$. Thus from the last result and the condition $X(1) = 0$, we get $D_0(1) = \lambda^{-2}B(1)$. So that, in this case we have $B(1) \neq 0$, since v is Laguerre-Hahn form of class s and so satisfies (24).

4) If $\Phi(1) = X(1) = X^{'}(1) = Y(1) = \Phi(0) = 0$, then it is possible to simplify $(32) - (43)$ by $z-1$. Thus u fulfills (32) with

$$\begin{cases} \tilde{\Phi}(z) = (z+1)(\theta_0\theta_1\Phi)(z^2) + (\theta_0\Phi)(z^2)\ , \\ \tilde{B}(z) = -2\lambda^{-1}B(z^2)\ , \\ \tilde{C}_0(z) = 2(z+1)\big(\theta_1\big(C_0 - \theta_0\theta_1\Phi - 2\lambda^{-1}B\big)\big)(z^2) - \\ \qquad\qquad -(z+2)(\theta_0\theta_1\Phi)(z^2)\ , \\ \tilde{D}_0(z) = -2\big(\theta_1(\lambda^{-1}B + \lambda D_0 - C_0)\big)(z^2) - \\ \qquad\qquad -4(z+1)\big(\theta_1^2(\lambda^{-1}B + \lambda D_0 - C_0)\big)(z^2)\ . \end{cases} \tag{45}$$

Then,the class of u verifies (40). ■

According to Proposition 5, the form u is also Laguerre-Hahn and its O.P sequence $\{Z_n\}_{n\geq 0}$ satisfies a structure relation. In general, $\{Z_n\}_{n\geq 0}$ fulfils

$$\begin{aligned} &\tilde{\Phi}(x)Z'_{n+1}(x) - \tilde{B}(x)Z_n^{(1)}(x) = \\ &\qquad \frac{1}{2}\big(\tilde{C}_{n+1}(x) - \tilde{C}_0(x)\big)Z_{n+1}(x) - \gamma_{n+1}\tilde{D}_{n+1}(x)Z_n(x)\ \ , n \geq 0, \end{aligned} \tag{46}$$

with

$$\begin{cases} \gamma_{n+1}\tilde{D}_{n+1}(x) = -\tilde{\Phi}(x) + \gamma_n\tilde{D}_{n-1}(x) - (x-(-1)^n)\tilde{C}_n(x) + (x-(-1)^n)^2\tilde{D}_n(x), \\ \tilde{C}_{n+1}(x) = -\tilde{C}_n(x) + 2(x-(-1)^n)\tilde{D}_n(x), \quad , n \geq 0, \end{cases} \tag{47}$$

where $\tilde{C}_0(x)$, $\tilde{D}_0(x)$ are given by (31) and $\gamma_0\tilde{D}_{-1}(x) = \tilde{B}(x)$.

We are going to establish the expression of $\tilde{C}_n$ and $\tilde{D}_n$, $n \geq 0$ in terms of those of the sequence $\{S_n\}_{n\geq 0}$.

Proposition 8. *The sequence $\{Z_n\}_{n\geq 0}$ fulfils (46) with (for $n \geq 0$)*

$$\begin{cases} \tilde{C}_{2n+1}(x) = \Phi(x^2) + 2x(x-1)\big(C_n(x^2) + 2\gamma_{2n+1}D_n(x^2)\big)\ , \\ \tilde{D}_{2n+1}(x) = 2x(x-1)^2 D_n(x^2). \end{cases} \tag{48}$$

$$\begin{cases} \tilde{C}_{2n+2}(x) = -\Phi(x^2) + 2x(x-1)\big(C_{n+1}(x^2) + 2\gamma_{2n+2}D_n(x^2)\big), \\ \tilde{D}_{2n+2}(x) = 2x\big(\gamma_{2n+2}D_n(x^2) + \gamma_{2n+3}D_{n+1}(x^2) + C_{n+1}(x^2)\big)\ . \end{cases} \tag{49}$$

$\tilde{C}_0(x)$ and $\tilde{D}_0(x)$ are given by (31) and γ_{n+1} by (17).

