QMI NOTES 1

→ Classically know $\vec{F} = m\ddot{\vec{r}}$ → know $\vec{r}(t)$; $\vec{v}(t)$, $\vec{a}(t)$

DETERMINISTIC

→ In QM know $H = KE + PE$ Get $\underbrace{\psi(\vec{r}, t)}_{\text{Wave function}}$ H → Hamiltonian

→ $\psi(\vec{r}, t)$ allows only probabilities, measurable physical quantities depend on H. Some may be measured precisely like energy others may only be known to a given certainty.

→ QM really talks about repeated measurements on identical systems.

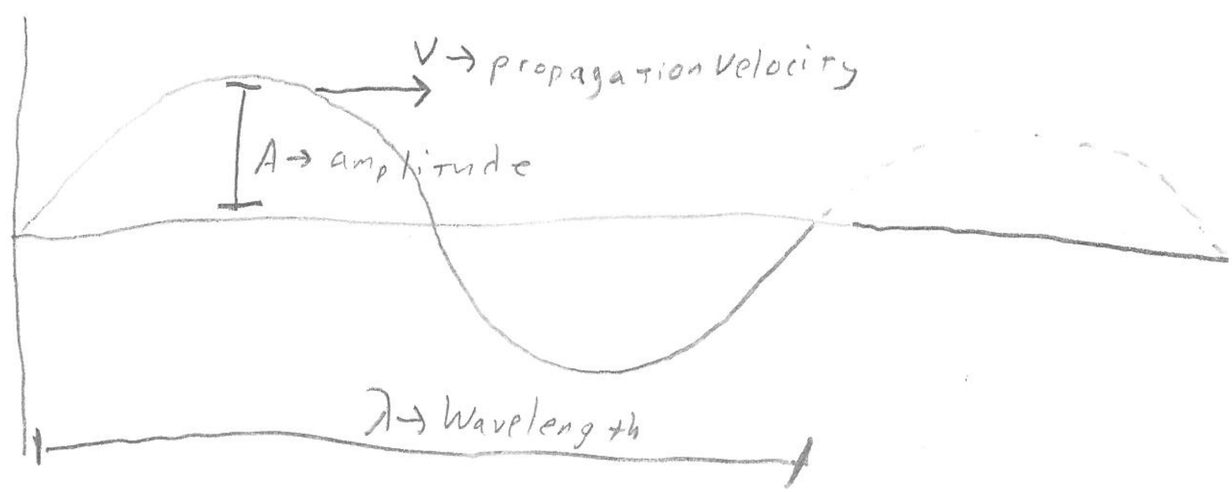
→ repeated measurements on identical systems will yield different results. Can only know averages. (Expectation Value)

Ex → System is this class → age? Average?

MATHEMATICAL INTERLUDE

$$y(x, t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt) + \delta\right]$$

δ → Shift from origin, 0 for Now



ν = wave frequency

$2\pi\nu = 2\pi \frac{v}{\lambda} = \omega$ = angular frequency (v ← velocity)

$v = \lambda\nu$ → check dimensions

$\frac{2\pi}{\lambda} = k$ → radians per unit distance

So $y(x,t) = A\,\mathrm{SIN}(kx - \omega t + \delta)$

3D $\psi(\vec{r}, t) = A\,\mathrm{SIN}(\vec{k}\cdot\vec{r} - \omega t)$

$\vec{k}\cdot\vec{r}$?

Example

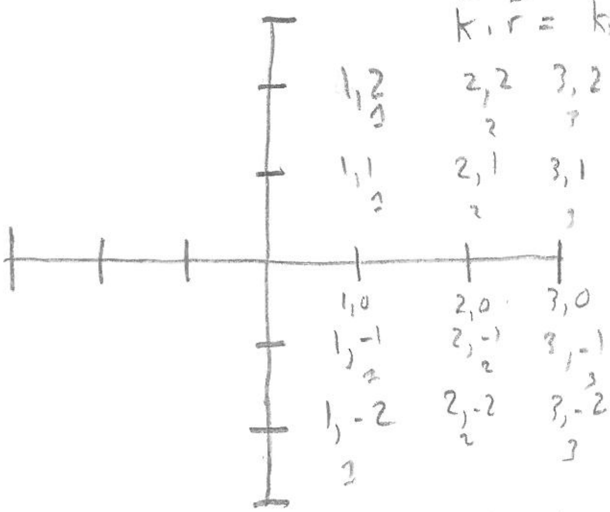
$\vec{r}$ is position vector $\vec{k}$ is wave vector

