Formalism continued

Eigenfunctions of a hermetian operator

2 Types Discrete, continuous

if continuous Not Normalizable

Some have both

Discrete

1: Eigenvalues are real

$$\hat{Q} f = q f$$

$$\langle f | Q f \rangle = \langle Q f | f \rangle = q \underbrace{\langle f | f \rangle}_{1} = q^* \underbrace{\langle f | f \rangle}_{1}$$

$$q = q^*$$

2: Eigenfunctions are orthonormal

$$\hat{Q} f = q f \quad \hat{Q} g = q' g \qquad q \neq q'$$

$$\langle f | \hat{Q} g \rangle = \langle Q f | g \rangle = \langle f | q' g \rangle = \langle q f | g \rangle$$

$$q' \langle f | g \rangle = q \langle f | g \rangle \qquad q, q' \text{ exist}$$

So $\langle f | g \rangle = 0$

3: Eigenfunctions form a complete set

any $f(x) = \sum c_n \psi_n$ or $\sum a_i |e_i\rangle$

where $c_n = \int \psi_m^* f(x)\,dx$ $\quad a_i = \langle e_j | \sum a_i |e_i\rangle$

$Ae^{-iq\phi}$? $\quad \langle f|g\rangle = ?$ $\displaystyle\int_0^{2\pi} A_q^* e^{iq\phi} A_{q'} e^{-iq'\phi}\,d\phi \quad q \neq q'$

$$= A_q A_{q'} \int_0^{2\pi} e^{i(q-q')\phi}\,d\phi = \frac{A_q A_{q'}}{i(q-q')} e^{i(q-q')\phi}\Big|_0^{2\pi} =$$

$q - q' =$ integer $\quad e^{2\pi i n}$

So
$$= \frac{A_q A_{q'}}{i(q-q')}\left(e^{i(q-q')2\pi} - 1\right)$$

$$= \text{'' ''}\,(1-1)$$

$$= 0$$

Continuous Spectrum

reality, orthogonality, completeness hold in a NEW way.

Example $-\hbar i \frac{d}{dx} = \hat{P} \rightarrow$ observable, hermetian

$$\hat{P} f_p = p f_p$$

$$-\hbar i \frac{d}{dx} f_p = p f_p \rightarrow \frac{d f_p}{f_p} = \frac{p}{-i\hbar} dx = \frac{ip}{\hbar} dx$$

$$\ln(f_p) = \frac{ip}{\hbar} x \qquad \underline{f_p = A e^{\frac{ipx}{\hbar}}}$$

Now $\langle f_{p'} | f_p \rangle = \int_{-\infty}^{+\infty} f_{p'}^* f_p \, dx = |A|^2 \int_{-\infty}^{+\infty} e^{\frac{i(p-p')x}{\hbar}} dx$

but $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} dk$

$\delta(q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iqx} dx$ and $\delta(\frac{q}{\hbar}) = \hbar \delta(q)$

So $\langle f_{p'} | f_p \rangle = A^2 2\pi\hbar \, \delta(p-p')$ Let $A = \frac{1}{\sqrt{2\pi\hbar}}$

Then $\langle f_{p'} | f_p \rangle = \delta(p-p')$

Dirac orthogonality

$$\langle \psi_m | \psi_n \rangle = \delta_{mn} \qquad \langle f_{p'} | f_p \rangle = \delta(p-p')$$

Completeness

$$f(x) = \int_{-\infty}^{+\infty} c(p)\, f_p(x)\, dp = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} c(p)\, e^{ipx/\hbar}\, dp$$

$$\updownarrow$$

$$\sum c_n \psi_n$$

$$\sum \rightarrow \int$$

$$c_n \rightarrow c(p)$$

Trick $\langle f_{p'} | f \rangle = \int_{-\infty}^{+\infty} c(p) \langle f_{p'} | f_p \rangle\, dp = \int_{-\infty}^{+\infty} c(p)\, \delta(p-p')\, dp = \underline{\underline{c(p')}}$

$\updownarrow$

$\langle \psi_m | f \rangle$

now $-i\hbar \frac{d}{dx} \cdot \left(\frac{1}{(2\pi\hbar)^{1/2}}\right) e^{\frac{ipx}{\hbar}} = \frac{-i\hbar}{\sqrt{2\pi\hbar}} \frac{ip}{\hbar} \cdot e^{\frac{ipx}{\hbar}} = p\, f_p$

with $\lambda = \frac{2\pi\hbar}{p}$

Space eigen Function

$$\hat{X} g_y(x) = y\, g_y(x) \qquad g_y(x) = \delta(x-y)$$

$$\int_{-\infty}^{+\infty} g_{y'}\, g_y \, dx = \int_{-\infty}^{+\infty} \delta(x-y')\,\delta(x-y)\, dx = \delta(y-y')$$

$$\langle g_{y'} | g_y \rangle = \delta(y-y')$$

$$F(x) = \int_{-\infty}^{+\infty} c(y)\, g_y(x)\, dx = \int_{-\infty}^{+\infty} c(y)\,\delta(x-y)\, dy$$

$$c(y) = F(y)$$

Generalized Statistical interpretation → which result do we measure?

