Phys 362
Spring 2016

Statistical and Thermal Physics: Homework 8

Due: 27 March 2018

1 Ensembles of spin-1/2 particles

Consider an ensemble of 16 spin-1/2 particles. MEDIUM

a) List the possible macrostates of the ensemble. Specifically, the number of macrostates where you'll find N particles in the spin up state.

b) Determine the total number of microstates possible.

c) Determine the number of microstates or the multiplicity that are associated with each of the macrostates. Your answer should be a list and you can use a calculator or Excel to compute the numbers. Use the form for multiplicity given in class, do not list by hand. There is a symmetry here you can exploit.

d) Determine the probability with which each macrostate could occur.

e) Which macrostate is most probable?

f) What is the probability that a macrostate will contain *at least* 4 spin up particles?

g) What is the probability that a macrostate will contain *at most* 5 spin up particles?

h) How does the probability of attaining the microstate

$$\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$$

compare to the probability of attaining the microstate

$$\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow?$$

Explain your answer.

2 Accessible and inaccessible microstates MEDIUM

Consider a system of non-interacting spin-1/2 particles. The energy of a single particle with spin up is $-\mu B$ and that with spin down is μB. If the total energy of the system is fixed, then an accessible microstate is one that gives this total energy. Suppose that the system consists of 6 particles and that the total energy of the system is $-2\mu B$.

a) Describe which macrostates are possible in terms of the number of particles with spin up. Write your answer in terms of $N_+(N_{total})$.

b) List all accessible microstates corresponding to this macrostate. If you use the form for multiplicity you should find that there are 15 possible configurations using 6 choose 4.

c) Determine the probability with which a given particle will be in a state with spin up. Hint, count the number of the leftmost particles that are up and divide by the total number of possibilities.

d) Determine the probability with which any two given particles will be in states with spin up. Hint, count the number of states where the first two particles are spin up.

3 Ensembles of spin-$1/2$ Particles in a magnetic field MEDIUM to HARD, depending on if you reinvent the wheel or extend previously derived results

The energy of a particle with spin up in a magnetic field with magnitude B oriented along the positive z axis is $-\mu B$. That for a particle with spin down is μB. Consider an ensemble of N spin-1/2 particles in this field. Let N_+ represent the number of particles with spin up. The probability with which any given particle will be in the spin up state is different to that in which it will be in the spin down state. It will emerge that the spin up state is favored over the spin down state. Denote the probability with which any particle is in the spin up state by $(1+\varepsilon)/2$ where $0 \leqslant \varepsilon \leqslant 1$.

a) Determine an expression for the energy of the ensemble in terms of N_+.

b) Determine the probability with which any particle is in the spin down state.

In the following, consider an ensemble of five spin half particles in this field

c) List all macrostates, the energy of each and the probability with which each occurs. Do this considering the total numbers possible for spin up. You'll need to calculate the multiplicity of each state to do this problem so list that as well. You'll need to consider a Bernoulli process to get the probabilities.

d) Determine an expression for the mean of the energy. This is easier than it seems, remember the connection between the mean of steps and the mean of average distance for a random walk. Hint $< N_+ >= \frac{1+\epsilon}{2} N$.

Now consider an ensemble of an arbitrary number of particles.

e) Determine expressions for the mean and standard deviation of the energy. Do not reinvent the wheel here, use previously derived results.

f) Show that the ratio of the standard deviation to the mean approaches zero as the size of the ensemble increases.

4 Gould and Tobochnik, *Statistical and Thermal Physics*, 4.3, page 182. STUNNINGLY EASY

5 Einstein solid multiplicity EASY

For the following Einstein solids, determine a formula for the multiplicity.

a) $N = 1$ and any q.

b) $N = 2$ and any q.

c) $q = 1$ and any N.

d) $q = 2$ and any N.

6 Einstein solid statistics EASY-MODERATE

Consider an Einstein solid consisting of five particles (these are labeled A, B, C, D and E).

a) Determine the multiplicities for the four lowest energy states, N=5 and q=0,1,2,3.

b) Suppose that there are three energy units, so q=0,1,2,3. Determine the probability with which one particle will have all three energy units.

c) Suppose that there are three energy units. Determine the probability with which particle A has two energy units.

7 Interacting Einstein solids: analytical approximations HARD

a) Use Stirling's approximation to show that for an Einstein solid don't forget N is much greater than 1

$$\Omega(N,q) \approx \left(1 + \frac{q}{N-1}\right)^{N-1} \left(1 + \frac{N-1}{q}\right)^{q} \sqrt{\frac{N+q-1}{2\pi(N-1)q}}$$
$$\approx \left(1 + \frac{q}{N}\right)^{N} \left(1 + \frac{N}{q}\right)^{q} \sqrt{\frac{N+q}{2\pi Nq}}$$

whenever $q \gg 1$ and $N \gg 1$.

b) Assume that $q \gg N$. Using the Taylor series approximation, (hint $(1 + \frac{q}{N})^N \approx (\frac{q}{N})^N$)

$$\left(1 + \frac{x}{n}\right)^{n} \approx e^x$$

whenever $n \gg 1$, show that

$$\Omega(N,q) \approx \left(\frac{eq}{N}\right)^{N} \frac{1}{\sqrt{2\pi N}}.$$

c) Suppose that two Einstein solids, labeled A and B, interact. Show that the multiplicity for the combined system is

$$\Omega = \kappa\, q_A^{N_A} q_B^{N_B}$$

where κ is a term that does not depend on q_A or q_B.

d) Determine an expression for the entropy for the combined system.

e) If the system is isolated, then $q = q_A + q_B$ is fixed. Use this to rewrite the entropy in terms of q_A only. Show that the entropy attains a maximum (partial derivative wrt to q_A) when

$$\frac{q_A}{q_B} = \frac{N_A}{N_B}.$$

Explain in words what this means for the way in which energy units are shared between the two subsystems when they are in thermal equilibrium.