Formalism PT #1, LINEAR ALGEBRA
Pt #1

Vectors $\rightarrow$ $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = |A\rangle_{3,1}$ Vector "ket"

row column

Arbitrary # Components $\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = |A\rangle_{n,1}$

Generalized Vector $|A\rangle \rightarrow |F\rangle$ Function can be treated in language of <u>Vector Spaces</u>

Rules

$+$

$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$ $\quad |\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$

$|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$

$|\alpha\rangle + |0\rangle = |\alpha\rangle$

$|\alpha\rangle + |-\alpha\rangle = |0\rangle$

$\times$

$a|\alpha\rangle = |\gamma\rangle$ $\quad a(|\alpha\rangle + |\beta\rangle) = a|\alpha\rangle + a|\beta\rangle$

$ab|\alpha\rangle = ba|\alpha\rangle$ $\quad 0|\alpha\rangle = |0\rangle$ $\quad \mathbb{1}|\alpha\rangle = |\alpha\rangle$

Linear independence and basis vectors

If every vector in a space can be written as a sum of n independent vectors these vectors span the space, are the basis of the space, and n is the dimension of the space.

Ex. $\hat{i}, \hat{j}, \hat{k}$ span 3D cartesian. All vectors

$$\vec{A} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$ → extend to n dimensions

$|e_1\rangle, |e_2\rangle, \ldots |e_n\rangle$ Then $|\alpha\rangle = \sum_i a_i |e_i\rangle$

Or, if each row is assumed to represent the e_i component

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad c|\alpha\rangle = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}$$

$$|\alpha\rangle + |\beta\rangle = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

PTG

Inner Products

$\langle\alpha|\alpha\rangle$? $\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2 = |\vec{A}|^2$

$= A_1^2 + A_2^2 + A_3^2 \ldots$ $\quad \langle\alpha|\beta\rangle = A_1 B_1 + A_2 B_2 + A_3 B_3 \ldots$

Properties $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$, $\langle\alpha|\alpha\rangle \geq 0$ $\quad \langle\alpha|\alpha\rangle = 0$ iff $|\alpha\rangle = |0\rangle$

$$\langle\alpha|\left[c_b|\beta\rangle + c_c|\gamma\rangle\right] = b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle$$

What if $|\alpha\rangle = \hat{i}$ $\quad |\beta\rangle = \hat{j} \rightarrow \langle\alpha|\beta\rangle = 0$

For $|\alpha_i\rangle \perp |\alpha_j\rangle$ $\quad \langle\alpha_i|\alpha_j\rangle = 0$

In general mutually orthogonal normalized vectors

$$\langle\alpha_i|\alpha_j\rangle = \delta_{ij} \rightarrow \text{look familiar?}$$

$$\int f^* g \, dx = \delta_{fg}$$

Hmm $\rightarrow$ $|\alpha\rangle = \sum a_i \hat{e}_i$ $\quad \langle \hat{e}_i|\alpha\rangle = a_i = a_i \sum \hat{e}_i \cdot \hat{e}_i = a_i$

$$\vec{A} = \underset{a_1\hat{e}_1}{3\hat{i}} + \underset{a_2\hat{e}_2}{4\hat{j}} + \underset{a_3\hat{e}_3}{5\hat{k}} \qquad a_2 = \hat{j}\cdot\vec{A} = \underbrace{3\,\hat{i}\cdot\hat{j}}_{0} + \underbrace{4\hat{j}\cdot\hat{j}}_{4} + \underbrace{5\,\hat{k}\cdot\hat{j}}_{0}$$

Hmm $\Psi(x,0) = \sum_n c_n \psi_n$ $\qquad c_n = \int \psi_n^* \Psi(x,0)\, dx$

$|\alpha\rangle = \sum a_i \hat{e}_i$ $\qquad c_n = \langle \psi_n^* | \Psi(x,0)\rangle$

Matrices

$\hat{T}$

$$T_{ij} \qquad \begin{array}{cc} & \xrightarrow{\hat{j}} \text{columns} \\ \text{rows } \hat{i}\downarrow & \begin{pmatrix} T_{11} & T_{21} & T_{31} & \cdots & T_{n1} \\ T_{12} & & & & \\ T_{13} & & & & \\ T_{14} & & & & \\ \vdots & & & & \\ T_{1n} & & & & \end{pmatrix} \end{array}$$

$$T_{ij}|\alpha\rangle = \sum_{j=1}^{n} a_j(\hat{T}|e_j\rangle) = \sum_{j=1}\sum_{i=1} a_j T_{ij}|e_i\rangle = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} T_{ij} a_j\right)|e_i\rangle$$

$$= \sum_{i=1} a_i'|e_i\rangle \rightarrow \text{Transforms } a_i \rightarrow a_i'$$

Matrices $\rightarrow$ Operators $\rightarrow$ just $\hat{Q}\psi = q\psi$

$\longleftrightarrow$ $\hat{T}|\alpha\rangle = |\alpha'\rangle$

We will see this concretely soon. RELAX

$$\hat{T}_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} & \cdots & T_{1j} \\ T_{21} & & & & \\ T_{31} & & & & \\ \vdots & & & & \\ T_{i1} & & & & T_{ij} \end{pmatrix}$$

$\hat{T}_{\text{row, column}}$

For orthonormal set of es $T_{ij} = \langle e_i | \hat{T} | e_j \rangle$

SEGUE

$\hat{A}\hat{B} = \hat{B}\hat{A}$? Only iff $[\hat{A},\hat{B}] = 0$

Things to be able to do

Matrices will now be able to be operators and vectors eigenfunctions.

