

Vector Calculus

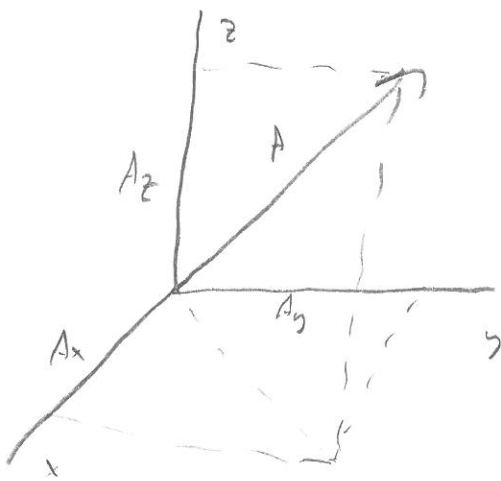
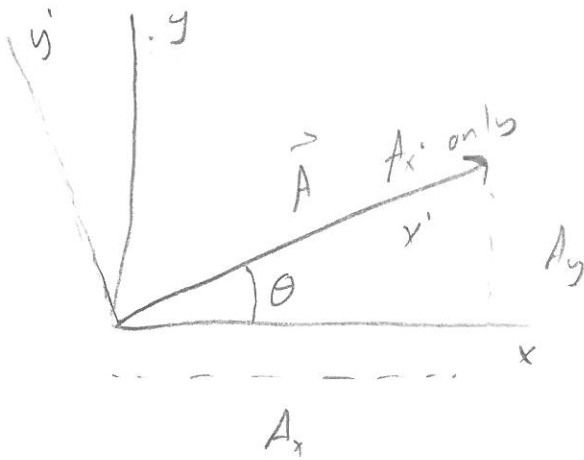
1. Scalar \rightarrow magnitude only

Invariants under coordinate transformations

2. Vector $\vec{A} = \sum_i x_i \hat{e}_i \rightarrow E_x (A_x, A_y, A_z)$

$\vec{r}, \vec{v}, \vec{a}, \vec{F}, \vec{p}$

depends on choice of coordinate system



Vector Algebra

#1. if $\vec{A} = \vec{B}$ then $A_x = B_x, A_y = B_y, A_z = B_z$

#2. $\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z)$

#3. $c\vec{A} = (cA_x, cA_y, cA_z)$

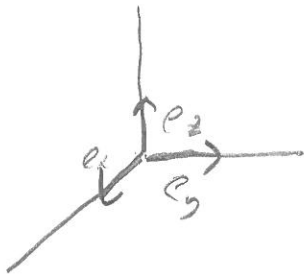
#4. $\vec{0} = (0, 0, 0)$

#5. $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

#6. $c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$

Unit Vectors

$$\hat{e}_x = (1, 0, 0) \quad \hat{e}_y = (0, 1, 0) \quad \hat{e}_z = (0, 0, 1)$$



Scalar Product

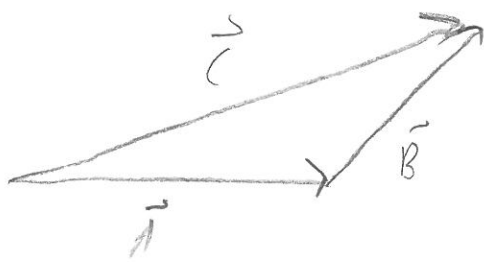
$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta = A_x B_x + A_y B_y + A_z B_z = (A_x, A_y, A_z) \cdot (B_x, B_y, B_z) = \sum_i A_i B_i$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{\vec{A} \cdot \vec{A}}$$

$$\rightarrow \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} = \cos \theta$$

USEFUL

$$(\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = |\vec{C}|^2$$



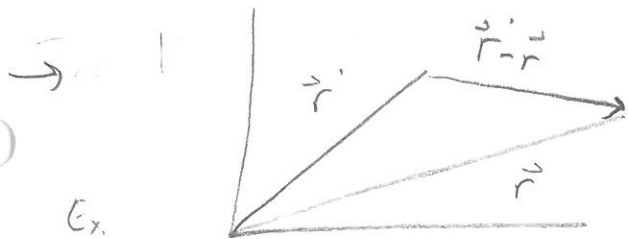
$$|\vec{A}|^2 + |\vec{B}|^2 + 2\vec{A} \cdot \vec{B} = |\vec{C}|^2$$

$$|\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta = |\vec{C}|^2$$

Pythagorean Theorem?

