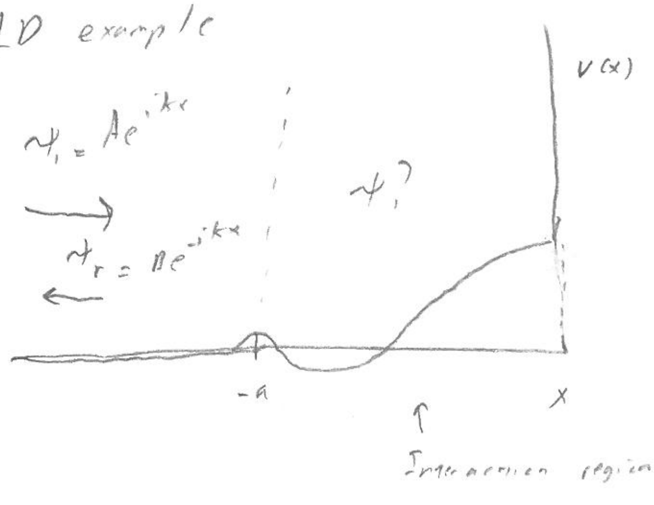
Phase Shifts

1D example



$|A| = |B|$ Conservation of Probability

For No $V(x)$ except wall → Pure bouncing

$B = -A$

$\psi_0(x) = A(e^{ikx} - e^{-ikx})$

For $V(x) \neq 0$

$\psi(x) = A\left[e^{ikx} - e^{i(2\delta - kx)}\right]$ Phase changes → δ

Need δ → Solve SE for $-a < x < 0$ impose boundary conditions

$Ae^{ikz} = A \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$ No z angular momentum $\underline{\underline{m=0}}$

all values of Total angular momentum $\ell = 0 \to \infty$

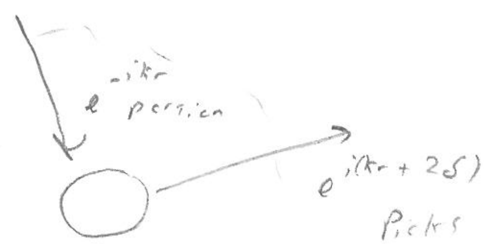
each Partial Wave → each member of Sum with different ℓ

For $V(r)=0$ $\psi_0 = Ae^{ikz}$ and $\psi_0^{(\ell)} = A i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$

but $j_\ell(x) = \frac{1}{2}\left[h^1(x) + h^2(x)\right] \approx \frac{1}{2x}\left[(-i)^{\ell+1} e^{ix} + i^{\ell+1} e^{-ix}\right]$ $x \gg 1$

So For large r $\psi_0^{(\ell)} = A\dfrac{(2\ell+1)}{2ikr}\left[e^{ikr} - (-1)^\ell e^{-ikr}\right] P_\ell(\cos\theta)$ $V(r)=0$

if $V(r) \neq 0$ $\psi^\ell \approx A\left[\dfrac{2\ell+1}{2ikr}\right]\left[e^{i(kr+2\delta)} - (-1)^\ell e^{-ikr}\right] P_\ell(\cos\theta)$



Picks up a δ upon going in
Then another going out

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before we talked in terms of a_ℓ

$$\psi^{(\ell)} \approx A\left[\frac{2\ell+1}{2ikr}\right]\left[e^{ikr} - (-1)^\ell e^{-ikr} + \frac{2\ell+1}{r} a_\ell e^{ikr}\right] P_\ell(\cos\theta)$$

Now $a_\ell = \dfrac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) = \dfrac{1}{k} e^{i\delta_\ell} \sin(\delta_\ell)$

and $F(\theta) = \sum (2\ell+1) a_\ell P_\ell(\cos\theta)$, $\sigma = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1)|a_\ell|^2$

Or

$$F(\theta) = \frac{1}{k} \sum_{\ell} (2\ell+1) e^{i\delta_\ell} \sin(\delta_\ell) P_\ell(\cos\theta)$$

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell} (2\ell+1) \sin^2(\delta_\ell)$$

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goal → Find δ_ℓ

Ex. δ_ℓ hard Sphere ..