$\vec{k} = \hat{i}$ →

$\vec{k} = \hat{j}$ ↑

$\hat{k} = \hat{i}, \hat{j}$ ↗

$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$

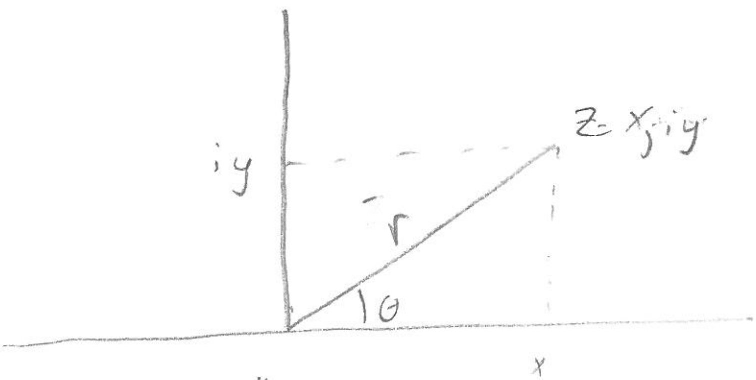


$\vec{k}$ → direction of propagation

COMPLEX NUMBERS

$Z = x + iy = re^{i\theta}$

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = (-1)(-1) = 1$



$r = (Z\bar{Z})^{1/2}$ $\quad \bar{Z}$ or Z^* $\quad i \to -i$ $\quad r = \sqrt{(x+iy)(x-iy)} = (re^{i\theta} re^{-i\theta})^{1/2}$

$= \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Eulers Identity $\quad re^{i\theta} = r(\cos\theta + i\sin\theta)$

$re^{-i\theta} = r(\cos\theta - i\sin\theta)$

$re^{i\theta} re^{-i\theta} = r^2\left[\cos^2\theta + \sin^2\theta\right] = r^2$

Waves $\rightarrow$ $Ae^{i(kx-wt)}$, $\quad Ae^{i(-kx+wt)}$

$\rightarrow$ Left $\quad \leftarrow$ Right

Easier than Sines and Cosines

USEFUL IDENTITIES

$\cos(-x) = \cos(x)$ Even
$\sin(-x) = -\sin(x)$ Odd $\rightarrow$ USE Even Odd in integrals

$\sin(u \pm v) = \sin u \cos v \pm \sin v \cos u$
$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

$$\sin^2 u = \frac{1-\cos(2u)}{2}, \quad \cos^2 u = \frac{1+\cos(2u)}{2}$$

GAUSSIAN INTEGRALS

$$I_{even} = \int_{-\infty}^{+\infty} x^{2n} e^{-ax^2} dx = 2\int_0^{\infty} x^{2n} e^{-ax^2} dx \quad \text{Even } x \rightarrow -x \ \underline{\underline{\text{Same}}}$$

$n = 0, 1, 2, 3 \ldots$

Example
$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = I_0$$

$$I_0 \cdot I_0 = \int_{-\infty}^{+\infty} e^{-ax^2} dx \int_{-\infty}^{+\infty} e^{-ay^2} \underset{\uparrow}{dy} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-a(x^2+y^2)} dx\,dy$$

dummy variable

Cartesian to polar $dx\,dy = r\,dr\,d\theta$

$$I_0^2 = \int_0^{\infty}\int_0^{2\pi} e^{-ar^2} r\,dr\,d\theta$$

Let $u = -ar^2$ $du = -2ar\,dr$
$r = 0$ $u = 0$
$r = \infty$ $u = -\infty$

$$I_0^2 = -\frac{1}{2}\int_0^{-\infty}\int_0^{2\pi} e^u\,du = -\frac{1}{2a}\,2\pi\,\int_0^{-\infty} e^u\,du = -\frac{\pi}{a}\left[e^{-\infty} - e^0\right] = -\frac{\pi}{a}[0-1]$$

$$= \frac{\pi}{a} \rightarrow I_0 = \sqrt{I_0^2} = \underline{\underline{\sqrt{\frac{\pi}{a}}}}$$