→ Work = $\int \vec{F} \cdot d\vec{r}$ → Component of force along direction of Travel

Ex.



$$|\vec{r}' - \vec{r}| = \sqrt{(\vec{r}' - \vec{r}) \cdot (\vec{r}' - \vec{r})}$$

Ex.

SCALAR PRODUCT

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \rightarrow \underline{\text{Show}}$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

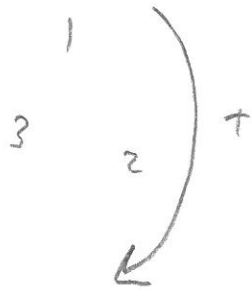
$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

USEFUL

$$\text{If } \vec{C} = \vec{A} \times \vec{B} \quad C_i = \sum_{j,k} \epsilon_{ijk} A_j B_k$$

$$\begin{aligned} \epsilon_{ijk} &= 0 \quad i=j, i=k, j=k \\ &= 1 \quad \epsilon_{123}, \epsilon_{231}, \epsilon_{312} \\ &= -1 \quad \epsilon_{132}, \epsilon_{213}, \epsilon_{321} \end{aligned}$$

Hint

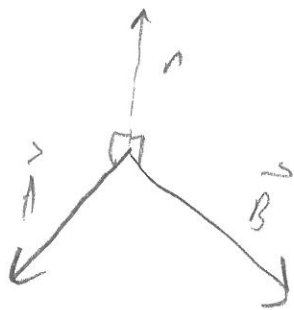


$$C_1 = \underset{+1}{\epsilon_{123}} A_2 B_3 + \underset{-1}{\epsilon_{132}} A_3 B_2 = A_2 B_3 - A_3 B_2$$

$$\begin{aligned} |\vec{A} \times \vec{B}|^2 &= (\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = (A_1^2 + A_2^2 + A_3^2)(B_1^2 + B_2^2 + B_3^2) - (A_1 B_2 + A_2 B_3 + A_3 B_1)^2 \\ &= \vec{A}^2 \vec{B}^2 - \vec{A}^2 \vec{B}^2 \cos^2 \theta \\ &= \vec{A}^2 \vec{B}^2 \sin^2 \theta \end{aligned}$$

So $|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta$

$$\vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta \hat{n}$$



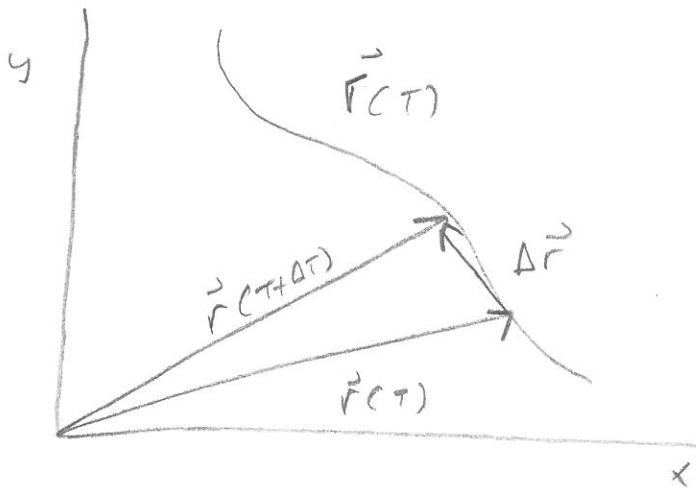
Finally

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

and

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Vector differentiation with respect to a scalar



$$\vec{r}(t) + \Delta \vec{r} = \vec{r}(t + \Delta t)$$

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \Rightarrow \frac{d\vec{r}(t)}{dt} = \frac{dx(t)}{dt} \vec{e}_x + \frac{dy(t)}{dt} \vec{e}_y$$

$$\frac{d}{dt} (\vec{A} + \vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt} (\vec{A} \cdot \vec{B}) = \frac{d\vec{A}}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt} (\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

$$\frac{d\phi \vec{A}}{dt} = \frac{d\phi}{dt} \vec{A} + \phi \frac{d\vec{A}}{dt} \rightarrow e_x T^3 (3t^2, 2t, 1)$$

