SIMPLE HARMONIC OSCILLATOR
ALGEBRAIC

$F = -kx = -\frac{dV}{dx} \Rightarrow \qquad kx\,dx = dV \rightarrow \quad V = \frac{1}{2}kx^2 + C$

who cares

Solution $\quad m\frac{d^2x}{dt^2} + kx = 0 \qquad (D^2 + \frac{k}{m})x = 0 \qquad (D + i\sqrt{\frac{k}{m}})(D - i\sqrt{\frac{k}{m}}) = 0$

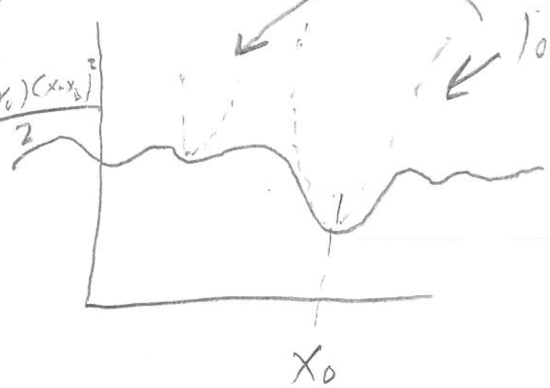
$$x(t) = A_1 e^{i\omega t} + B_1 e^{-i\omega t} \rightarrow \omega = \sqrt{\frac{k}{m}}$$

$$\rightarrow A_2 \cos(\omega t) + B_2 \sin(\omega t)$$

Utility $\rightarrow$ $\quad V(x)$

locally $V(x) = \frac{1}{2}kx^2$

$$V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{V''(x_0)(x-x_0)^2}{2}$$



Now $V(x_0) = C$ ignore

$V'(x_0)(x-x_0) = 0$

So $\quad V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$

Now $\quad \hat{H}\psi = E\psi \qquad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2x^2$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

Hold on

Algebraic

Re write

$$\frac{\hat{p}^2}{2m}\psi + V\psi = E\psi \rightarrow \frac{1}{2m}\left[\hat{p}^2 + (m\omega x)^2\right]\psi = E\psi$$

if this was $(u^2+v^2) \rightarrow (u+iv)(u-iv)$

but it is NOT.

why? $\hat{p}, x$ are operators

$$\hat{p}x = -\hbar i \frac{\partial}{\partial x}[x(\ )] \neq \hat{x}\hat{p} = -\hbar i\, x\frac{\partial}{\partial x}[\ ]$$

Write $a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x)$

$$a_- a_+ = \frac{1}{2\hbar m\omega}(i\hat{p} + m\omega x)(-i\hat{p} + m\omega x)$$

$$= \frac{1}{2\hbar m\omega}(\hat{p}^2 + (m\omega x)^2 - im\omega(x\hat{p} - \hat{p}x))$$

commutator of $[x, \hat{p}]$

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

what is $[x, \hat{p}]$? Important

To determine, feed a trial function

$$[x, \hat{p}]f(x) = \left[x\frac{\hbar}{i}\frac{df}{dx} - \frac{\hbar}{i}\frac{d}{dx}(xf)\right] = \frac{\hbar}{i}\left[x\frac{df}{dx} - x\frac{df}{dx} - f\right] = i\hbar f(x)$$

$[x, \hat{p}] = i\hbar$ Important, for non commuting operators

$\rightarrow [\hat{A}, \hat{B}] \neq 0$ Operators don't commute

So $a_- a_+ = \frac{1}{\hbar\omega}\left[\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega x^2 + \frac{1}{2}\right]$ $\leftarrow -\frac{im\omega}{2m} \cdot i\hbar \frac{1}{\hbar\omega} = \frac{1}{2}$

Or $a_- a_+ = \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}$ and $\hat{H} = \hbar\omega\left[a_- a_+ - \frac{1}{2}\right]$

$a_+ a_- = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}$ and $\hat{H} = \hbar\omega\left[a_+ a_- + \frac{1}{2}\right]$

RESULT

$[a_-, a_+] = a_- a_+ - a_+ a_- = \left(\frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}\right) - \left(\frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}\right)] = 1$