MULTIPLY

$$\boxed{A_{ij} B_{jk} = C_{ik}}$$

For repeated indices → sum j $\quad A_{ij} B_{jk}$. For $i=3, j=3$

column row

column row

$$C_{11} = A_{1j} B_{j1} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} = C_{11}$$

$$C_{23} = A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33} = C_{23}$$

$$\begin{pmatrix} C_{11} & & \\ & & \\ & C_{23} & \end{pmatrix}$$

ith row, jth column

$$\hat{A} \qquad \hat{B}$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = C \begin{pmatrix} C_{11} & & \\ & & C_{23} \end{pmatrix}$$

Ex. $\quad \hat{A}_{22} = \begin{pmatrix} 4 & 2 \\ -3 & 1 \end{pmatrix} \qquad \hat{B}_{23} = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 7 & -4 \end{pmatrix}$

$$\hat{A}\hat{B} = \begin{pmatrix} 4+4 & 20+14 & 12-8 \\ -3+2 & -15+7 & -9-4 \end{pmatrix} = \hat{C}_{23} = \begin{pmatrix} 8 & 34 & 4 \\ -1 & -8 & -13 \end{pmatrix}$$

$\hat{B}\hat{A} = \hat{B}_{23} A_{22}$ no

Introduce $|\alpha\rangle \to$ KET $= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ $\langle\alpha| = \overline{a_1^* a_2^* \cdots a_n^*}$

Inner Product

$\langle\alpha|\alpha\rangle = \alpha_{1n}\alpha_{n1} = C_{11} = \overline{a_1^* a_2^* a_3^*} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ 3 For concreteness

$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$= a_1^* a_1 + a_2^* a_2 + \cdots a_n^* a_n = |\alpha|^2$

$|\alpha\rangle\langle\alpha|$?

$\alpha_{n1}\alpha_{1n} \to \hat{A}_{nn}$

$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \overline{a_1^* a_2^* a_3^*} = \begin{pmatrix} a_1a_1^* + a_1a_2^* + a_1a_3^* & , & , \\ & & \\ & & a_3a_1^* + a_3a_2^* + a_3a_3^* \end{pmatrix}$

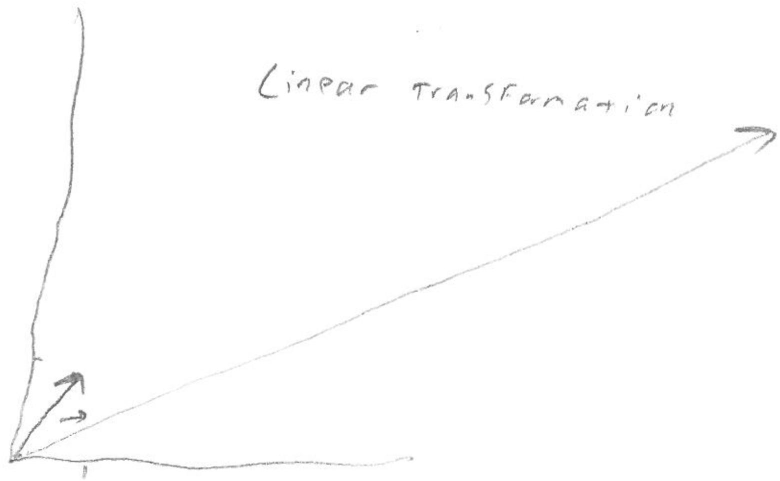
Matrix

Back to $\hat{T}|\alpha\rangle$ $\begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = A_{33}B_{31} = C_{31}$

$= \begin{pmatrix} 3 + 8 + 15 \\ 1 + 4 + 9 \\ 0 + 2 + 12 \end{pmatrix} = \begin{pmatrix} 27 \\ 14 \\ 14 \end{pmatrix}$

$\hat{T}|\alpha\rangle \to |\alpha'\rangle$ Transformed $\begin{pmatrix} 1\hat{e}_1 \\ 2e_2 \\ 3e_3 \end{pmatrix} \to \begin{pmatrix} 27\hat{e}_1 \\ 14\hat{e}_2 \\ 14e_3 \end{pmatrix}$

Look at first 2 dimensions



$$\hat{T}|\alpha\rangle \to \qquad a_i \to \sum_{j=1} \left(T_{ij}\, a_j \right) = a_i'$$

What if $\hat{T}|\alpha\rangle = c|\alpha\rangle \to \hat{T}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix} = 2\begin{pmatrix}1\\2\\3\end{pmatrix}$

$\hat{T}|\alpha\rangle = c|\alpha\rangle \to \underbrace{\hat{Q}\Psi = q\Psi}$

$|\alpha\rangle$ eigenvector of $\hat{T}$