Mechanics

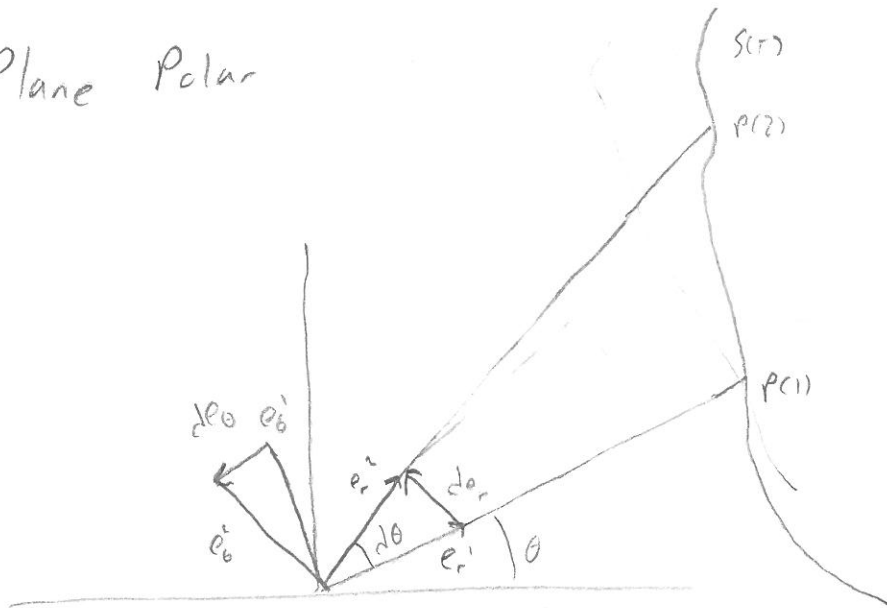
$$\vec{r} = \vec{r}(t) \quad \vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \quad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = \ddot{\vec{r}} = \dot{\vec{v}}$$

$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \dot{z}\hat{e}_z$$

$$\vec{a} = \ddot{x}\hat{e}_x + \ddot{y}\hat{e}_y + \ddot{z}\hat{e}_z$$

Plane Polar



$\hat{i}, \hat{j}, \hat{k} \rightarrow$ Static

$\hat{e}_r, \hat{e}_\theta$ NOT!!

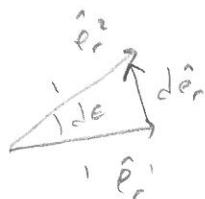
$$\hat{e}_r \perp \hat{e}_\theta$$

$$d\hat{e}_r = \hat{e}_r^2 - \hat{e}_r^1 \rightarrow \parallel \hat{e}_\theta \quad \text{magnitude } |d\theta|$$

$$\text{So } d\hat{e}_r = d\theta \hat{e}_\theta$$

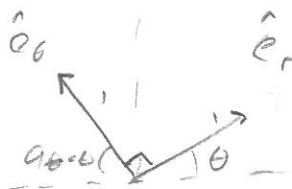
$$d\hat{e}_\theta = \hat{e}_\theta^2 - \hat{e}_\theta^1 \parallel -\hat{e}_r$$

$$\text{So } d\hat{e}_\theta = -d\theta \hat{e}_r$$



$$|d\hat{e}_r| = |d\theta|$$

or



$$\hat{e}_r = \cos\theta \hat{i} + \sin\theta \hat{j}$$

$$\hat{e}_\theta = -\sin\theta \hat{i} + \cos\theta \hat{j}$$

$$\hat{e}_\theta = -\sin\theta \hat{i} + \cos\theta \hat{j}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$d\hat{e}_r = -\sin\theta d\theta \hat{i} + \cos\theta d\theta \hat{j} = d\theta \hat{e}_\theta$$

$$d\hat{e}_\theta = -\cos\theta d\theta \hat{i} - \sin\theta d\theta \hat{j} = -d\theta \hat{e}_r$$

$$\dot{\hat{e}}_r = \frac{d\hat{e}_r}{dt} = \dot{\theta} \hat{e}_\theta, \quad \dot{\hat{e}}_\theta = \frac{d\hat{e}_\theta}{dt} = -\dot{\theta} \hat{e}_r$$

