

## Statistical and Thermal Physics: Homework 2

Due: 30 January 2018

### 1 Thermal expansion

The linear thermal expansion coefficient describes the increase in length of a homogenous material as its temperature changes and is defined by

$$\alpha_1 := \frac{1}{L} \frac{\partial L}{\partial T}$$

where  $L$  is the length of the material. *The subscript in  $\alpha_1$  is not standard notation; it is included to distinguish the linear thermal expansion coefficient from the volume thermal expansion coefficient.*

- a) For copper,  $\alpha_1 = 1.65 \times 10^{-5} \text{ K}^{-1}$ . Determine the amount by which a copper rod of length 10 m will expand if it is heated from  $-10^\circ \text{ C}$  to  $30^\circ \text{ C}$ , assuming that  $\alpha$  is independent of temperature.
- b) A rectangular sheet of any material will expand as its temperature increases. Here the coefficient of area expansion is defined by

$$\gamma := \frac{1}{A} \frac{\partial A}{\partial T}$$

where  $A$  is the area of the material (Hint - write  $A = L_1 L_2$ ). Show that

$$\gamma = 2\alpha_1.$$

Determine the amount by which the area of a rectangular copper roof, whose sides are 10 m and 4 m increases if it is heated from  $-10^\circ \text{ C}$  to  $30^\circ \text{ C}$ .

### 2 Thermal expansion coefficient for a van der Waals gas

Determine an expression for the thermal expansion coefficient for a van der Waals gas. *Hint: note that differentiation of  $V$  w.r.t.  $T$  will be difficult. There is an identity that we encountered in class that will make this easier.* Simplify until your answer looks like  $\frac{NK}{\text{term1} + \text{term2} + \text{term3}}$

### 3 Work done on an ideal gas: specific examples

An ideal gas which undergoes various processes, with initial volume  $V_i$  and pressure  $P_i$ .

- a) Suppose that the gas expands to double its initial volume and during this process the pressure is  $P = aV^2$  where  $a$  is a constant with units of  $\text{Pa}/\text{m}^6$ . Sketch the process on a  $PV$  diagram. Determine an expression for the work done on the gas for this process. Literally, the pressure goes up parabolically. Does your answer make sense?

- b) Suppose that the gas undergoes a cyclic process in three stages. First it expands to double its initial volume while the pressure remains constant. Then the volume is held constant and the pressure increases to three times the original pressure. Then it returns to its initial state via a process which can be represented as a straight line on a  $PV$  diagram. Sketch the entire process on a  $PV$  diagram. Determine an expression for the work done on the gas in for the entire process. Hint, do you need an integral for leg 3? What does the integral mean geometrically?

#### 4 Work done on a gas: general considerations

The work done on any gas can be computed via

$$W = - \int P \, dV.$$

- a) There is only one type of process where it is legitimate to compute the work via

$$W = -P \int dV.$$

Describe what conditions this process must satisfy for this to give the correct expression for work. What curve would represent this process on a  $PV$  diagram?

- b) For ideal gases  $P = NkT/V$  and thus

$$W = - \int_{V_i}^{V_f} \frac{NkT}{V} \, dV.$$

It may appear that this gives

$$W = -NkT \int_{V_i}^{V_f} \frac{1}{V} \, dV = -NkT \ln \left( \frac{V_f}{V_i} \right).$$

But there is also only one type of process for which this is true. What condition must  $T$  satisfy during the process for this to be true? What curve would represent this process on a  $PV$  diagram?

- c) During an isobaric process, is  $T$  independent of  $V$ ? Would it be true for this process that

$$W = -NkT \int_{V_i}^{V_f} \frac{1}{V} \, dV = -NkT \ln \left( \frac{V_f}{V_i} \right) ?$$

Explain your answer.

*In general one must be careful when calculating work. One must ask: Does  $P$  (or  $T$ ) depend on  $V$  during the process? If it does, then the process of integration will be more complicated than extracting a constant and integrating with respect to  $V$  as done in the cases above.*

## 5 Monoatomic ideal gas process

A monoatomic ideal gas undergoes a cyclical process, starting at an initial volume  $V_1$  and initial pressure  $P_1$  and eventually ending at the same pressure and volume. During the first stage of the process, the pressure and volume are related by

$$P = aV^2$$

where  $a$  is a constant with units of  $\text{Pa}/\text{m}^6$ . During this process the volume is reduced to  $V_1/2$ . During the second and third stages either the pressure or else the volume remains constant. At the end of the second stage the gas reaches a pressure of  $P_1$ .

- Sketch the process on a  $PV$  diagram.
- Determine the work done during each stage of the process. Write your answer only in terms of  $V$ .
- Determine the change in internal energy during each stage of the process. Write everything in terms of the initial pressure and volume, look at how they are related
- Determine the heat that leaves or enters the gas during each stage of the process. Write your answer in terms of  $V$ .
- Over the entire cycle, does the gas do work on its surroundings or is work done on the gas?
- Consider the second stage. As the gas evolves during this stage is heat *constantly* supplied or *constantly* removed? Or is heat supplied during some portions of this stage and removed during other portions of this stage? Repeat this for the third stage.
- Suppose that the only cost to running the gas through this cycle is the heat added (the heat removed is just lost). Determine what fraction of the heat added over the entire cycle is converted to work.

## 6 Energy for a van der Waals gas process.

A monoatomic van der Waals gas satisfies the equation of state,

$$\left(P + a\frac{N^2}{V^2}\right)(V - Nb) = NkT$$

and has thermal energy given by

$$E = \frac{3}{2} NkT - a\frac{N^2}{V}$$

where  $a$  and  $b$  are constants that depend on the particular gas. Suppose that the gas undergoes a constant volume process in which the pressure doubles. Determine expressions for the work done, the change in internal energy and the heat supplied during this process. Hint, rewrite  $E$  in terms of  $P$  and  $V$ .

## 7 Equations of state for a van der Waals gas.

The particle number density of a gas is defined as  $\rho := N/V$ . Rewrite the equation of state and the equation for the energy for a van der Waals gas in terms of particle number density rather than volume.

- 8 Gould and Tobochnik, *Statistical and Thermal Physics*, 2.57, page 104. When you calculate the work done, you will have to use the expression for the pressure in terms of volume and not particle number density. After you have computed the work done, you can rewrite the resulting expression in terms of particle number density. Ultimately you should obtain an expression for  $W_{\text{vdW}} - W_{\text{ideal}}$  in terms of particle number densities  $\rho_i = N/V_i$ , temperature and the parameters  $a, b$ . Use a Taylor series to approximate the expression when  $\rho_i, b \ll 1$ . Then discuss the difference in works for various temperatures – this is actually more easily done for the case where the gas is compressed. Hint  $(1 - \frac{bN}{V})^{-1} \approx 1 + \frac{bN}{V}$ . Also, your discussion should focus on the  $kTb - a$  term.

## 9 Differentiation of multivariable functions

Consider an ideal gas and suppose that a function of interest satisfies

$$F = PV^n$$

where  $n$  is an integer.

- a) Determine  $\left(\frac{\partial F}{\partial T}\right)_V$  and  $\left(\frac{\partial F}{\partial T}\right)_P$ . Rewrite the results in terms of  $P$  and  $V$ .
- b) Determine values of  $n$  for which  $\left(\frac{\partial F}{\partial T}\right)_V = \left(\frac{\partial F}{\partial T}\right)_P$ .