

Ch. #7 Lagrangian and Hamiltonian Dynamics

Side Note $\rightarrow \frac{\partial J}{\partial x} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \eta(x) dx$ recall $\frac{\partial y}{\partial x} = \eta(x)$

Then $\frac{\partial J}{\partial x} dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \frac{\partial y}{\partial x} dx dx$

or $\delta J = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y dx$

Where $\frac{\partial J}{\partial x} dx = \delta J$ δ variation

$\frac{\partial y}{\partial x} dx = \delta y$

Then $\delta J = \delta \int_{x_1}^{x_2} F(y, y'; x) dx = 0 \rightarrow \text{extremum}$

$\delta J = \int_{x_1}^{x_2} \delta F dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx$

$\delta y' = \delta \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\delta y)$

So $\delta J = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \frac{d}{dx} (\delta y) \right] dx$

Integrate 2 by parts

$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y dx$ AS before

Hamilton's Principle

Of all possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.

$$\text{or } \delta \int_{T_1}^{T_2} (T-U) dt = 0 \quad \text{where } T = T(x, \dot{x}), U = U(x)$$

$$\text{Let } L = T - U = L(x_i, \dot{x}_i)$$

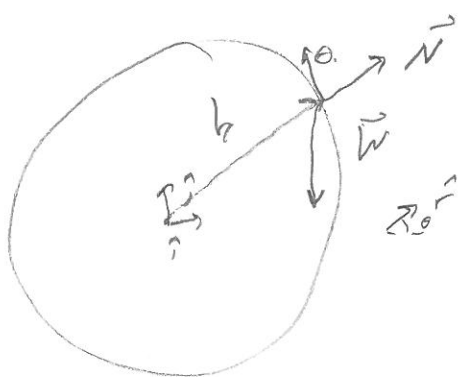
$$\text{Then } \delta \int_{x_1}^{x_2} F(y, y'; x) dx = \delta \int_{T_1}^{T_2} L(x_i, \dot{x}_i) dt \quad \begin{array}{l} x \rightarrow T \\ y_i(x) \rightarrow x_i(t) \\ y'_i(x) = \dot{x}_i(t) \end{array}$$

And Lagrange's Equations become

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \quad i = 1, 2, 3$$

Why? We have $\vec{F} = m\vec{a}$

Ex: Constrained Motion



$$\vec{W} = (\vec{W} \cdot \hat{r}) \hat{r} + (\vec{W} \cdot \hat{\theta}) \hat{\theta}$$

$$\vec{W} = (-mg \hat{j} \cdot \hat{r}) \hat{r} + (-mg \hat{j} \cdot \hat{\theta}) \hat{\theta}$$

$$\vec{W} = - [mg \sin \theta \hat{r} + mg \cos \theta \hat{\theta}]$$

$$\vec{N} = N \hat{r}$$

$$\vec{F} = m(\ddot{r} - v\dot{\theta}^2) \hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{\theta} = -mg[\sin \theta + N] \hat{r} + (-mg \cos \theta) \hat{\theta}$$

$r = b \quad \dot{r} = \ddot{r} = 0$

$$\text{So } -mb\dot{\theta}^2 = N - mg \sin \theta \quad \underline{\underline{1}}$$

$$\dot{\theta} [b\ddot{\theta}] = -[g \cos \theta] \dot{\theta} \quad \underline{\underline{2}} \rightarrow b \frac{d}{dt} \left(\frac{\dot{\theta}^2}{2} \right) = -g \frac{d}{dt} \sin \theta$$

$$\text{Or } \frac{b\dot{\theta}^2}{2} = -g \sin \theta + C \quad \text{when } \theta = \frac{\pi}{2} \quad \dot{\theta} = 0 \quad (C = g)$$

$$b\dot{\theta}^2 = 2g(1 - \sin \theta)$$

$$\text{So } N = mg(3 \sin \theta - 2) = 0 \quad \text{For } \theta = \sin^{-1} \left(\frac{2}{3} \right)$$