Proof. Change $x \longrightarrow x^2$, $n \longrightarrow n-1$ in (30) and multiply by $2x(x-1)^2$, we obtain by taking $(15)-(16)$, (19) and (33) into account,

$$\begin{aligned} \tilde{\Phi}(x)Z^{'}_{2n+1}(x) - \tilde{B}(x)Z_{2n}^{(1)}(x) = &\big\{x(x-1)\big(C_n(x^2) - C_0(x^2) + 2\lambda^{-1}B(x^2)\big) \\ &+ \Phi(x^2)\big\}Z_{2n+1}(x) - 2\rho_n x(x-1)D_n(x^2)Z_{2n-1}(x), \quad n \geq 1. \end{aligned}$$

Using (14) and (21) where $n \longrightarrow 2n-1$, the last equation becomes

$$\begin{aligned} \tilde{\Phi}(x)Z^{'}_{2n+1}(x) - \tilde{B}(x)Z_{2n}^{(1)}(x) = &\big\{x(x-1)\big(C_n(x^2) - C_0(x^2) + 2\lambda^{-1}B(x^2) + \\ &+2\gamma_{2n+1}D_n(x^2)\big) + \Phi(x^2)\big\}Z_{2n+1}(x) - \end{aligned}$$

$$-2\gamma_{2n+1}x(x-1)^2D_n(x^2)Z_{2n}(x),\ n\geq 0.$$

From (46) and the above equation, we have for $n\geq 0$

$$\left\{\frac{\tilde{C}_{2n+1}(x)-\tilde{C}_0(x)}{2}-X_n(x)\right\}Z_{2n+1}(x)=\gamma_{2n+1}\left\{\tilde{D}_{2n+1}-Y_n(x)\right\}Z_{2n}(x)$$

with

$X_n(x)=\left(C_n(x^2)-C_0(x^2)+2\gamma_{2n+1}D_n(x^2)-2\lambda^{-1}B(x^2)\right)x(x-1)+\Phi(x^2)$

and $Y_n(x)=2x(x-1)^2D_n(x^2)$.

Z_{2n+1} and Z_{2n} have no common zeros, then Z_{2n+1} divides $Y_n(x)-\tilde{D}_{2n+1}(x)$, which is a polynomial of degree at most equal to $2s+5$.

Then, we have necessarily $Y_n(x)-\tilde{D}_{2n+1}(x)=0$ for $n>s+2$, and also $X_n(x)=\dfrac{\tilde{C}_{2n+1}(x)-\tilde{C}_0(x)}{2}$, $n>s+2$. Therefore, $\tilde{C}_{2n+1}(x)=2X_n(x)+\tilde{C}_0(x)$ and $\tilde{D}_{2n+1}=Y_n(x)$, $\quad n>s+2$. Then, by (31), we get (48) for $n>s+2$. By virtue of the recurrence relation (47) and (31), we can easily prove by induction that the system (48) is valid for $0\leq n\leq s+2$. Hence (48) is valid for $n\geq 0$.

Finally, from (47) and (48), we give (49). ■

3. Illustrative example

We study the problem (10), with $v:=\mathcal{J}^{(1)}(\alpha,\beta)$ where $\mathcal{J}^{(1)}(\alpha,\beta)$ is the associated form of the first order of Jacobi form. Here [6, 9]

$$\begin{cases}\xi_n=\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)}\ ,\quad n\geq 0\ ,\\ \rho_{n+1}=4\frac{(n+2)(n+\alpha+2)(n+\beta+2)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)^2(2n+\alpha+\beta+5)}\ ,\quad n\geq 0\ .\end{cases}\tag{50}$$

The regularity condition is $\alpha\,,\ \beta\neq -n\,,\ \alpha+\beta\neq -n\,,\ n\geq 2.$

$$\begin{cases}\Phi(x)=x^2-1\ ,\quad B(x)=4\frac{(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\ ,\\ \Psi(x)=-(\alpha+\beta+4)x-\frac{\alpha^2-\beta^2}{\alpha+\beta+2}\ ,\end{cases}\tag{51}$$

$$\begin{cases}D_n(x)=2n+\alpha+\beta+3\ ,\\ C_n(x)=(2n+\alpha+\beta+2)x+\frac{\alpha^2-\beta^2}{2n+\alpha+\beta+2}\ ,\quad n\geq 0\ .\end{cases}\tag{52}$$

We assume $(\alpha+\beta+1)(\alpha+1)(\beta+1)\neq 0$, then v is Laguerre-Hahn form of class $s=0$.