$$\text{Now } \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r = \underline{\underline{\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta}}$$

$$\dot{r}\hat{e}_r = \vec{v}_r, \quad r\dot{\theta}\hat{e}_\theta \rightarrow r\omega\hat{e}_\theta \rightarrow \text{circular motion spin velocity}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta]$$

$$\vec{a} = \underset{1}{\ddot{r}\hat{e}_r} + \underset{2}{\dot{r}\dot{\hat{e}}_r} + \underset{3}{\dot{r}\dot{\theta}\hat{e}_\theta} + \underset{4}{r\ddot{\theta}\hat{e}_\theta} + \underset{5}{r\dot{\theta}\dot{\hat{e}}_\theta}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

\nwarrow
 $\frac{v^2}{r} = \omega^2 r$
 Centripetal

\nwarrow
 Coriolis

\nwarrow $r\alpha \rightarrow$ Spin up Spin down

Cartesian

$$d\vec{s} = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z \quad \leftarrow \text{Line element}$$

$$d\vec{s} \cdot d\vec{s} = |d\vec{s}|^2 = dx^2 + dy^2 + dz^2$$

$$v^2 = \frac{|d\vec{s}|^2}{|dt|^2} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \dot{z}\hat{e}_z$$

Important

Cylindrical

$$d\vec{s} = dr\hat{e}_r + r d\theta\hat{e}_\theta + dz\hat{e}_z$$

$$|d\vec{s}|^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$v^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z$$

Spherical

$$d\vec{s} = dr\hat{e}_r + r d\theta\hat{e}_\theta + r \sin\theta d\phi\hat{e}_\phi$$

$$|d\vec{s}|^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$$v^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\phi}\hat{e}_\phi$$

Important Later

Gradient Operator

→ $\Phi = \Phi(x, y, z) \rightarrow$ Scalar Function

$$\nabla = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}$$

$$\nabla \Phi = \frac{\partial \Phi}{\partial x} \hat{e}_x + \frac{\partial \Phi}{\partial y} \hat{e}_y + \frac{\partial \Phi}{\partial z} \hat{e}_z \rightarrow \text{if there was a hill } \nabla \text{ would pt uphill}$$

$$\vec{\nabla} \cdot \vec{A} = \sum_i \hat{e}_i \frac{\partial}{\partial x_i} \cdot \vec{A}_i$$

Normal to Φ : constant
at point where change is maximum

$$\vec{\nabla} \cdot \vec{A} = \sum_i \frac{\partial \vec{A}_i}{\partial x_i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \rightarrow \text{How much "Flow" into or out of Area}$$

$$\vec{\nabla} \times \vec{A} = \sum_{j,k} \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} \hat{e}_i \quad \text{or} \quad \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Measure of
Rotation in Field

$$\nabla^2 \Phi = \vec{\nabla} \cdot \vec{\nabla} \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Ex: $d\phi \rightarrow$ Perfect differential

$$d\phi = \sum \frac{\partial \phi}{\partial x_i} dx_i = \sum (\nabla \phi)_i dx_i$$

$$\text{or } d\phi = (\nabla \phi) \cdot d\vec{s} \quad d\vec{s} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

Vector Integration

$$\text{Let } \vec{A} = A_1(x, y, z) \hat{e}_1 + A_2(x, y, z) \hat{e}_2 + A_3(x, y, z) \hat{e}_3$$

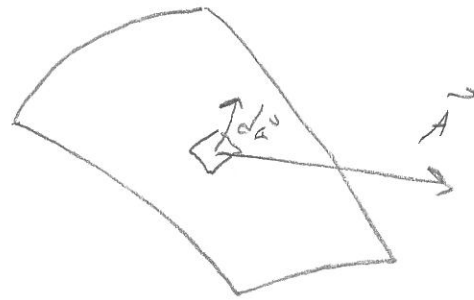
$$\text{Then } \int \vec{A} d\vec{r} = \int A_1 d\vec{r} + \int A_2 d\vec{r} + \int A_3 d\vec{r}$$