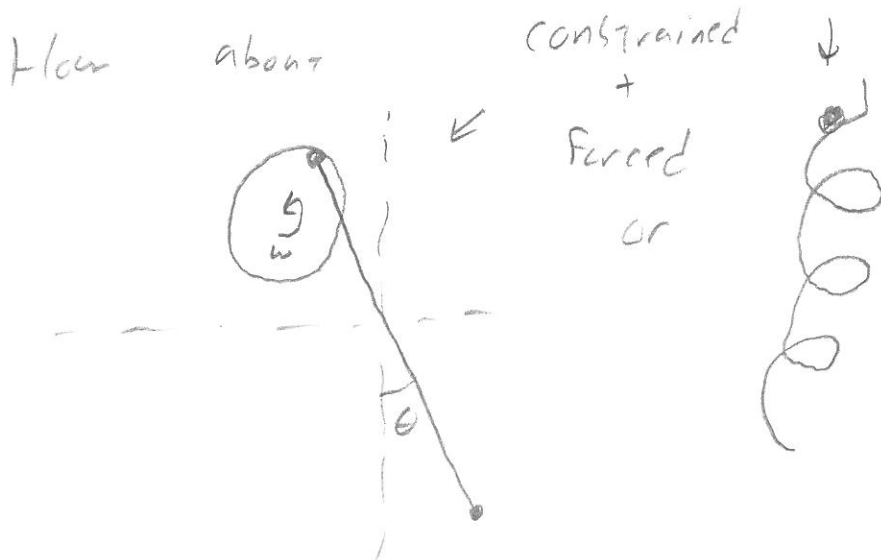
or $E_i = E_f$

$\rightarrow mgb = mgb \sin \theta + \frac{1}{2}mv^2$

So $v^2 = 2gb(1 - \sin \theta)$

and $\vec{F} = m\vec{a} = \frac{v^2}{b} \hat{r} - \frac{dv}{dt} \hat{e} = \vec{W} + \vec{N} = (N - mg \sin \theta) \hat{r} - mg \cos \theta \hat{e}$
 $\frac{v^2}{r} \hat{N} + \frac{dv}{dt} \hat{t}$

$\rightarrow \frac{v^2}{b} = N - mg \sin \theta$ ✓



Energy is much easier

Find $d\vec{s} \rightarrow \frac{d\vec{s} \cdot d\vec{s}}{dt^2}, U(q_i)$

Reduce to $\mathcal{L} = T - U$ Stick in

$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0 \quad \text{gives } \vec{F} = m\vec{a}$$

$\nwarrow \quad \nearrow$
 $\mathcal{L} \sim U \quad \sim m\vec{a}$

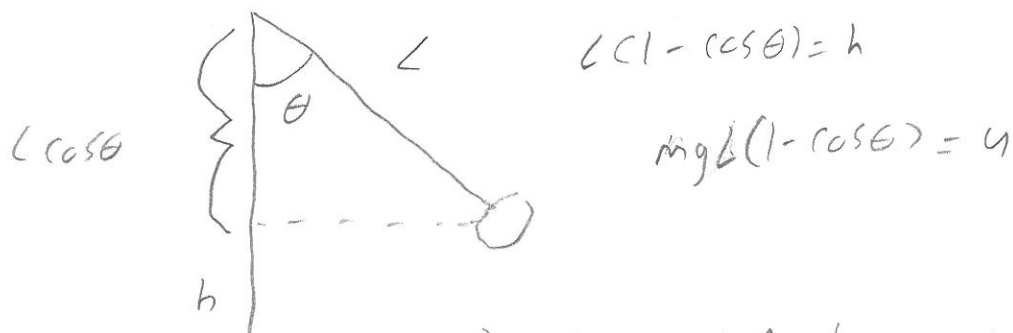
Ex. #1. $U = \frac{1}{2} kx^2 \quad v = \dot{x} \quad d\vec{s} = x \vec{i}$

$$\mathcal{L} = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{d}{dt} m \dot{x} = m \ddot{x}$$

$$m \ddot{x} + kx = 0 \quad \rightarrow \quad m \ddot{x} = -kx$$

Ex #2.



$$d\vec{s} = dr \hat{r} + r d\theta \hat{\theta} \quad \text{but } r = L \quad dr = 0$$

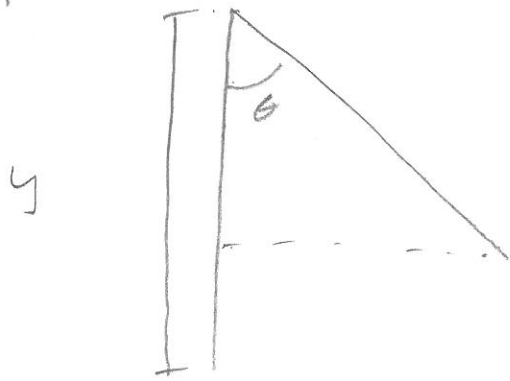
$$d\vec{s} \cdot d\vec{s} = r^2 d\theta^2 \quad \rightarrow \quad \left(\frac{ds}{dt} \right)^2 = r^2 \dot{\theta}^2$$

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 - mg(1 - \cos\theta) \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0$$

$$\frac{\partial L}{\partial \theta} = + (mg) (-\sin\theta) = -mg \sin\theta$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = m l^2 \ddot{\theta} \quad m l^2 \ddot{\theta} + mg \sin\theta = 0$$

Ex:



$$d\vec{s} = dr\hat{r} + r d\theta\hat{\theta}$$

$$r=l \quad d\vec{s} = l d\theta\hat{\theta}$$

$$d\vec{s} \cdot d\vec{s} = l^2 d\theta^2$$

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgl(1 - \cos\theta)$$

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \theta} = -mgl \sin\theta \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m l^2 \ddot{\theta}$$

$$\text{or } m l^2 \ddot{\theta} + mgl \sin\theta = 0$$

Note $I\alpha = \tau \quad \tau = \vec{r} \times \vec{F} = -mgl \sin\theta$

$$I\vec{\alpha} = I \frac{d\vec{\omega}}{dt} = \frac{d}{dt} (I\vec{\omega}) = \frac{d\vec{L}}{dt} \quad L = m l^2 \omega$$

Generalized Coordinates

For n particles there are $3n$ possible coordinates

However there are possibly m equations of constraint

leaving $S = 3n - m$ total coordinates.