Using (5) and (50), we get

$$S_n(1)=2^n\frac{2^n}{\Gamma(\alpha+\beta+2n+3)}b_n(\alpha,\beta),\quad n\geq 0\ ,\tag{53}$$

where for $n\geq 0$

$$b_n(\alpha,\beta)=\begin{cases}\frac{1}{\alpha}\left(\frac{\Gamma(\alpha+n+2)\Gamma(\alpha+\beta+n+2)}{\Gamma(\alpha+1)}-\frac{\Gamma(\alpha+\beta+1)\Gamma(n+2)\Gamma(\beta+n+2)}{\Gamma(\beta+1)}\right)\ ,\ \alpha\neq 0\ ,\\ \Gamma(n+2)\Gamma(n+\beta+2)\displaystyle\sum_{k=0}^{n}\left(\frac{1}{k+1}+\frac{1}{\beta+k+1}\right)\ ,\ \alpha=0\ .\end{cases}\tag{54}$$

From (6) and (50), we obtain by induction

$$S_n^{(1)}(1) = 2^n \frac{(\alpha+\beta+2)^2(\alpha+\beta+3)}{(\alpha+1)(\beta+1)\Gamma(\alpha+\beta+2n+5)} c_n(\alpha,\beta) \ , \quad n \geq 0 \ , \tag{55}$$

where for $n \geq 0$

$$c_n(\alpha,\beta) = \frac{(\alpha+\beta+1)(\beta+1)}{(\alpha+\beta+2)} b_{n+1}(\alpha,\beta) - \frac{\Gamma(\beta+n+3)\Gamma(\alpha+\beta+n+3)}{\Gamma(\beta+1)}. \tag{56}$$

By virtue of (8) , (53) and (55), we deduce

$$S_n(1,\lambda) = \frac{2^n}{\Gamma(\alpha+\beta+2n+3)} d_n(\lambda,\alpha,\beta), \quad n \geq 0 \tag{57}$$

where for $n \geq 0$

$$d_n(\lambda,\alpha,\beta) = (\alpha+\beta+1) b_n(\alpha,\beta) - \lambda \frac{(\alpha+\beta+2)^2(\alpha+\beta+3)}{2(\alpha+1)(\beta+1)} c_{n-1}(\alpha,\beta), \ c_{-1}(\alpha,\beta) = 0. \tag{58}$$

Then, u is regular for every $\lambda \neq 0$ such that

$$\lambda \neq 2 \frac{(\alpha+1)(\beta+1)(\alpha+\beta+1) b_n(\alpha,\beta)}{(\alpha+\beta+3)(\alpha+\beta+2)^2 c_{n-1}(\alpha,\beta)}, \quad n \geq 1 \ . \tag{59}$$

(18) and (57) give

$$a_n = -\frac{2}{(\alpha+\beta+2n+3)(\alpha+\beta+2n+4)} \frac{d_{n+1}(\lambda,\alpha,\beta)}{d_n(\lambda,\alpha,\beta)}, \quad n \geq 0 \ . \tag{60}$$

Then,with (17), we obtain for $n \geq 0$

$$\begin{cases} \gamma_1 = -\lambda \ , \\ \gamma_{2n+2} = -\dfrac{2}{(\alpha+\beta+2n+3)(\alpha+\beta+2n+4)} \dfrac{d_{n+1}(\lambda,\alpha,\beta)}{d_n(\lambda,\alpha,\beta)} \ , \\ \gamma_{2n+3} = -2 \dfrac{(n+2)(\alpha+n+2)(\beta+n+2)(\alpha+\beta+n+2)}{(\alpha+\beta+2n+4)(\alpha+\beta+2n+5)} \dfrac{d_n(\lambda,\alpha,\beta)}{d_{n+1}(\lambda,\alpha,\beta)} \ . \end{cases} \tag{61}$$

Taking into account that the form v is Laguerre-Hahn and by virtue of Proposition 5, the form u is also Laguerre-Hahn. It satisfies (32) and (35) with

$$\begin{cases} \tilde{\Phi}(x) = (x-1)(x^4-1) \ , \ \tilde{B}(x) = -8\lambda^{-1} \frac{(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2} x(x-1)^2 \ , \\ \tilde{\Psi}(x) = -2x(x-1)\Big((\alpha+\beta+4)x^2 + \frac{\alpha^2-\beta^2}{\alpha+\beta+2} - 16\lambda^{-1} \frac{(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2} \Big) \ , \end{cases} \tag{62}$$

$$\begin{cases} \tilde{C}_0(x) = -x^4 + 2x(x-1)\Big((\alpha+\beta+2)x^2 + \frac{\alpha^2-\beta^2}{\alpha+\beta+2} - 16\lambda^{-1} \frac{(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2} \Big) + 1 \ , \\ \tilde{D}_0(x) = -2x\Big(-(\alpha+\beta)x^2 - \frac{\alpha^2-\beta^2}{\alpha+\beta+2} + 4\lambda^{-1} \frac{(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2} + \lambda(\alpha+\beta+3) \Big) \ . \end{cases} \tag{63}$$