Surface Integral

$$\oiint \vec{A} \cdot d\vec{a} = \oiint \vec{A} \cdot \hat{n} da$$

$$\int_S \int_S$$

$$= \oiint [A_1 da_1 + A_2 da_2 + A_3 da_3]$$



\hat{n} is normal to surface (unit normal)

da , projection of vector component on surface normal

$$da_1 = dx_2 dx_3$$

$$\begin{array}{l} \uparrow d\vec{a} \quad \vec{A} \quad \vec{A} \cdot d\vec{a} = 0 \\ \uparrow d\vec{a} \quad \uparrow \vec{A} \quad \vec{A} \cdot d\vec{a} = 1 \\ \uparrow d\vec{a} \quad \downarrow \vec{A} \quad \vec{A} \cdot d\vec{a} = -1 \end{array}$$

|A|cosθ

$$\hat{E}_1 \quad \text{Let } \vec{r} = \vec{A}$$

FIND $\oint \vec{A} \cdot d\vec{a}$ over Sphere

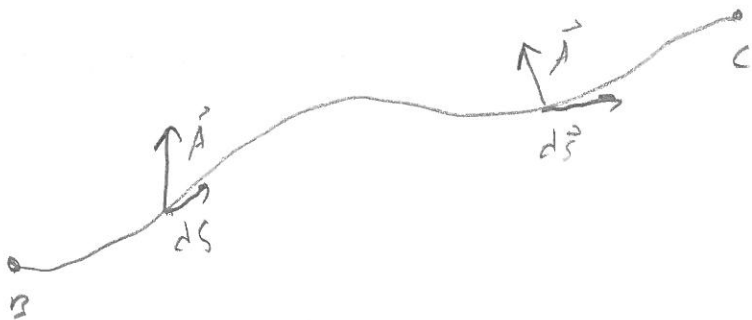
$$d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$\int_0^{2\pi} \int_0^{\pi} r^3 \sin\theta \, d\theta \, d\phi = R^3 \frac{4\pi}{3}$$

Fricky \rightarrow generally need to form \hat{N} via gradients
or cross products. Then need to parameterize
the surface, will use \oint primarily in cases
with high degrees of symmetry

Line integrals

$$\int_B^C \vec{A} \cdot d\vec{s} = \int_B^C (A_x dx + A_y dy + A_z dz)$$



Let $\vec{F} = -\nabla\phi \rightarrow$ work done is independent of path

$$\int_{P_1}^{P_2} \left[\frac{\partial\phi}{\partial x} \hat{e}_x + \frac{\partial\phi}{\partial y} \hat{e}_y + \frac{\partial\phi}{\partial z} \hat{e}_z \right] \cdot [dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z]$$

$$= - \int_{P_1}^{P_2} \left[\frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \right] = - \int_{P_1}^{P_2} d\phi = - [\phi(P_2) - \phi(P_1)]$$

and = 0 if $P_1 = P_2 \rightarrow$ closed path

Gauss Theorem

$$\iiint_V \nabla \cdot \vec{A} \, dV = \oiint_S \vec{A} \cdot d\vec{a}$$

↑
divergence

\rightarrow essentially what
is made or lost
in a volume
is how much "Flows"
in or out

Stokes Theorem

$$\oiint_S (\nabla \times \vec{A}) \cdot d\vec{a} \rightarrow \text{"Swirl on Surface"}$$

$$= \oint \vec{A} \cdot d\vec{s} \quad \text{Flowing along boundary}$$

Examples

Find the normal to the surface $z = x^2 + y^2$

→ if $F(x, y, z) = 0$ Then $\frac{\nabla F}{|\nabla F|} = \hat{n}$

$$F(x, y, z) = x^2 + y^2 - z \quad \nabla F = 2x\hat{i} + 2y\hat{j} - 1\hat{k}$$

$$|\nabla F| = \sqrt{\nabla F \cdot \nabla F} = \sqrt{4x^2 + 4y^2 + 1}$$

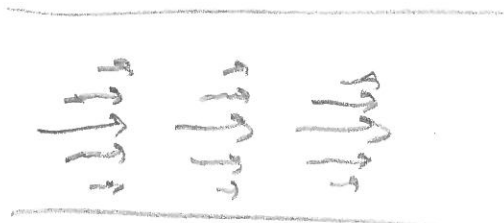
$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j} - \hat{k}}{\sqrt{4x^2 + 4y^2 + 1}} \quad \text{at } (1, -1, 2) \quad \frac{2\hat{i} - 2\hat{j} + 1\hat{k}}{\sqrt{3}}$$