Generalized Coordinates \rightarrow you pick

$$X_i \rightarrow q \quad X(q, T), \quad \dot{X}(q, T)$$

really $X_{\alpha, i}(q_1, q_2, q_3, \dots, q_n)$

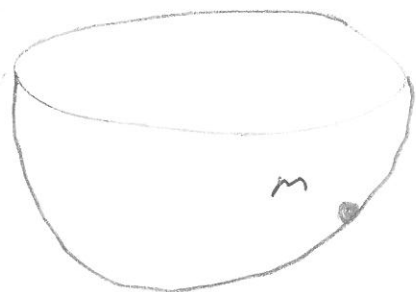
$\alpha \in [1, n]$

$i = 1, 2, 3$

$X_{\alpha, i}(q_j, T) \quad j \in [1, S]$

How to pick? Experience

Example



hemisphere radius R



$$x^2 + y^2 + z^2 = R^2 \quad \text{let } x^2 + y^2 + z^2 - R^2 = 0$$

$$\text{Then } q_1 = \frac{x}{R} \quad q_2 = \frac{y}{R} \quad q_3 = \frac{z}{R} \quad \text{and } q_1^2 + q_2^2 + q_3^2 = 1$$

doesn't work if $q_i = q_i(q_1, q_2)$ Needs to be independent

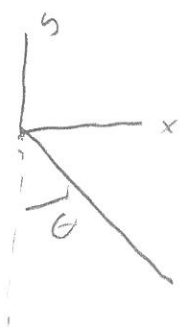
2 q 's, 1 equation of constraint

$$\underline{3, 2} - 1 = 2$$

Better \rightarrow $d\vec{s} = dr\hat{r} + r d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$

$$T = \left(r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) \frac{1}{2} m$$

Example



$$d\vec{s} = dx\hat{i} + dy\hat{j}$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2$$

$$U = mgy$$

$$L = T - U$$

$$x = l \sin \theta \quad y = -l \cos \theta$$

$$\dot{x}^2 = l^2 \dot{\theta}^2 \cos^2 \theta \quad \dot{y}^2 = l^2 \dot{\theta}^2 \sin^2 \theta$$

$$T = \frac{1}{2} m (l^2 \dot{\theta}^2 [\cos^2 \theta + \sin^2 \theta]) = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = -m g l \cos \theta$$

$\mathcal{L} \rightarrow$ Same Equations $\ddot{\theta} + \frac{g}{l} \sin \theta$

MUST BE !! $S = 3n - m \rightarrow$ need S, q 's

New $\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = 0 \quad j = 1, 2, 3, \dots, S$

2 conditions

1 Forces must be derivable from potentials

2 Equations of constraint $F_k(x_{ij}, T)$

connect coordinates and may be functions of time

2 \rightarrow holonomic

if constraint not $F(T) \rightarrow$ scleronomous

moving constraints \rightarrow rheonomic

Projectile Motion

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 \quad U = mgy$$

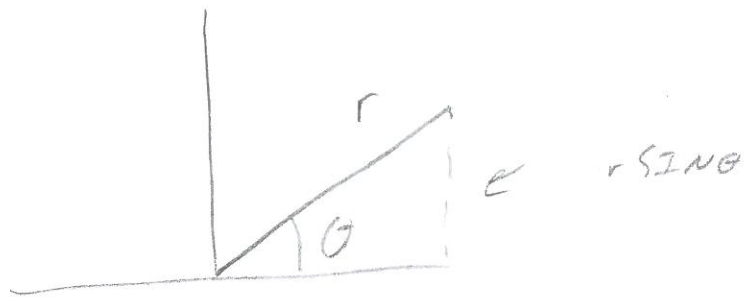
$$L = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$$\frac{\partial L}{\partial x} = 0 \rightarrow m \ddot{x} = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{d}{dt} m \dot{q} = m \ddot{q}$$

$$\frac{\partial L}{\partial y} = -mg \rightarrow m \ddot{y} + mg = 0$$

How about



$$T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

$$U = +mgy \sin \theta$$

$$L = T - U = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - mgr \sin \theta$$

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 - mg \sin \theta \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m \ddot{r}$$