From (62) − (63), we have

$$\begin{cases} \Phi(0) = -1 \ , \quad \Phi(1) = 0 \ , \\ X(1) = -(\alpha+\beta+3)\lambda^{-1}\Big(\lambda + \frac{2\beta+2}{(\alpha+\beta+2)(\alpha+\beta+3)}\Big)\Big(\lambda + \frac{2(\alpha+1)(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)}\Big) \ , \\ X'(1) = \alpha+\beta+2 \ , \quad Y(1) = 2\frac{(\alpha+1)(\alpha+\beta)}{\alpha+\beta+2} - 8\lambda^{-1}\frac{(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2} \ . \end{cases}$$

Now it is enough to use Proposition 7 in order to obtain the following results:
(i) If λ satisfies (13) and $\lambda \notin E = \{-\frac{2\beta+2}{(\alpha+\beta+2)(\alpha+\beta+3)}, -\frac{2(\alpha+1)(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)}\}$, then the class of u is $\tilde{s} = 3$.
(ii) If $\lambda \in E$, then the class of u is $\tilde{s} = 2$ since $X'(1) \neq 0$.

Now, we are going to give the elements of the structure relation of the sequence $\{Z_n\}_{n\geq 0}$.
Using (52), (61) and Proposition 8., we obtain for $n \geq 0$

$$\begin{cases} \tilde{C}_0(x) = -x^4 + 2x(x-1)\Big((\alpha+\beta+2)x^2 + \frac{\alpha^2-\beta^2}{\alpha+\beta+2} - 16\lambda^{-1}\frac{(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\Big) + 1 \ , \\ \tilde{C}_1(x) = 2x(x-1)\Big((\alpha+\beta+2)x^2 + \frac{\alpha^2-\beta^2}{\alpha+\beta+2} - 2\lambda(\alpha+\beta+3)\Big) + x^4 - 1 \ , \\ \tilde{C}_{2n+2}(x) = 2x(x-1)\Big((\alpha+\beta+2n+4)x^2 + \frac{\alpha^2-\beta^2}{\alpha+\beta+2n+4} \\ \qquad -\frac{4}{(\alpha+\beta+2n+4)}\frac{d_{n+1}(\lambda,\alpha,\beta)}{d_n(\lambda,\alpha,\beta)}\Big) - x^4 + 1 \ , \\ \tilde{C}_{2n+3}(x) = 2x(x-1)\Big((\alpha+\beta+2n+4)x^2 + \frac{\alpha^2-\beta^2}{\alpha+\beta+2n+4} \\ \qquad -4\frac{(n+2)(\alpha+n+2)(\beta+n+2)(\alpha+\beta+n+2)}{(\alpha+\beta+2n+4)}\frac{d_n(\lambda,\alpha,\beta)}{d_{n+1}(\lambda,\alpha,\beta)}\Big) + x^4 - 1 \ , \\ \tilde{D}_0(x) = -2x\Big(-(\alpha+\beta)x^2 - \frac{\alpha^2-\beta^2}{\alpha+\beta+2} + 4\lambda^{-1}\frac{(\alpha+\beta+1)(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2} + \lambda(\alpha+\beta+3)\Big) \ , \\ \tilde{D}_{2n+1}(x) = 2x(x-1)^2(\alpha+\beta+2n+3) \ , \\ \tilde{D}_{2n+2} = 2x\Big((\alpha+\beta+2n+4)x^2 + \frac{\alpha^2-\beta^2}{\alpha+\beta+2n+4} - \\ \qquad 2\frac{(n+2)(\alpha+n+2)(\beta+n+2)(\alpha+\beta+n+2)}{(\alpha+\beta+2n+4)}\frac{d_{n+1}(\lambda,\alpha,\beta)}{d_n(\lambda,\alpha,\beta)} - \frac{2}{(\alpha+\beta+2n+4)}\frac{d_n(\lambda,\alpha,\beta)}{d_{n+1}(\lambda,\alpha,\beta)}\Big) \ . \end{cases} \tag{64}$$

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