Or let $r = x\hat{i} + y\hat{j} + (x^2 + y^2)\hat{k}$

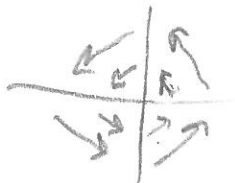
Then $\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y}$ yields same

Divergences and curls

$$\vec{F}_1 = (1 - y^2, 0)$$



$$\vec{F}_2 = (-y, x)$$



google rotation
vector field

Radial Field $\vec{F}_3 = (x, y, z)$

$$\nabla \cdot \vec{F}_3 = \underline{\underline{3}} \quad \nabla \times \vec{F}_3 ? \quad \frac{\partial(1-y^2)}{\partial z} e_1 - \frac{\partial(1-y^2)}{\partial y} e_2 = 2y e_2$$

Note $\nabla \times \nabla \phi = 0$ Always

and $\nabla \cdot \nabla \times \vec{F} = 0$ Always

Ex. Let $\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$

evaluate $\int \vec{A} \cdot d\vec{r}$ on the line joining $0,0,0$ and $1,1,1$

$$d\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad (\text{let } x=y=z=t \text{ then } t \in [0,1])$$

$$\vec{A} \cdot d\vec{r} = (3t^2 + 6t) - 14t^2 + 20t^3 = (t^3 - 11t^2 + 20t)$$

$$\int \vec{A} \cdot d\vec{r} = \int_0^1 (t^3 - 11t^2 + 20t) dt$$

Ex. Find the work done moving

a particle around the origin on a circle with $R=2$

and $\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$

In the $z=0$ plane

$$\vec{F} = (2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\vec{F} \cdot d\vec{r} = (2x-y)dx + (x+y)dy$$

Parameterize $x = 3\cos(t)$, $y = 3\sin(t)$

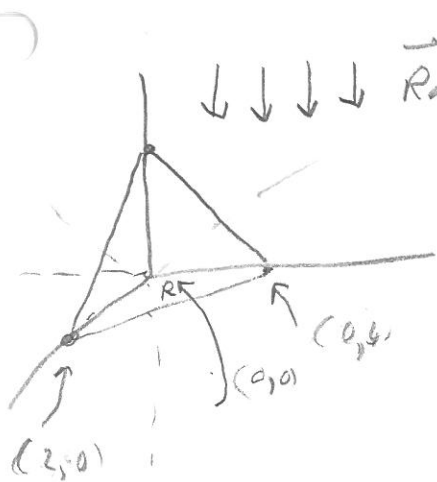
$$dx = -3\sin(t) dt \quad dy = 3\cos(t) dt$$

$$\oint_0^{2\pi} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [(6\cos(t) - 3\sin(t))(-3\sin(t)) + (3\cos(t) + 3\sin(t))3\cos(t)] dt$$

$$= \int_0^{2\pi} 9 - 9\sin(t)\cos(t) dt = 18\pi$$

I'll leave 2 examples of Stokes and Gauss's Theorems For HW

Surface integral \rightarrow rain on the roof



$$\vec{R}_{rain} = 0\hat{i} + 0\hat{j} - A\hat{k}$$

$$z = 4 - 2x - y$$

$$\vec{r} = u\hat{i} + v\hat{j} + (4 - 2u - v)\hat{k}$$

$$d\vec{r} = du\hat{i} + dv\hat{j} + (-2du - dv)\hat{k}$$

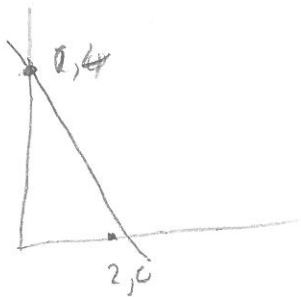
$$(\hat{i} - 2\hat{k})du + (\hat{j} - \hat{k})dv$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

only \hat{k} matters $\rightarrow 1\hat{k}$

but $\hat{n} = 2\hat{i} + \hat{j} + \hat{k}$
 $\downarrow \hat{n} = -2\hat{i} - \hat{j} - \hat{k}$