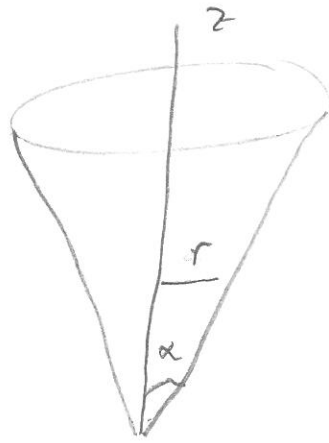
$$\text{or } m r \dot{\theta}^2 - mg \sin \theta - m \ddot{r} = 0$$

$$\frac{\partial L}{\partial \theta} = -mgr \cos \theta \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt} (m r^2 \dot{\theta}) = m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta}$$

$$-mgr \cos \theta - 2 m r \dot{r} \dot{\theta} - m r^2 \ddot{\theta} = 0$$

~

Example



$$d\vec{s} = dr \hat{r} + r d\theta \hat{\theta} + dz \hat{z}$$

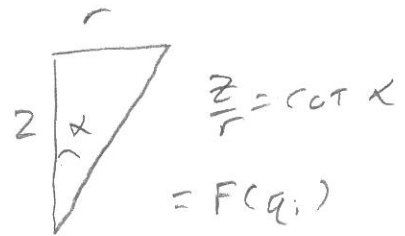
$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2)$$

$$U = mgrz$$

how many degrees of freedom?

$$3 - 1 = 2$$

eliminate either z or r



We'll kill z

$$U = mgr \cot \alpha$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{r}^2 \cot^2 \alpha) = \frac{1}{2} m (\dot{r}^2 \csc^2 \alpha + r^2 \dot{\theta}^2)$$

$$\sin^2 \alpha + \cos^2 \alpha = 1 \rightarrow \text{divide by } \sin^2 \alpha$$

$$\text{Now } L = \frac{1}{2} m (\dot{r}^2 \csc^2 \alpha + r^2 \dot{\theta}^2) - mgr \cot \alpha$$

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{So } \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta}' = 0 \quad \text{and} \quad mr^2 \dot{\theta} = \frac{\partial L}{\partial \dot{\theta}} = \text{constant}$$

or $I\omega = L = \text{constant} \rightarrow$ angular momentum

i) conserved

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 + mg \cot \alpha$$

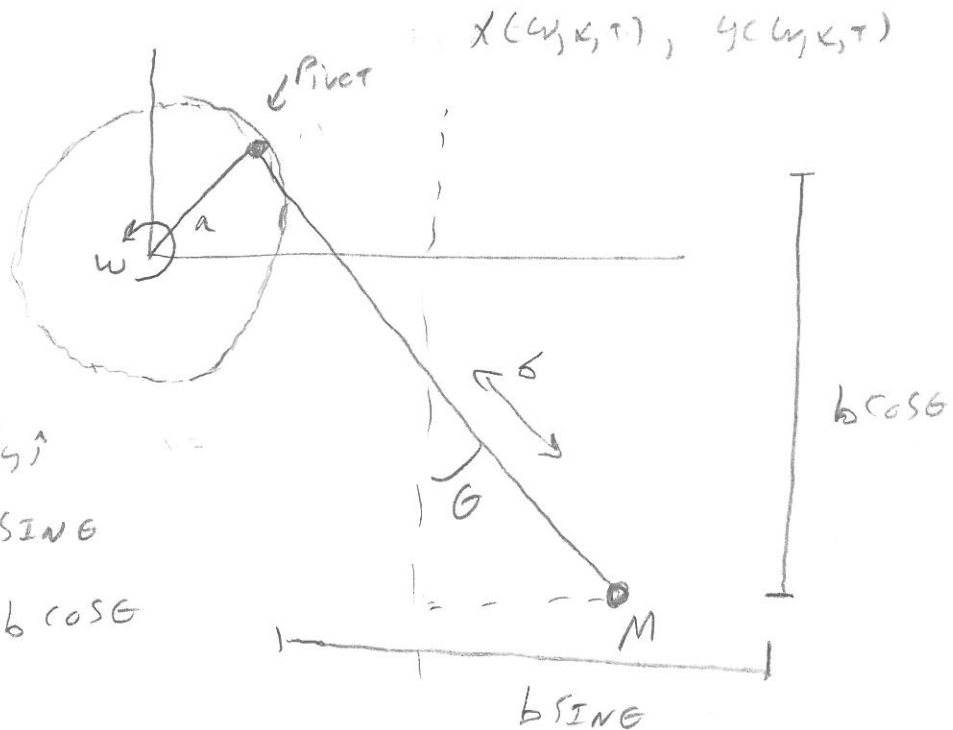
$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = mr'' \csc^2 \alpha$$

$$mr'' \csc^2 \alpha + mr\dot{\theta}^2 + mg \frac{\cos(\alpha)}{\sin(\alpha)} = 0$$

$$mr'' - mr\dot{\theta}^2 \sin^2 \alpha + mg \cos(\alpha) \sin(\alpha) = 0$$

We'll come back to this

Example



$$d\vec{s} = dx\hat{i} + dy\hat{j}$$

$$x = a \cos(\omega t) + b \sin \theta$$

$$y = a \sin(\omega t) - b \cos \theta$$

$$dx = -a\omega \sin(\omega t) + b\dot{\theta} \cos \theta$$

$$dy = a\omega \cos(\omega t) + b\dot{\theta} \sin \theta$$

$\rightarrow \theta$ is the only free coordinate

$$T = \frac{1}{2}m(-a\omega \sin(\omega t) + b\dot{\theta} \cos \theta)^2 + \frac{1}{2}m(a\omega \cos(\omega t) + b\dot{\theta} \sin \theta)^2$$