$$I_2 ? \rightarrow \int_{-\infty}^{+\infty} x^2 e^{-ax^2}\,dx \rightarrow \text{Trick}$$

$$I_0 = \sqrt{\pi}\,a^{-1/2} \rightarrow \left[\frac{dI_0}{da} = -\frac{1}{2}\sqrt{\pi}\,a^{-3/2} = \frac{d}{da}\int_{-\infty}^{+\infty} e^{-ax^2}\,dx = \int_{-\infty}^{+\infty} -x^2 e^{-ax^2}\,dx = -I_2\right]$$

$$\underline{I_2 = \frac{\sqrt{\pi}}{2}\,a^{-3/2}}$$

$$\frac{dI_2}{da} = -\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}\cdot a^{-5/2} = \frac{d}{da}\int_{-\infty}^{+\infty} x^2 e^{-ax^2}\,dx = \int_{-\infty}^{+\infty} -x^4 e^{-ax^2}\,dx = -I_4$$

$$\underline{I_4 = \frac{\sqrt{\pi}}{2}\cdot\frac{3}{2}\cdot a^{-5/2}}$$

or $$\int_{-\infty}^{+\infty} x^n e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(n+1)\,\pi^{1/2}}{2^{n/2}\,a^{\frac{n+1}{2}}} \qquad n = 0, 2, 4, \ldots$$

Odd?

$$I_1 = \int_{-\infty}^{+\infty} x e^{-ax^2}\,dx = 0$$ ODD FUNCTION SAME FOR x^3, x^5, x^7, etc

BE CAREFUL

HOW ABOUT

$$\text{LIMIT IS CHANGED} \rightarrow \int_0^{\infty} xe^{-ax^2}dx \qquad \text{Let} \quad u=-ax^2 \quad du=-2ax\,dx$$

$$x=0 \quad u=0$$
$$x=\infty \quad u=-\infty$$

$$-\int_0^{-\infty} \frac{1}{2a} e^{u}\,du = -\frac{1}{2a}\left[e^{-\infty}-1\right] = \frac{1}{2a} = \frac{1}{2}a^{-1}$$

$$\frac{dI_1}{da} = -\frac{1}{2}a^{-2} = \frac{d}{da}\int_0^{\infty} x e^{-ax^2}dx = \int_0^{\infty} -x^3 e^{-ax^2}dx = -I_3$$

$$I_3 = \frac{a^{-2}}{2}$$

$$\frac{dI_3}{da} = -\frac{2}{2}a^{-3} = \frac{d}{da}\int_0^{\infty} x^3 e^{-ax^2}dx = \int_0^{\infty} -x^5 e^{-ax^2}dx = -I_5$$

$$I_5 = a^{-3}$$

$$\frac{dI_5}{da} = -3a^{-4} = \frac{d}{da}\int_0^{\infty} x^5 e^{-ax^2}dx = \int_{-\infty}^{+\infty} -x^7 e^{-ax^2}dx = -I_7$$

$$I_7 = 3a^{-4}$$

$$\int_0^{\infty} x^{2n+1} e^{-ax^2}dx \quad n=0,1,2,3 \quad = \frac{n!}{2a^{n+1}}$$

What about $\int_{-\infty}^{+\infty} e^{-(ax^2+bx+c)} dx$?

$$= \int e^{-a\left(x^2+\frac{b}{a}x+\frac{c}{A}\right)} dx$$

Want $x^2+\frac{b}{a}x+\frac{c}{A} = ((x+D))^2 + F$

$$x^2+\frac{b}{a}x+\frac{c}{a} = x^2+2Dx+D^2+F$$

$$2D = \frac{b}{a} \quad D = \frac{b}{2a}$$

$$D^2+F = \frac{c}{a} \quad F = \frac{c}{a} - \frac{b^2}{4a^2}$$

$$\int_{-\infty}^{+\infty} e^{-a(x^2+bx+c)} dx = \int_{-\infty}^{+\infty} e^{-a\left[\left(x+\frac{b}{2a}\right)^2 + \left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]}$$

$$= e^{-\left(c-\frac{b^2}{4a}\right)} \underbrace{\int e^{-a\left(x+\frac{b}{2a}\right)^2} dx}_{\sqrt{\frac{\pi}{a}}}$$

$$= \sqrt{\frac{\pi}{a}}\, e^{\left(\frac{b^2}{4a}-c\right)}$$