RESULT

Now $\hbar\omega\left[a_\pm a_\mp \pm \frac{1}{2}\right]\psi = E\psi$

So I now claim $\rightarrow$ if ψ satisfies $\underline{\hat{H}\psi = E\psi}$

$$\underline{\hat{H}(a_+\psi) = (E+\hbar\omega)(a_+\psi)}$$

$\psi \rightarrow$ eigenvector with Energy E $\quad a_+\psi$ Eigenvector with energy $E+\hbar\omega$

RESULT

$$\hat{H}(a_+\psi) = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)a_+\psi = \hbar\omega\left(a_+a_-a_+ + \frac{1}{2}a_+\right)\psi$$

Order matters

$$= \hbar\omega\,(a_+)\left(a_-a_+ + \frac{1}{2}\right)\psi$$

but $[a_-, a_+] = a_-a_+ - a_+a_- = 1 \rightarrow a_-a_+ = a_+a_- + 1$

So $$= \hbar\omega\,(a_+)\left(a_+a_- + 1 + \frac{1}{2}\right) = a_+\hbar\omega\,\underbrace{\left(a_+a_- + \frac{1}{2}\right.}_{\hat{H}}\left. + 1\right)\psi$$

$$= a_+\left(\hat{H} + \hbar\omega\right)\psi = a_+(\underset{\uparrow \text{number}}{E} + \hbar\omega)\psi = \boxed{(E+\hbar\omega)a_+\psi = \hat{H}(a_+\psi)}$$

Similarly $\boxed{\hat{H}(a_-\psi) = (E-\hbar\omega)a_-\psi}$

$a_+ \rightarrow$ raising operator $\quad \hat{H}\psi = E\psi \rightarrow \hat{H} a_+\psi = (E+\hbar\omega) a_+\psi$

$a_+\psi \rightarrow \psi_{+1}$

$a_- \rightarrow$ Lowering operator $\quad \hat{H}\psi = E\psi \quad \hat{H} a_-\psi = (E-\hbar\omega) a_-\psi$

$a_-\psi \rightarrow \psi_{-1}$

$\vdots$

$E+2\hbar\omega \rightarrow (a_+)^2\psi$

$E+\hbar\omega \rightarrow a_+\psi$

$E \rightarrow \psi$

$E-\hbar\omega \rightarrow a_-\psi$

$E-2\hbar\omega \rightarrow (a_-)^2\psi$

$\vdots$

$E_0 \rightarrow \psi_0$

So $a_-\psi_0$ kills everything

$$a_-\psi_0 = \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\psi_0 = 0 \rightarrow \hbar\frac{d}{dx}\psi_0 = -m\omega x\psi_0$$

$$\frac{d\psi_0}{\psi_0} = -\frac{m\omega x}{\hbar}dx \rightarrow \ln(\psi_0) = -\frac{1}{2}\frac{m\omega x^2}{\hbar} + C$$

$$\psi_0 = Ae^{-\frac{m\omega x^2}{2\hbar}}$$

$$1 = A^2 \int_{-\infty}^{+\infty} e^{-\frac{m\omega x^2}{\hbar}} dx = \left(\frac{\pi\hbar}{m\omega}\right)^{1/2} A^2$$

$$\boxed{\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}}$$

E_0 ? $\quad \hbar\omega \left[a_+a_- + \frac{1}{2}\right]\psi_0 = E_0\psi$

$$\hbar\omega\, a_+ \underbrace{a_- \psi_0}_{0} + \frac{1}{2}\hbar\omega\,\psi_0 = E_0\psi$$

$$\boxed{\frac{\hbar\omega}{2} = E_0}$$

$$\boxed{\psi_n = A_n (a_+)^n \psi_0(x) \qquad E_n = \left(n + \frac{1}{2}\right)\hbar\omega}$$