$$= \frac{1}{2}m [a^2\omega^2 \sin^2(\omega t) + b^2\dot{\theta}^2 \cos^2 \theta - 2ab\omega \dot{\theta} \sin(\omega t) \cos \theta]$$

$$+ \frac{1}{2}m [a^2\omega^2 \cos^2(\omega t) + b^2\dot{\theta}^2 \sin^2 \theta + 2ab\omega \dot{\theta} \cos(\omega t) \sin \theta]$$

$$T = \frac{1}{2} m [a^2 \dot{\omega}^2 + b^2 \dot{\theta}^2 + 2b\dot{\theta} a \omega \sin(\theta - \omega t)]$$

$$U = mgy = mg(a \sin \omega t - b \cos \theta)$$

$$L = T - U$$

$$\frac{\partial L}{\partial \theta} = -mgb \sin \theta + mb\dot{\theta} a \omega \cos(\theta - \omega t)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mb^2 \dot{\theta} + b a \omega \sin(\theta - \omega t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = mb^2 \ddot{\theta} + mb a \omega \cos(\theta - \omega t) (\dot{\theta} - \omega)$$

$$mb^2 \ddot{\theta} + mb a \omega (\dot{\theta} - \omega) \cos(\theta - \omega t) - mb\dot{\theta} a \omega \cos(\theta - \omega t) + mgb \sin \theta = 0$$

$$mb^2 \ddot{\theta} - mb a \omega^2 \cos(\theta - \omega t) + mgb \sin \theta = 0$$

$$\ddot{\theta} - \frac{a \omega^2}{b} \cos(\theta - \omega t) + \frac{g}{b} \sin \theta = 0$$

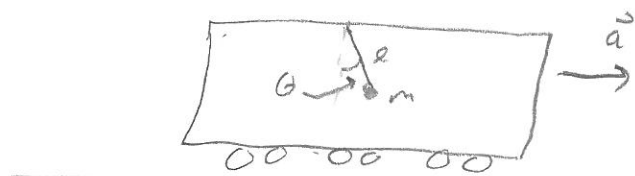
$$\ddot{\theta} = \frac{a \omega^2}{b} \cos(\theta - \omega t) - \frac{g}{b} \sin \theta = 0$$

This is a DRIVEN Pendulum

Super duper non-linear

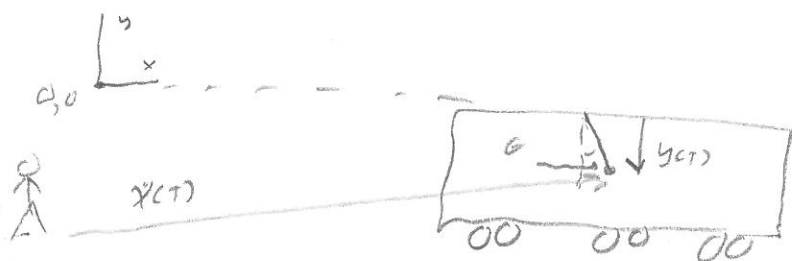
Example

begins with v_0



for θ small

Must go into inertial reference frame



$$d\vec{s} = dx\hat{i} + dy\hat{j}$$

$$x(t) = v_0 t + \frac{1}{2} a t^2 + l \sin \theta \quad \frac{dx}{dt} = v_0 + at + l \dot{\theta} \cos \theta = \dot{x}(t)$$

$$y(t) = -l \cos \theta \quad \frac{dy}{dt} = l \dot{\theta} \sin \theta = \dot{y}(t)$$

$$T = \frac{m}{2} (v_0 + at + l \dot{\theta} \cos \theta)^2 + \frac{m}{2} l^2 \dot{\theta}^2 \sin^2 \theta$$

$$T = \frac{m}{2} (v_0^2 + a^2 t^2 + l^2 \dot{\theta}^2 \cos^2 \theta + 2v_0 at + 2arl \dot{\theta} \cos \theta + 2v_0 l \dot{\theta} \cos \theta) + \frac{m}{2} l^2 \dot{\theta}^2 \sin^2 \theta$$

$$T = \frac{1}{2} m v_0^2 + \frac{1}{2} m a^2 t^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{m}{2} (2v_0 at + 2arl \dot{\theta} \cos \theta + 2v_0 l \dot{\theta} \cos \theta)$$

$$L = T - U \quad U = -mgl \cos \theta \quad \text{dropped } a^2 r^2 \text{ doesn't matter}$$

$$\mathcal{L} = \frac{1}{2} m (v_0^2 + l^2 \dot{\theta}^2) + m [v_0 a t + a t l \dot{\theta} \cos \theta + v_0 l \dot{\theta} \cos \theta] + m a^2 t^2$$

$$+ mgl \cos \theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = m a t l \dot{\theta} \sin \theta - m v_0 l \dot{\theta} \sin \theta - mgl \sin \theta$$

$$1 \quad \frac{\partial \mathcal{L}}{\partial \theta} = -\sin(\theta) [m a t l \dot{\theta} + m v_0 l \dot{\theta} + mgl] = -\sin(\theta) [m (v_0 + a t) l \dot{\theta} + mgl]$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right] = \frac{d}{dt} [m l^2 \dot{\theta} + (a t + v_0) m l \cos \theta]$$