$a_\ell = \frac{1}{k} e^{i\delta_\ell} \sin\delta_\ell$ $\quad a_\ell = \frac{i\, j_\ell(ka)}{k\, h_\ell^{(1)}(ka)}$ $\quad$ So $e^{i\delta_\ell}\sin(\delta_\ell) = \frac{i\, j_\ell(ka)}{h_\ell^{(1)}(ka)}$

but $\frac{i\, j_\ell(ka)}{j_\ell(ka) + i n_\ell(ka)} = \frac{i}{1 + i\left(\frac{n}{j}\right)} = i\,\frac{(1 - i(n/j))}{1+\left(\frac{n}{j}\right)^2} = \frac{\frac{n}{j}+i}{1+\left(\frac{n}{j}\right)^2}$

So $e^{i\delta_\ell} = \cos(\delta_\ell) + i\sin(\delta_\ell)$

and $e^{i\delta_\ell}\sin(\delta_\ell) = \sin(\delta_\ell)\cos(\delta_\ell) + i\sin^2(\delta_\ell) = \frac{\frac{n}{j}+i}{1+\left(\frac{n}{j}\right)^2}$

Equate $\cos(\delta_\ell)\sin(\delta_\ell) = \frac{\left(\frac{n}{j}\right)}{1+\left(\frac{n}{j}\right)^2}$ $\quad \sin^2(\delta_\ell) = \frac{1}{1+\left(\frac{n}{j}\right)^2}$

Or $\tan\delta_\ell = \frac{1}{n/j}$ and $\delta_\ell = \tan^{-1}\left(\frac{j}{n}\right) = \tan^{-1}\left(\frac{j(ka)}{n(ka)}\right)$

Born Approximation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \rightarrow \overset{1}{(\nabla^2+k^2)\psi = Q\psi}$$

where $k^2 = \frac{2mE}{\hbar^2}$ and $Q = \frac{2m}{\hbar^2}V\psi$

Suppose we could find $G(r)$ ST

$$(\nabla^2+k^2)G(r) = \delta^3(r)$$

Multiply 1 by $G(r-r_0)$

$$(\nabla^2+k^2)\psi\, G(r-r_0) = Q\, G(r-r_0)$$

or $\psi(r)\cdot\delta^3(r-r_0) = Q\, G(r-r_0)$

and $\int \psi(r)\,\delta^3(r-r_0)\,d^3r_0 = \int Q\, G(r-r_0)\,d^3r_0$

$$\boxed{\text{So } \psi(r) = \int Q(r_0)\, G(r-r_0)\, d^3r_0}$$

Need $G(r)$

Fourier Transform $G(r)$

differential → algebraic

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$$G(r) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{s}\cdot\vec{r}} g(s)\, d^3s$$

$$(\nabla^2 + k^2)\, G(r) = \frac{1}{(2\pi)^{3/2}} \int [\nabla^2 + k^2] e^{i\vec{s}\cdot\vec{r}} g(s)\, d^3s$$

$= \delta^3 r$

$$\nabla \sim \frac{d}{dr}$$

$$\nabla^2 e^{i\vec{s}\cdot\vec{r}} = -s^2 e^{i\vec{s}\cdot\vec{r}}$$

and $\delta^3(r) = \frac{1}{(2\pi)^3} \int e^{i\vec{s}\cdot\vec{r}}\, d^3s$

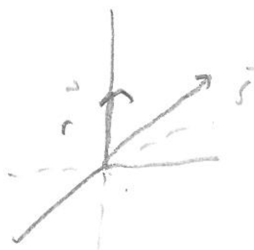
So $\frac{1}{(2\pi)^{3/2}} \int (k^2 - s^2)\, e^{i\vec{s}\cdot\vec{r}} g(s)\, d^3s = \frac{1}{(2\pi)^3} \int e^{i\vec{s}\cdot\vec{r}}\, d^3s$

So $g(s) = \frac{1}{(2\pi)^{3/2} (k^2 - s^2)}$

or $G(r) = \frac{1}{(2\pi)^3} \int e^{i\vec{s}\cdot\vec{r}} \frac{1}{(k^2 - s^2)}\, d^3s$

Now r is Fixed $\int$ over s

Pick



$$\vec{s}\cdot\vec{r} = sr\cos\theta$$

Now $$G(r) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^{\pi} e^{isr\cos\theta}\cdot\frac{1}{k^2-s^2}\, s^2\sin\theta \, d\theta\, d\phi\, ds$$

$$G(r) = \frac{1}{(2\pi)^3}\cdot 2\pi \int_{-\infty}^{+\infty}\int_0^{\pi} e^{isr\cos\theta} \sin\theta \, \frac{s^2}{k^2-s^2}\, d\theta\, ds$$

$$G(r) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\left[\int_0^{\pi} e^{isr\cos\theta}\sin\theta\, d\theta\right]\frac{s^2}{k^2-s^2}\, ds$$

$u = isr\cos\theta \quad du = -isr\sin\theta\, d\theta$

$$\frac{1}{isr}\int_{-isr}^{isr} e^{u}\, du = \frac{e^{isr}-e^{-isr}}{isr} = \frac{2\sin(sr)}{sr}$$

