Time dependent Perturbation Theory

Till now $\hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t}$

but $V = V(r)$ only

So $\hat{H}\psi = E\psi$ and $\psi(r,t) = \psi(r)e^{-iEt/\hbar}$

Now let $\hat{H} = \hat{H}_i^0 + \hat{H}'(t)$ $\quad \hat{H}' \ll \hat{H}_i^0$

$$\downarrow$$

$$E_i^0 \ll E'$$

Start with just 2 states on $\hat{H}_0$

$\hat{H}^0\psi_a = E_a\psi_a$ and $\hat{H}^0\psi_b = E_b\psi_b$

$\langle\psi_i|\psi_j\rangle = \delta_{ij}$

$\psi(t=0) = C_a\psi_a + C_b\psi_b$

$\psi(t) = C_a\psi_a e^{-iE_at/\hbar} + C_b\psi_b e^{-iE_bt/\hbar}$

$|C_a|^2 + |C_b|^2 = 1$

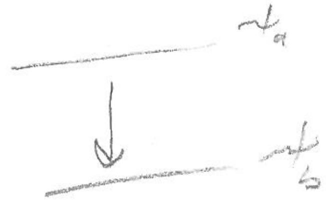
Now add $H'(t)$

$\psi(t) = C_a(t)\psi_a e^{-iE_at/\hbar} + C_b(t)\psi_b e^{-iE_bt/\hbar}$

Want $C_a(t)$ and $C_b(t)$

Suppose $C_a(0) = 1$ and $C_b(0) = 0$ and later $C_a(T_1) = 0$ and $C_b(T_1) = 1$

State went from $\psi_a \to \psi_b$ Transition



Require $\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$ where $\hat{H} = H^0 + H'(t)$

Now $\overset{1}{C_a [H^0 \psi_a] e^{-iE_a t/\hbar}} + \overset{2}{C_b [H^0 \psi_b] e^{-iE_b t/\hbar}}$

$+ \overset{3}{C_a [H' \psi_a] e^{-iE_b t/\hbar}} + \overset{4}{C_b [H' \psi_b] e^{-iE_b t/\hbar}}$

$= i\hbar \left[\overset{5}{\dot{C}_a \psi_a e^{-iE_a t/\hbar}} + \overset{6}{\dot{C}_b \psi_b e^{-iE_b t/\hbar}} \right]$

$+ i\hbar \left[\overset{7}{C_a \psi_a \left(-\frac{iE_a}{\hbar}\right) e^{-iE_a t/\hbar}} + \overset{8}{C_b \psi_b \left(-\frac{iE_b}{\hbar}\right) e^{-iE_b t/\hbar}} \right]$

$1 + 2 = 7 + 8$

So

$C_a [H' \psi_a] e^{-iE_a t/\hbar} + C_b [H' \psi_b] e^{-iE_b t/\hbar} = i\hbar \left[\dot{C}_a \psi_a e^{-iE_a t/\hbar} + \dot{C}_b \psi_b e^{-iE_b t/\hbar} \right]$

Take IP with ψ_a

$$c_a \langle \psi_a | H' | \psi_a \rangle e^{-iE_a t/\hbar} + c_b \langle \psi_a | H' | \psi_b \rangle e^{-iE_b t/\hbar} = i\hbar \dot{c}_a e^{-iE_a t/\hbar}$$

call $H'_{ij} = \langle \psi_i | H' | \psi_j \rangle$ and multiply by $\frac{1}{i\hbar} e^{iE_a t/\hbar}$

Note $H'_{ij} = (H'_{ji})^*$

$$\boxed{\dot{c}_a = \frac{-i}{\hbar}\left[c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \right]}$$

Similiarly

$$\boxed{\dot{c}_b = -\frac{i}{\hbar}\left[c_b H'_{bb} + c_a H'_{ba} e^{i(E_b - E_a)t/\hbar} \right]}$$

Typically $H'_{aa} = H'_{bb} = 0$ NOT ALWAYS !!!

