Statistical and Thermal Physics: Homework 10

Due: 17 April 2018

1 Thermodynamics of spin systems

Consider a system of N spin-1/2 particles, each with magnetic dipole moment μ and in a magnetic field of magnitude B. Let n_+ the number of particles with spin up. The entropy of the system is

$$S = k \ln \left[\Omega(n_+)\right]$$

where $\Omega(n_+)$ is the multiplicity of the macrostate with n_+ particles with spin up. LONG but not HARD

a) Determine an expression for the energy of the system given that there are n_+ particles with spin up.

b) Show that the temperature of the system satisfies

$$\frac{1}{T} = \frac{k}{2\mu B} \ln \left[\frac{N - E/\mu B}{N + E/\mu B}\right]$$

and use the result to determine an expression for the energy equation of state $E = E(T, N)$.

c) List the range of possible values of E (in terms of μB and N) and plot the temperature as a function of E over the entire range. Describe when the temperature is positive and when it is negative.

d) Determine the probability with which a single particle is in the spin up state in terms of T, N, μ and B. Repeat this for spin down.

e) Using these probabilities, determine the mean energy $\overline{E}$ for a single particle. How does this compare to the expression for the energy, E, of the entire system that you obtained earlier?

2 Chemical potential for a system of spin-1/2 particles

Consider a system of N spin-1/2 particles, each with magnetic dipole moment μ and in a magnetic field of magnitude B. Let n_+ the number of particles with spin up. LONG MEDIUM

a) Determine an expression for the chemical potential of the system. Express your answer in terms of E,N,B,T and μ. Do not expand out T, or E just call it T or E.

b) Determine conditions on n_+ that give a positive or a negative chemical potential. Your answer should be in terms of N and n_+. There are two regimes to answer. One for positive T and one for negative T.

c) Suppose that $n_+ = N/3$. Determine an expression for the chemical potential. If particles are added to the system in such a way that this ratio is preserved, will the system gain or lose energy?

3 Einstein solid: low temperature limit

For any Einstein solid

$$\Omega(N,q) \approx \left(1+\frac{q}{N-1}\right)^{N-1}\left(1+\frac{N-1}{q}\right)^{q}\sqrt{\frac{N+q-1}{2\pi(N-1)q}}$$
$$\approx \left(1+\frac{q}{N}\right)^{N}\left(1+\frac{N}{q}\right)^{q}\sqrt{\frac{N+q}{2\pi Nq}}$$

whenever $q \gg 1$ and $N \gg 1$.

In the low temperature limit $q \ll N$. LONGISH MEDIUM

a) Show that if $q \ll N$ then

$$\Omega(N,q) \approx \left(\frac{eN}{q}\right)^{q}\frac{1}{\sqrt{2\pi q}}$$

and

$$S = kq\left[\ln\left(\frac{N}{q}\right) - 1\right].$$

You will discard one term in the entropy formula as q is large.

b) Use this to determine an expression for the temperature of the solid in terms of energy. If possible, invert the expression to get energy in terms of temperature.

4 Partition functions for artificial systems

Consider three systems, each with a single particle at temperature 10^5 K. System A has two states, one with energy 0 eV and the other with 10 eV. System B has three states, one with energy 0 eV and the other two each with 10 eV. System C has four states, two each with energy 0 eV and the other two each with 10 eV. Let Z_A be the partition function for system A, etc,.... EASY

a) Which of the following is true? Explain your answer.

i) $Z_A = Z_B = Z_C$.
ii) $Z_A = Z_B \neq Z_C$.
iii) $Z_A = Z_C \neq Z_B$.
iv) $Z_B = Z_C \neq Z_A$.
v) None of the partition functions are the same.

Now suppose that system A has a single particle and two states, one with energy 0 eV and the other with 10 eV. System B has two distinguishable particles and each could be in one of the two states of system A.

b) Is $Z_A = Z_B$ in this case? Explain your answer.

5 Gould and Tobochnik, *Statistical and Thermal Physics*, 4.24, page 211.EASY

6 Interstellar heat bath

The molecule CN is often found in interstellar molecular clouds. This molecule has many states associated with rotational motion. Observations indicate that about 10% of all such molecules are in any one of the first three excited states, each of which has the same energy, 4.7×10^{-4} eV, above the ground state. Assuming that the molecules are in thermal equilibrium with a heat bath, determine the temperature of the heat bath. EASY

This addresses a famous issue in cosmology and is discussed in detail in P. Thaddeus, *Annual Review of Astronomy and Astrophysics,* vol. 10, p. 305 (1972).

7 Systems with degenerate energy levels

Some systems have degenerate energy levels, meaning that there are different states that have the same energy. Consider a system that has five states with energies $0, \epsilon, \epsilon, 2\epsilon, 2\epsilon$ (electrons in a hydrogen atom have this type of degeneracy). If $kT = 2\epsilon$, determine the probabilities with which the system will have energy 0, energy ϵ and energy 2ϵ.