So $$G(r) = \frac{2}{r}\cdot\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\frac{s\sin(sr)}{k^2-s^2}\, ds$$

or $$G(r) = \frac{i}{8\pi^2 r}\left[\int_{-\infty}^{+\infty} \frac{s e^{isr}}{(s-k)(s+k)} ds - \int_{-\infty}^{+\infty} \frac{s e^{-isr}}{(s-k)(s+k)}\right]$$

$$I_1 \quad - \quad I_2$$

Cauchy integral Formula

$$\oint \frac{F(z)}{(z-z_0)} dz = 2\pi i F(z_0)$$

We'll skip

$I_1 = i\pi e^{ikr} \quad I_2 = -i\pi e^{ikr}$

$$\boxed{G(r) = \frac{i}{8\pi^2 r}\left(i\pi e^{ikr} + i\pi e^{ikr}\right) = -\frac{e^{ikr}}{4\pi r}}$$

$$\boxed{\psi(r) = \psi_0(r) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik(r-r_0)}}{|r-r_0|} \psi(r_0) V(r_0) d^3 r_0}$$

Born Approximation Continued

HW 11.8, 11.10, 11.11, 11.18

Suppose $U(r_0)$ - localized at $V(0)$ and ~ 0 for $|r| \gg |r_0|$

$$|r-r_0|^2 = |r|^2 - |r_0|^2 - 2\vec{r}\cdot\vec{r_0} = |r|^2\left(1 - \frac{r_0^2}{r^2} - 2\frac{\vec{r}\cdot\vec{r_0}}{|r|^2}\right) \sim |r|^2\left(1 - 2\frac{\vec{r}\cdot\vec{r_0}}{|r|^2}\right)$$

Scalar

or $|r-r_0| \cong |r|\left(1 - \frac{2\vec{r}\cdot\vec{r_0}}{|r|^2}\right)^{1/2}$

This is $\sim (1-x)^{1/2}$ which goes as $1 - \frac{1}{2}x$ for $x \ll 1$

or $1 - \frac{2}{2}\frac{\vec{r}\cdot\vec{r_0}}{|r|^2}$ so $|r|\left(1 - \frac{\vec{r}\cdot\vec{r_0}}{|r|^2}\right) = |r| - \frac{\vec{r}\cdot\vec{r_0}}{|r|} = |r| - \hat{r}\cdot\vec{r_0}$

Scalar

Let $k = k\hat{r}$ Then $e^{ik|r-r_0|} \simeq e^{ikr} e^{-i\vec{k}\cdot\vec{r_0}}$

and $\frac{e^{ik|\vec{r}-\vec{r_0}|}}{|r-r_0|} \cong \frac{e^{ikr}}{r} e^{-i\vec{k}\cdot\vec{r_0}}$

$|r-r_0| \sim r$ keeping terms of order $\frac{r_0}{r}$

Want $\psi_0(r) = Ae^{ikz}$ incident wave

For radiation zone $\psi(r) \cong Ae^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}\cdot\vec{r_0}} V(r_0)\psi(r_0)\,d^3r_0$

Function of r but depends on localized $V(r_0)$

but $\psi_{r \gg} = Ae^{ikz} + A\frac{e^{ikr}}{r}F(\theta,\phi)$

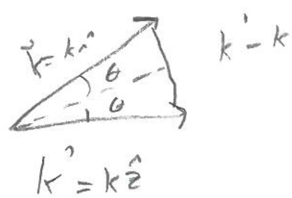
$$\boxed{\text{So} \quad F(\theta,\phi) \cong \frac{-m}{2\pi\hbar^2 A}\int e^{-i\vec{k}\cdot r_0}\, V(r_0)\,\psi(r_0)\, d^3(r_0)}$$

For weak potential, one which redirects but does not substantially alter ψ $\quad \psi(r_0) \sim \psi_0(r_0) = Ae^{ikz_0} = Ae^{ik'\cdot r_0}$

where $k' = k\hat{z}$ BORN APPROXIMATION

$$\boxed{\text{And} \quad F(\theta,\phi) \cong \frac{-m}{2\pi\hbar^2}\int e^{i(k'-k)\cdot r_0}\, V(r_0)\, d^3 r_0}$$

Now



essentialy $\hbar(k-k')$ is momentum change

And low energy, long wavelength scattering says $e^{i(k'-k)\cdot r_0} \sim$ constant

or $\boxed{F(\theta,\phi) \cong \frac{-m}{2\pi\hbar^2}\int V(r)\, d^3 r}$!!!