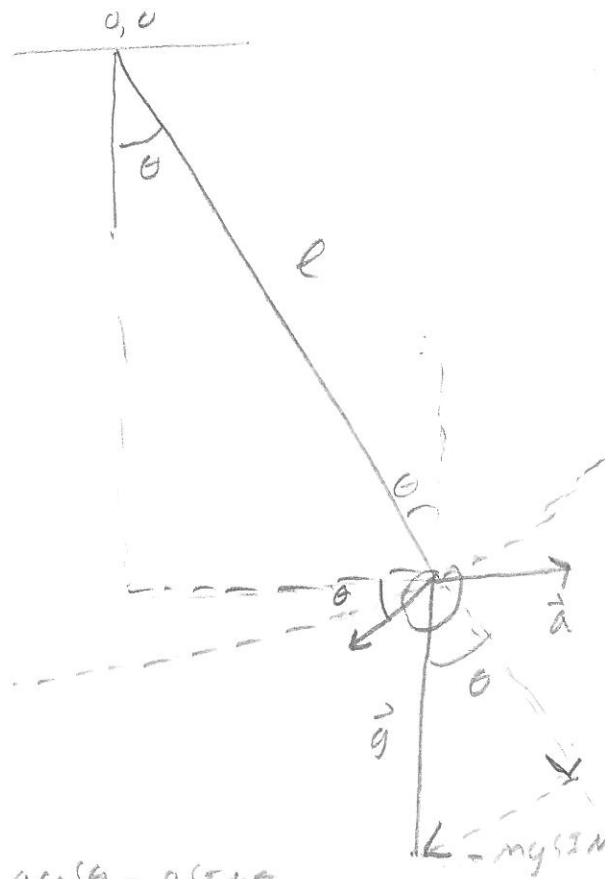
$$2 \quad \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right] = m l^2 \ddot{\theta} - (a t + v_0) m l \dot{\theta} \sin \theta + m a l \cos \theta$$

$$1 = 2 \quad -\sin \theta [m (v_0 + a t) l \dot{\theta} + mgl] = m l^2 \ddot{\theta} - m l \dot{\theta} \sin \theta (a t + v_0) + m a l \cos \theta$$

$$-\sin \theta mgl = m l^2 \ddot{\theta} + m a l \cos \theta$$

$$m l^2 \ddot{\theta} = -m a l \cos \theta - \sin \theta (mgl)$$

$$\ddot{\theta} = -\frac{a}{l} \cos \theta - \frac{g}{l} \sin \theta$$



and $m \frac{d^2 s}{dt^2} = -a \cos \theta - g \sin \theta$

$$\frac{d^2 s}{dt^2} = \frac{d^2 (l\theta)}{dt^2} = l \ddot{\theta}$$

$$m l \ddot{\theta} = -m a \cos \theta - m g \sin \theta \rightarrow \ddot{\theta} = -\frac{a}{l} \cos \theta - \frac{g}{l} \sin \theta$$

Now θ_e when $m a \cos \theta_e = -m g \sin \theta_e$

$$\text{or } \tan^{-1}\left(\frac{-a}{g}\right) = \theta_{eq}$$

Frequencies of small oscillations?

$$\text{Let } \theta = \theta_e + \eta \quad \ddot{\theta} = \ddot{\eta}$$

$$\ddot{\eta} = -\frac{g}{l} \sin(\theta_e + \eta) - \frac{a}{l} \cos(\theta_e + \eta)$$

$$\ddot{\eta} = -\frac{g}{l} [\sin\theta_e \cos\eta + \cos\theta_e \sin\eta] - \frac{a}{l} (\cos\theta_e \cos\eta - \sin\theta_e \sin\eta)$$

$$\cos(\eta) \sim 1 \quad \sin(\eta) \sim \eta$$

$$\ddot{\eta} = -\frac{g}{l} [\sin\theta_e + \eta \cos\theta_e] - \frac{a}{l} [\cos\theta_e - \eta \sin\theta_e]$$

$$\ddot{\eta} = -\frac{1}{l} \left[\underbrace{(g \sin\theta_e + a \cos\theta_e)}_0 + \eta (g \cos\theta_e - a \sin\theta_e) \right]$$

0

by definition

$$g \sin\theta_e + a \cos\theta_e = 0$$

$$\text{So } \ddot{\eta} = -\frac{1}{l} \eta (g \cos\theta_e - a \sin\theta_e)$$

$$\text{but } \theta_e = \tan^{-1} \left(\frac{-a}{g} \right)$$



$$\ddot{\eta} = -\frac{1}{l} \eta \left[\frac{g^2}{\sqrt{a^2 + g^2}} + \frac{a^2}{\sqrt{a^2 + g^2}} \right] = -\frac{1}{l} \eta \left[\frac{g^2 + a^2}{\sqrt{a^2 + g^2}} \right]$$