$Ex \rightarrow \quad \psi_1 = A_1 a_+ \psi_0 = \frac{A_1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$

$$= \frac{A_1}{\sqrt{2\hbar m\omega}}\left[\left(-\hbar\left(-\frac{2m\omega x}{2\hbar}\right)\right)e^{''} + m\omega x\, e^{''}\right]\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$= \frac{A_1}{\sqrt{2\hbar m\omega}}\left(2m\omega x\, e^{''}\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$= \boxed{A_1 \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot \sqrt{\frac{2m\omega}{\hbar}} \; x \, e^{-\frac{m\omega x^2}{2\hbar}}}$$

Normalization

Now $a_\pm \psi_n \propto \psi_{n\pm 1}$

So $a_+ \psi_n = C_n \psi_{n+1}$ $\quad a_- \psi_n = D_n \psi_{n-1}$

Now $\int_{-\infty}^{+\infty} f^* (a_\pm g) \, dx = \int_{-\infty}^{+\infty} (a_\mp f)^* g \, dx$ Operators in QM are Hermetian

$$\int_{-\infty}^{+\infty} f^* (a_\pm g) = \frac{1}{\sqrt{2\hbar m\omega}} \int f^* \left(\mp \hbar \frac{d}{dx} + m\omega x\right) g \, dx$$

$$= \frac{1}{\sqrt{2\hbar m\omega}} \int f^* \left(\mp \hbar \frac{d}{dx} g\right) + m\omega x f^* g \, dx$$

$$\rightarrow \int f^* \frac{dg}{dx} = - \int \left(\frac{df}{dx}\right)^* g \, dx$$

$$= \frac{1}{\sqrt{2\hbar m\omega}} \int \left(\left(\pm \hbar \frac{d}{dx} + m\omega x\right) f\right)^* g \, dx$$

$$= \int_{-\infty}^{+\infty} (a_\mp f)^* g \, dx$$

$$\text{So} \quad \int_{-\infty}^{+\infty} (a_\pm \psi_n)^* (a_\pm \psi_n) \, dx = \int_{-\infty}^{+\infty} (a_\mp a_\pm \psi_n)^* \psi_n \, dx$$

Pull back $a_\pm$ and make it $a_\mp$

$$\hbar\omega \left(a_+ a_- + \frac{1}{2}\right)\psi_n = \overset{\nearrow \left(\frac{\hbar\omega}{2} + n\hbar\omega\right)}{E_n}\psi_n \rightarrow \left(a_+ a_- + \frac{1}{2}\right)\psi_n = \left(n + \frac{1}{2}\right)\psi_n$$

$$\boxed{\rightarrow \quad a_+ a_- \psi_n = (n)\psi_n}$$

$$\hbar\omega \left(a_- a_+ - \frac{1}{2}\right)\psi_n = \left(n + \frac{1}{2}\right)\hbar\omega \, \psi_n$$

$$\boxed{a_- a_+ \psi_n = (n+1)\psi_n}$$

Now

$\downarrow$

$$\int_{-\infty}^{+\infty} (a_+\psi_n)^*(a_+\psi_n)\,dx = \int_{-\infty}^{+\infty} (a_-a_+\psi_n)^*(\psi_n)\,dx = |c_n|^2 \int |\psi_{n+1}|^2 dx = (n+1)\underbrace{\int_{-\infty}^{+\infty} |\psi_n|^2 dx}_{1}$$

$$\boxed{c_n = \sqrt{n+1}}$$

$$\int_{-\infty}^{+\infty} (a_-\psi_n)^*(a_-\psi_n)\,dx = |d_n|^2 \int |\psi_{n-1}|^2 dx = n \underbrace{\int |\psi_n|^2 dx}_{1}$$

$$\boxed{d_n = \sqrt{n}}$$

$$a_+\psi_n = c_n\psi_{n+1} \rightarrow \psi_1 = a_+\psi_0 = \sqrt{1}\,\psi_1$$

$$\psi_1 = \frac{a_+\psi_0}{1}$$

$$\psi_2 = a_+\psi_1 = (a_+)^2\psi_0 = \sqrt{2}\,\psi_2 \rightarrow \psi_2 = \frac{(a_+)^2}{\sqrt{1}\,\sqrt{2}}\psi_0$$