Example

$$V(r) = \begin{cases} V_0 & r \le a \\ 0 & r > a \end{cases}$$

not ∞ ← This substantially alters the wavefunction

$$\int_0 \quad F(\theta,\phi) = \left[\int_0^a \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \, dr \, d\phi \, d\theta \right] \frac{-m}{2\pi\hbar^2} V_0$$

$$F(\theta,\phi) = \frac{4}{3}\pi a^3 \cdot \left(\frac{-m}{2\pi\hbar^2}\right) V_0$$

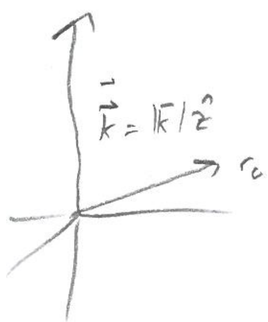
$$\frac{d\sigma}{d\Omega} = |F|^2 \cong \frac{2mV_0 a^3}{3\hbar^2}$$

$$\sigma = 4\pi\left(\frac{2mV_0a^3}{3\hbar^2}\right)$$ → just integrate $\int_0^{2\pi}\int_0^{\pi} |F|^2 \, d\Omega$

$\sin\theta \, d\theta \, d\phi$

Let $\bar{k} = k' - k$

SPHERICAL Symmetry but not necessarily low energy

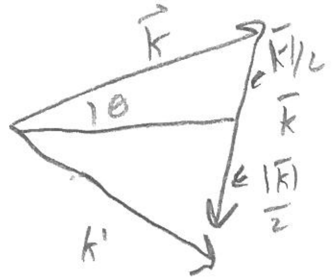


$$(\vec{k}' - \vec{k}) \cdot \vec{r}_0 = |\bar{k}| \, r_0 \cos\theta_0$$

Then $$F(\theta) \cong \frac{-m}{2\pi\hbar^2} \int e^{i|\bar{k}| r_0 \cos\theta_0} V(r_0) r_0^2 \sin\theta_0 \, dr_0 \, d\theta_0 \, d\phi_0$$

Or $$\boxed{F(\theta) = \frac{-2m}{\hbar^2 |\bar{k}|} \int_0^\infty r V(r) \sin(|\bar{k}| r) \, dr}$$!!

and



$$|\bar{K}| = 2k\,\mathrm{SIN}\left(\frac{\theta}{2}\right)$$

Example Yukawa Potential $V(r) = \dfrac{\beta e^{-\mu r}}{r}$

$$F(\theta) \cong \frac{-2m\beta}{\hbar^2 |\bar{K}|} \int_0^\infty e^{-\mu r}\,\mathrm{SIN}(|\bar{K}|r)\,dr = \frac{-2m\beta}{\hbar^2(\mu^2 + K^2)}$$

if $\mu = 0$ and $\beta = \dfrac{q_1 q_2}{4\pi\epsilon_0}$ This reduces to

$$F(\theta) \cong \frac{-2m\, q_1 q_2}{4\pi\epsilon_0 \hbar^2 |\bar{K}|^2} \rightarrow \text{Rutherford Scattering}$$

Formally

σ

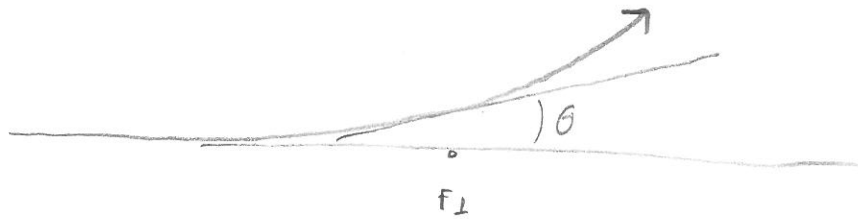
$$F(\theta) \cong \frac{-q_1 q_2}{16\pi\epsilon_0 E\,\mathrm{SIN}^2\left(\frac{\theta}{2}\right)}$$ using $|\bar{K}|^2 = 4k^2\,\mathrm{SIN}^2\left(\frac{\theta}{2}\right)$ and $k^2 = \dfrac{2mE}{\hbar^2}$

$$\frac{d\sigma}{d\Omega} = |F(\theta)|^2 = |\quad|^2$$

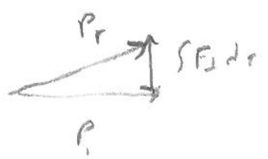
Compare to the classical derivation

The Born Series

Impulse approximation



$I = \int F_\perp d\tau$ minimal deflection, changes direction but not momentum (by too much)



$$\theta = \tan^{-1}\left(\frac{\int F_\perp d\tau}{P_i}\right) = \tan^{-1}\left(\frac{I}{P_i}\right)$$

Now $\psi = \psi_0(r) + \int g(r - r_0) V(r_0)\, \psi(r_0)\, d^3 r_0$

$\uparrow$ Incident + $\uparrow$ Scattered Part

$$g(r) = \frac{-m}{2\pi\hbar^2} \frac{e^{ikr}}{r}$$

So $\psi_1 = \psi_0 + \int g V \psi_0$ Solve plug back in

$$\psi_2 = \psi_0 + \int g V \psi_0 + \iint g V g V \psi_1$$

$$\psi_2 = \psi_0 + \int g V \psi_0 + \iint g V g V \psi_0 + \iiint g V g V g V \psi_0$$