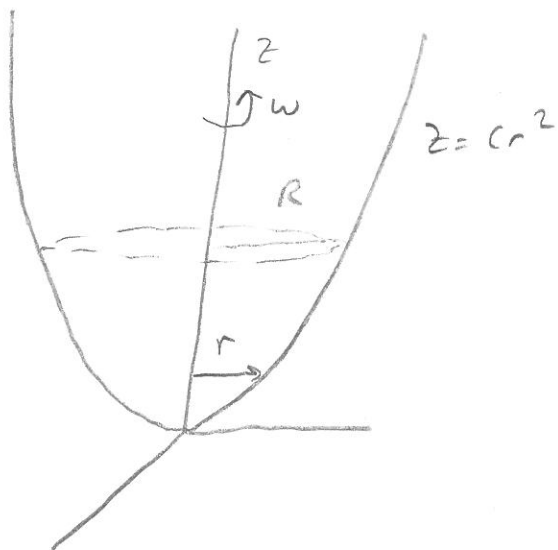
$$\ddot{\eta} = -\frac{\eta}{\ell} \left(\frac{g^2 r a^2}{\sqrt{a^2 r^2}} \right) = -\frac{\eta}{\ell} \frac{((g^2 r a^2)^{1/2})^2}{(g^2 r a^2)^{1/2}} = -\frac{\eta}{\ell} \sqrt{g^2 r a^2}$$

$$\ddot{\eta} = \eta \left[\underbrace{\frac{\sqrt{g^2 r a^2}}{\ell}}_{\omega^2} \right] \quad \omega = \frac{g}{\ell} \text{ For } \vec{a} = 0$$

Ex. Bead on a rotating wire

$z = cr^2$, $\omega = \text{rotation frequency}$, bead stays at $r = R$

For $c = ?$



$$d\vec{s} = dr\hat{r} + r d\theta\hat{\theta} + dz\hat{z}$$

$$T = \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right) \frac{1}{2} m$$

$\nwarrow \omega^2$

$$U = mgz$$

$$U = mgr^2$$

$$\dot{z} = 2cr\dot{r}$$

$$L = T - U = \frac{m}{2} \left(\dot{r}^2 + 4c^2 r^2 \dot{r}^2 + r^2 \omega^2 \right) - mgr^2$$

$$\frac{\partial \mathcal{L}}{\partial r} = 4mc^2 r \dot{r}^2 + m\omega^2 r - 2mgcr$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} + 4mc^2 r^2 \dot{r}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = m[\ddot{r} + 4c^2 r^2 \ddot{r} + 8c^2 r \dot{r}^2]$$

$$\ddot{r} + 4c^2 r^2 \ddot{r} + 8c^2 r \dot{r}^2 = 4c^2 r \dot{r}^2 + m\omega^2 r - 2mgcr$$

$$\ddot{r} [1 + 4c^2 r^2] + (4c^2 r) \dot{r}^2 + (\omega^2 - 2gc)r = 0$$

$$\text{If } r = \text{constant} \quad \ddot{r} = \dot{r}^2 = 0$$

$$(\omega^2 - 2gc)R = 0 \quad c = \frac{\omega^2}{2g}$$

Lagrange's Equations with Undetermined multipliers

Holonomic \rightarrow constraints that are algebraic relationships amongst the coordinates, integrable (solvable)

If constraints are of the form $F(x_i, \overset{\text{Velocity}}{\dot{x}_i}, t) = 0$ Non-Holonomic
Not generally integrable

Consider a constraint of the form $\sum_i A_i \dot{x}_i + B = 0$

$$\text{if } A_i = \frac{\partial F}{\partial x_i} \text{ and } B = \frac{\partial F}{\partial t}$$

$$\text{The } \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial t} = \frac{dF}{dt} = 0 \rightarrow \text{integrable}$$

$$\text{of } F(x, t) - C = 0 \quad \checkmark \quad \text{makes } F(x, \dot{x}, t) \rightarrow F(x, t) - C$$

$$\text{So constraints of } \sum_j \frac{\partial F_j}{\partial q_j} dq_j + \frac{\partial F}{\partial t} dt = 0$$