$$\psi_3 = a_+\psi_2 = (a_+)^3\psi_0 = \sqrt{3}\,\psi_3 \rightarrow \psi_3 = \frac{1}{\sqrt{1\cdot 2\cdot 3}}(a_+)^3\psi_0$$

$$\boxed{\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0}$$

Also $\int_{-\infty}^{+\infty} \psi_m^* \psi_n \, dx = \delta_{mn}$ complete set

$$\int \psi_m^* (a_+ a_-) \psi_n \, dx = n \int \psi_m^* \psi_n \, dx = \int_{-\infty}^{+\infty} (a_- \psi_m)^* (a_- \psi_n) \, dx$$

$$= \int_{-\infty}^{+\infty} (a_+ a_-) \psi_m^* \psi_n \, dx = m \int \psi_m^* \psi_n \, dx$$

$= 0$ unless $n = m$

complete

$\rightarrow \Psi(x,0) = \sum c_n \psi_n(x)$

$c_n = \int \psi_n^* \Psi(x,0) \, dx$

Ex.)

$$\langle V \rangle = ? \quad \langle \tfrac{1}{2} m\omega^2 x^2 \rangle = \tfrac{1}{2} m\omega^2 \langle x^2 \rangle = \tfrac{1}{2} m\omega^2 \int \psi_n^* x^2 \psi_n \, dx$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-) \qquad x^2 = \frac{\hbar}{2m\omega} (a_+^2 + a_+ a_- + a_- a_+ + a_-^2)$$

$$\langle V \rangle = \frac{1}{2} \frac{m\omega^2 \hbar}{2m\omega} \int \psi_n^* (a_+^2 + a_+ a_- + a_- a_+ + a_-^2) \psi_n \, dx$$

$$= \frac{\hbar\omega}{4} \int \psi_n^* \left[\psi_{n+2} + (n)\psi_n + (n+1)\psi_n + \psi_{n-2} \right] dx$$

$$= \frac{\hbar\omega}{4} [n + n + 1] = \frac{\hbar\omega}{4}(2n+1) = \underline{\frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right)}$$

Now $\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) e^{-\frac{m\omega}{2\hbar}x^2}$

$H_n \rightarrow$ Hermite Polynomials odd $\rightarrow$ even

$H_0 = 1$ $\quad H_1 = 2x$ $\quad H_2 = 4x^2 - 2$ $\quad H_3 = 8x^3 - 12x$ $\quad H_4 = 16x^4 - 48x^2 + 12$

Rodriguez $\rightarrow$ $(-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$

$H_0 \rightarrow (-1)^0 e^{x^2} e^{-x^2} = 0$

$H_1 \rightarrow (-1)^1 e^{x^2} \frac{d}{dx} e^{-x^2} = -1 e^{x^2} \cdot (-2x) e^{-x^2} = 2x$

$H_2 \rightarrow (-1)^2 e^{x^2} \frac{d}{dx}\left(\frac{d}{dx} e^{-x^2}\right) = e^{x^2} \frac{d}{dx}\left(-2x e^{-x^2}\right) = -2e^{x^2}\left[e^{-x^2} - 2x^2 e^{-x^2}\right]$

$= 4x^2 - 2$

Recursion $\quad H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$

need H_n, H_{n-1}

$H_2 = 2x H_1 - 2n H_0 = 2x(2x) - 2 \cdot 1 \cdot 1 = \underline{4x^2 - 2}$

$H_3 = 2x(4x^2 - 2) - 2 \cdot 2 \cdot 2x$

$= 8x^3 - 4x - 8x = 8x^3 - 12x$

Generating function

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) = e^{-z^2+2zx} \rightarrow H_n(x) = \left(\frac{d}{dz}\right)^n e^{-z^2+2zx}\Big|_{z=0}$$

$$H_0 = e^0$$

$$H_1 = -2z+2x\Big|_{z=0} e^0 = 2x \quad \ldots$$

Final

$$\frac{dH_n}{dx} = 2n\, H_{n-1}(x) \qquad H_n = \int 2n\, H_{n-1}(x)\, dx$$