$$\iint \vec{R} \cdot d\vec{a} = \iint + A \, du \, dv$$



$$y = 2x + 4$$

$$y = 4 - 2x$$

$$\int_0^2 \int_0^{4-2x} A \, dy \, dx$$

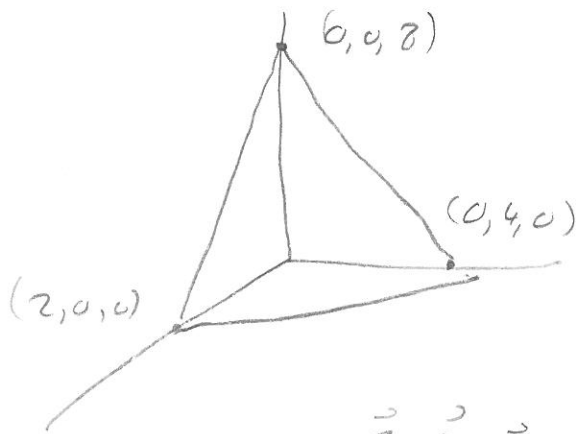
$$\int_0^2 [2x + 4] A \, dx \rightarrow [x^2 + 4x]_0^2 A$$

$$\underline{\underline{-4 + 8 = 4A}}$$

Same as no roof

What is $\int \vec{F} \cdot d\vec{r}$ For $\vec{F} = z\hat{i} - z\hat{j} + (x^2 - y^2)\hat{k}$ For boundary

of $z = 8 - 4x - 2y$



$$z - 8 + 4x + 2y = 0$$

$$+4\hat{i} + 2\hat{j} + \hat{k} = \vec{n}$$

$$\vec{\nabla} \times \vec{F} = (1-2y)\hat{i} + (1-2x)\hat{j}$$

$$\begin{aligned} \vec{\nabla} \times \vec{F} \cdot \vec{n} &= 4(1-2y) + 2(1-2x) = 4 + 2 - 4y - 8x \\ &= 6 - 4x - 8y \end{aligned}$$

$$\iint \vec{\nabla} \times \vec{F} \cdot \vec{n} \, dA = \int_0^2 \int_0^{4-2x} (6 - 4x - 8y) \, dx \, dy$$

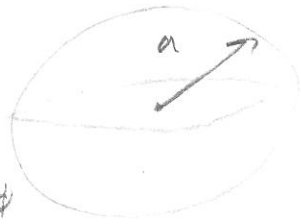
$$= -\frac{88}{3} \quad \text{or could do 3 separate line integrals}$$

$$\vec{F} = \frac{\vec{r}}{(\vec{r} \cdot \vec{r})^{3/2}} \quad ? \quad \frac{1}{r^2} = \frac{1}{\vec{r} \cdot \vec{r}} \quad , \quad \vec{r} = \frac{\vec{r}}{(\vec{r} \cdot \vec{r})^{1/2}}$$

$$\frac{1}{r^2} \vec{r} = \frac{\vec{r}}{(\vec{r} \cdot \vec{r})^{1/2} \cdot (\vec{r} \cdot \vec{r})^{2/2}} = \frac{\vec{r}}{(\vec{r} \cdot \vec{r})^{3/2}} = \frac{\vec{r}}{r^3}$$

$$\oint \vec{F} \cdot \vec{n} \, dA \quad ?$$

$$a^2 \sin \theta \, d\theta \, d\phi$$



$$\oint \left(\frac{\vec{r}}{r^3} \cdot \frac{\vec{r}}{r} \right) dA = \frac{\vec{r} \cdot \vec{r}}{r^4} = \frac{\vec{r} \cdot \vec{r}}{(\vec{r} \cdot \vec{r})^2}$$

$$= \frac{1}{\vec{r} \cdot \vec{r}} = \frac{1}{a^2}$$

$$= \frac{4\pi a^2}{a^2} = 4\pi$$

$$\vec{E}(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} \quad \oint \vec{E} \cdot \vec{n} \, dA = \frac{Q}{\epsilon_0}$$

$$\iiint \nabla \cdot \vec{E} \, dV = \frac{Q}{\epsilon_0}$$

$$Q = \iiint \rho(x, y, z) \, dx \, dy \, dz$$