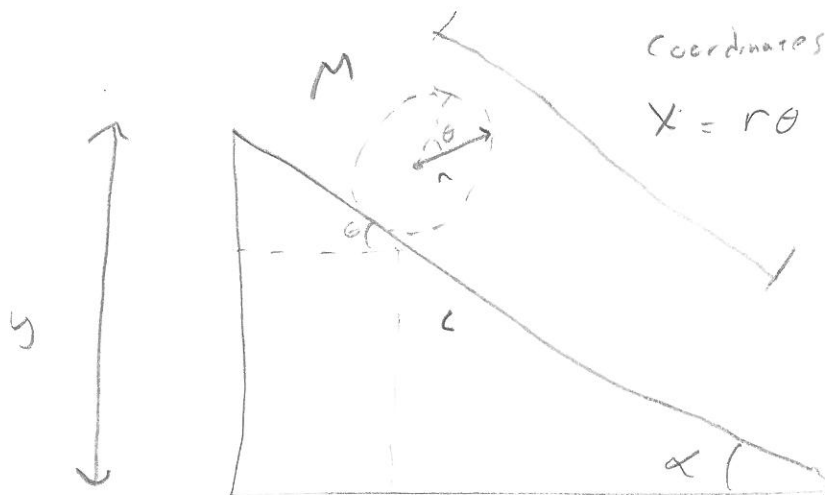
We can add this to $\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = 0$ without changing it

$$\text{as } \frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} + \sum_k \lambda_k(t) \frac{\partial F_k}{\partial q_j} = 0$$

Allows For the determination of Forces of constraint



$$Q_j = \sum_k \lambda_k \frac{\partial F_k}{\partial q_j} \quad k \rightarrow \perp \text{ eq's constraint}$$



Coordinates

$$x = r\theta \rightarrow x - r\theta = 0$$

$$d\vec{s} = dx \hat{i} \rightarrow T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 \quad u = mgy$$

$$u = mg(L-x) \sin \alpha$$

$$L = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 - mg(L-x) \sin \alpha$$

$$F(x, \theta) = x - r\theta = 0$$

$$\text{Now} \quad \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \lambda \frac{\partial F}{\partial x} = 0 \quad 1$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} + \lambda \frac{\partial F}{\partial \theta} = 0 \quad 2$$

$$mg \sin \alpha - M \ddot{x} + \lambda = 0$$

$$\text{Also } \ddot{x} = r \ddot{\theta} \text{ so } \frac{\ddot{x}}{r} = \ddot{\theta}$$

$$-\frac{I \ddot{\theta}}{2} - \lambda r = 0$$

$$\text{or } \frac{M r^2}{2} \frac{\ddot{x}}{r} = -\lambda r \rightarrow \lambda = -\frac{1}{2} M \ddot{x}$$

$$\text{For which } Mg \sin(\alpha) - \frac{5}{2} M \ddot{x} = 0$$

$$\boxed{\frac{2}{3} g \sin(\alpha) = \ddot{x}}$$

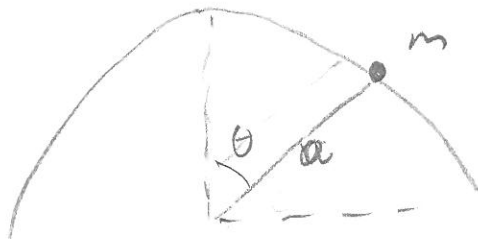
$$\text{and } \boxed{\lambda = -\frac{1}{3} Mg \sin(\alpha)}$$

$$\boxed{\ddot{\theta} = \frac{2g}{3r} \sin(\alpha)}$$

$$Q_x = \lambda \frac{\partial F}{\partial x} = \lambda = -\frac{Mg}{3} \sin(\alpha)$$

$$Q_\theta = \lambda \frac{\partial F}{\partial \theta} = -\lambda r = \frac{Mg r}{3} \sin(\alpha)$$

} rolling w/out slipping



$$F(r, \theta) = r = a \quad \frac{\partial F}{\partial r} = 1 \quad r = a \quad \dot{r} = 0 \quad \ddot{r} = 0$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) = \frac{1}{2} m r^2 \dot{\theta}^2$$

$$U = mgr \cos \theta$$

$$L = \frac{1}{2} m r^2 \dot{\theta}^2 - mgr \cos \theta$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0$$

$$mgr \sin \theta - mr^2 \ddot{\theta} = 0$$

$$mga \sin \theta - ma^2 \ddot{\theta} = 0$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} + \lambda = 0 \quad mr \dot{\theta}^2 - mg \cos \theta + \lambda = 0$$

$$ma \dot{\theta}^2 - mg \cos \theta + \lambda = 0$$

$$\ddot{\theta} = \frac{g}{a} \sin \theta \quad \ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta} \frac{d\theta}{dt} = \dot{\theta} \frac{d\dot{\theta}}{d\theta}$$

$$\dot{\theta} \frac{d\dot{\theta}}{d\theta} = \frac{g}{a} \sin \theta \rightarrow \left(\dot{\theta} d\dot{\theta} = \frac{g}{a} \sin \theta d\theta \right)$$

$$\frac{\dot{\theta}^2}{2} = -\frac{g}{a} \cos \theta + \frac{g}{a} \quad \text{and} \quad \lambda = mg \cos \theta - 2ma \left(\frac{g}{a} \right) [1 - \cos \theta]$$

$$\lambda = 3mg \cos \theta - 2mg = mg(3 \cos \theta - 2)$$

$$\frac{\partial F}{\partial r} = 1 \quad \lambda = \text{Normal Force}$$