If so $\dot{c}_a = \frac{-i}{\hbar} H'_{ab} e^{-i\omega_0 t}$ $\dot{c}_b = \frac{-i}{\hbar} H'_{ba} e^{i\omega_0 t}$

$\omega_0 = \frac{E_b - E_a}{\hbar}$ where $E_b \geq E_a$ so $\omega_0 \geq 0$

How To Solve?

Successive Approximations

Suppose $C_a(0)=1$ and $C_b(0)=0$ No $\hat{H}'$ No change

1: 0th order $\rightarrow$ $C_a^{(0)}(t)=1$ $C_b^{(0)}(t)=0$ Starts in one State

2: 1st order $\rightarrow$ use 0th Order

$$\dot{C}_a(t) = \frac{-i}{\hbar}\hat{H}'_{ab}e^{-i\omega_0 t}\underset{\nwarrow 0}{C_b^{(0)}} \quad \text{and} \quad \dot{C}_b(t) = \frac{-i}{\hbar}\hat{H}'_{ba}e^{i\omega_0 t}\underset{\nwarrow 1}{C_a^{(0)}}$$

where $\omega_0 = \dfrac{E_b - E_a}{\hbar}$

Now $\dfrac{dC_a^{(1)}(t)}{dt} = 0$ or $C_a^{(1)}(t)=1$

and $\dfrac{dC_b^{(1)}(t)}{dt} = \frac{-i}{\hbar}\hat{H}'_{ba}(t)e^{i\omega_0 t} \rightarrow C_b^{(1)}(t) = \frac{-i}{\hbar}\int_0^t \hat{H}'_{ba}(t')e^{i\omega_0 t'}dt'$

3: 2nd Order

$$\dot{C}_a^{(2)}(t) = \frac{-i}{\hbar}\hat{H}'_{ab}e^{-i\omega_0 t}C_b^{(1)} \rightarrow \frac{dC_a^{(2)}(t)}{dt} = \frac{-i}{\hbar}\hat{H}'_{ab}(t)e^{-i\omega_0 t}\left(\frac{-i}{\hbar}\right)\int_0^t \hat{H}'_{ba}(t')e^{i\omega_0 t'}dt'$$

$$C_a^{(2)}(t) = 1 - \frac{1}{\hbar^2}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}\left[\int_0^{t'}\hat{H}'_{ba}(t'')e^{i\omega_0 t''}dt''\right]dt'$$

and $C_b^{(2)}(t) = C_b^{(1)}(t)$ Since $C_a^{(1)}(t)=1$

Ex.) $H'_{aa} \neq 0 \quad H'_{bb} \neq 0$

use $0 \to 1$
$1 \to 2$
etc.

0th $\quad C_a^0(0) = 1 \quad C_b^0(0) = 0$

$$\dot{C}_a = -\frac{i}{\hbar}\left[C_a H'_{aa} + C_b H'_{ab} e^{-i\omega_0 t}\right]$$

$$\dot{C}_b = -\frac{i}{\hbar}\left[C_b H'_{bb} + C_a H'_{ba} e^{i\omega_0 t}\right]$$

$$\dot{C}_a^{(1)} = -\frac{i}{\hbar}\left[H'_{aa}\right] \to C_a^{1}(t) = 1 - \frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt'$$

$$\dot{C}_b^{(1)} = -\frac{i}{\hbar}\left[H'_{ba}(t)\,e^{i\omega_0 t}\right] \to C_b^{(1)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ba}(t')\,e^{i\omega_0 t'}\,dt'$$

$$|C_a^{(1)}(t)|^2 = 1 + \left[\frac{1}{\hbar}\int_0^t H'_{aa}(t')\,dt'\right]^2 \to 1 \text{ to 1st order}$$

$$|C_b^{(1)}(t)|^2 = \left[-\frac{i}{\hbar}\int_0^t H'_{ba}(t')\,e^{i\omega_0 t'}\,dt'\right]\left[\frac{i}{\hbar}\int_0^t H'_{ba}(t')\,e^{-i\omega_0 t'}\,dt'\right] = 0 \text{ to 1st order}$$