For discrete eigenvalues

$P(q_n) = |c_n|^2$ where $c_n = \langle f_n | \Psi \rangle$

Continuous

$P(q) = |c(z)|^2 dz$ $\quad P(q)$ in range $z, z+dz$ $\quad c(z) = \langle f_z | \Psi \rangle$

Upon measuring wavefunction collapses to corresponding eigenfunction

$$\Psi(x,t) = \sum c_n(t) f_n(x)$$

$$c_n = \langle f_n | \Psi \rangle, \quad \sum |c_n|^2 = 1$$

$$\langle Q \rangle = \sum \underset{\uparrow \text{value}}{q_n} \underset{\uparrow \text{probability}}{|c_n|^2} \longrightarrow \sum q_n \cdot \text{amt of } n$$

$$\langle Q \rangle = \langle \Psi | \hat{Q} \Psi \rangle = \left\langle \sum c_{n'} f_{n'} \middle| \hat{Q} \sum c_n f_n \right\rangle$$

$$= \left\langle \sum c_{n'} f_{n'} \middle| q_n \sum c_n f_n \right\rangle = q_n \sum \sum c_{n'}^* c_n \underset{\uparrow \delta_{n'n}}{\langle f_{n'} | f_n \rangle}$$

$$= \sum q_n |c_n|^2$$

Now $g_y(x)$ or $\hat{x}$ Operator ← continuous

$= \delta(x-y)$

$$C(y) = \langle g_y | \psi \rangle = \int_{-\infty}^{+\infty} \delta(x-y)\, \psi(x,t)\, dx = \psi(y,t)$$

$$P(y) = |\psi(y,t)|^2 dy$$

$$y \in [a,b] = \int_a^b |\psi(y,t)|^2 dy$$

---

momentum ↓

$$C(p) = \langle f_p | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-\frac{ipx}{\hbar}} \psi(x,t)\, dx = \phi(p,t)$$

Momentum representation

$$P(p) = |\phi(p,t)|^2 dp$$

Ex.

$\hat{A}$ has ψ_1, ψ_2 with a_1, a_2 eigenvalues

$\hat{B}$ has ϕ_1, ϕ_2 with b_1, b_2 eigenvalues

$$\psi_1 = \frac{(3\phi_1 + 4\phi_2)}{5} \qquad \psi_2 = \frac{4\phi_1 - 3\phi_2}{5}$$

$\hat{A}$ is measured, a_1 is obtained. What is the state?

$\rightarrow \psi_1$

$\hat{B}$ is measured now what are the outcomes and probabilities

$$\phi_1, \phi_2 \rightarrow \frac{9}{25}, \frac{16}{25}$$

$\hat{A}$ is measured again what is the probability of a_1?

$$\phi_1 = \frac{3\psi_1 + 4\psi_2}{5} \qquad \phi_2 = \frac{4\psi_1 - 3\psi_2}{5}$$

$$P(\phi_1) = \frac{9}{25} \rightarrow P(\psi_1) = \frac{9}{25}$$

$$P(\phi_2) = \frac{16}{25} \rightarrow P(\psi_1) = \frac{16}{25}$$

$$P(a_1) = P(\phi_1)P(\psi_1) + P(\phi_2)P(\psi_1)$$

$$= \left(\frac{9}{25}\right)^2 + \left(\frac{16}{25}\right)^2 = .5392$$

Ex.

Particle bound in delta function $V(x) = -\alpha\,\delta(x)$

$$P\left(p > \frac{m\alpha}{\hbar}\right)? \quad \Psi(x,t) \to \frac{\sqrt{m\alpha}}{\hbar}\, e^{-\frac{m\alpha|x|}{\hbar^2}}\, e^{-iEt/\hbar} \quad E = -\frac{m\alpha^2}{2\hbar^2}$$

need to find $\phi(p,t) = C(p) = \langle f_p | \Psi \rangle$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{\sqrt{m\alpha}}{\hbar} e^{-iEt/\hbar} \int_{-\infty}^{+\infty} e^{-ipx/\hbar} e^{-m\alpha|x|/\hbar^2} dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{p_0^{3/2} e^{-iEt/\hbar}}{p^2 + p_0^2}$$

$$\int_{p_0}^{\infty} |\phi(p,t)|^2 dp = \frac{2}{\pi} p_0^3 \int_{p_0}^{\infty} \frac{1}{(p^2+p_0^2)^2} dp = \frac{1}{\pi}\left[\frac{p p_0}{p^2+p_0^2} + \tan^{-1}\left(\frac{p}{p_0}\right)\right]\Bigg|_{p_0}^{\infty}$$

$$= \frac{1}{4} - \frac{1}{2\pi} = .0908$$