$$k = \left(\frac{me^2}{4\pi\epsilon_0 \hbar^2}\right) \cdot \frac{1}{an} \rightarrow \quad a = \frac{4\pi\epsilon_0\hbar^2}{me^2} = .529 \cdot 10^{-10}\,m$$

Now $\rho = kr = \frac{r}{an}$

$$\psi_{nlm} \rightarrow R_{nl}(r)\, Y_l^m(\theta,\phi)$$

where $R_{nl}(r) = \frac{1}{r}u(\rho) = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho)$

$v(\rho) = P(\rho^n)$ where $c_{j+1} = \frac{2(j+1+l-n)}{(j+1)(j+2l+2)}c_j$

$$E_1 = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] = -13.6\,eV$$

$\psi_{100}(r,\theta,\phi)$ $\quad l=0, m=0, n=1$

$$c_1 = \frac{2(0+1+0-1)c_0}{(\ )(\ )} \rightarrow \quad c_1 = 0$$

$$R_{10}(r) = \frac{1}{r}\rho e^{-\rho}c_0 = \frac{c_0 e^{-\rho}}{a}$$

$$\int_0^\infty |R_{10}(r)|^2 r^2\, dr = \frac{c_0^2}{a^2}\int_0^\infty r^2 e^{-2r/a}\, dr = |c_0|^2\frac{a}{4} = 1 \qquad c_0 = \frac{2}{\sqrt{a}}$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \quad \rightarrow \psi_{1,0,0} = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

$n=2$ $\quad \ell=0 \quad m=0$ $\qquad E_2 = \dfrac{-13.6\,eV}{4} = -3.4\,eV$

$\ell=1 \quad m=\pm1, 0$

$c_1 = -c_0 \qquad c_2 = 0 \qquad v(\rho) = c_0(1-\rho) \qquad \ell=0$

$$R_{20}(r) = \frac{c_0}{2a}\left(1-\frac{r}{2a}\right)e^{-r/2a}$$

$$R_{21} = \frac{c_0}{4a^2}\, r e^{-r/2a} \qquad c_1 = 0 \qquad \ell=1$$

$v(\rho) =$ Associated Laguerre Polynomial

$$v(\rho) = L^{2\ell+1}_{n-\ell-1}(2\rho)$$

$$L^{p}_{q-p}(x) = (-1)^p\left(\frac{d}{dx}\right)^p L_q(x) \qquad \text{So } 2\ell+1 = p$$

$$q-2\ell+1 = n-\ell-1$$

$$L_q(x) = e^x\left(\frac{d}{dx}\right)^q (e^{-x}x^q) \qquad q = n-\ell+2\ell-1-1$$

$$q = n+\ell-2$$

$$\psi_{n\ell m} = \sqrt{\left(\frac{2}{na}\right)^3\frac{(n-\ell-1)!}{2n\,[(n+\ell)!]^3}}\; e^{-r/na}\left(\frac{2r}{na}\right)^{\ell} L^{2\ell+1}_{n-\ell-1}(2r/na)\, Y^m_\ell(\theta,\phi)$$

Note $\rightarrow$ $E \rightarrow E(n) = -\dfrac{13.6\,eV}{n^2}$ but there are $n-1$ ℓ's

Transition Energies

$$E_\gamma = E_i - E_F = -13.6\,eV\left(\frac{1}{n_i^2} - \frac{1}{n_F^2}\right) = \left(\frac{1}{n_F^2} - \frac{1}{n_i^2}\right) 13.6\,eV$$

$n_F < n_i$

$$\left[\left(\frac{1}{n_F^2}\right) - \left(\frac{1}{n_i^2}\right)\right] 13.6\,eV = h\nu = \frac{hc}{\lambda}$$

$$\frac{1}{\lambda} = \frac{13.6\,eV}{hc}\left[\frac{1}{n_F^2} - \frac{1}{n_i^2}\right] = 1.097 \cdot 10^7\,m^{-1}\left[\frac{1}{n_F^2} - \frac{1}{n_i^2}\right]$$

Lyman, Balmer, Paschen, brackett, pfund

$L\alpha$, $H\alpha$, $B\gamma$? → Segue Optical Depth

$\vec{L} = \vec{r} \times \vec{p}$ Angular Momentum

$L_x = yP_z - zP_y$ $\quad L_y = zP_x - xP_z$ $\quad L_z = xP_y - yP_x$

$P_x = -i\hbar\frac{\partial}{\partial x}$ $\quad P_y = -i\hbar\frac{\partial}{\partial y}$ $\quad P_z = -i\hbar\frac{\partial}{\partial z}$ → $[r_i, P_j] = i\hbar\delta_{ij}$, $[r_i, r_j] = [P_i, P_j] = 0$

ALGEBRA

$[L_x, L_y]$? $\rightarrow$ $[yP_z - zP_y, zP_x - xP_z] = \overset{1}{[yP_z, zP_x]} - \overset{2}{[yP_z, xP_z]}$

$- \overset{3}{[zP_y, zP_x]} + \overset{4}{[zP_y, xP_z]}$

Now $[AB, C] = A[B,C] + [A,C]B$

#2, $-\left[xyP_z^2 - xyP_z^2\right] \searrow 0$ $\quad xy = yx \quad \frac{\partial}{\partial z}(x) = \frac{\partial}{\partial z} y = 0$

#3, $-\left[-z^2 P_y P_x - z^2 P_y P_x\right] \searrow 0$ $\quad \frac{\partial}{\partial y}(z) = \frac{\partial}{\partial x}(z) = 0 \quad \frac{\partial^2}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial y}$

$[L_x, L_y] = [yP_z, zP_x] + [zP_y, xP_z]$

#1 $(yP_z zP_x - zP_x yP_z] = yP_x \underbrace{[P_z, z]}_{-i\hbar}$

#4 $(zP_y xP_z - xP_z zP_y) = xP_y \underbrace{[z, P_z]}_{i\hbar}$

$= i\hbar[xP_y - yP_x]$

$= i\hbar L_z$

$[L_y, L_z] = [zP_x - xP_z, xP_y - yP_x] = i\hbar L_x \cdots$

$[L_z, L_x] = i\hbar L_y$

Now $[L_i, L_j] = i\hbar L_k$

So $\sigma_{L_i}^2 \, \sigma_{L_j}^2 \geq \frac{\hbar^2}{4}$ NOT Commuting

$L^2 = L_x^2 + L_y^2 + L_z^2$

$[L^2, L_x]$? $= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x]$

$= \underbrace{L_x^3 - L_x^3}_{0} + L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$

$= L_y(-i\hbar L_z) - i\hbar L_z L_y + L_z(i\hbar L_y) + i\hbar L_y L_z$

$= -i\hbar\left[-L_y L_z - L_z L_y + L_z L_y + L_y L_z\right]$

$= 0$

Now $[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$

Pick one. CAN MEASURE L^2, L_i

So L^2, L_z

$[L^2, \vec{L}] = 0$

Eigenvalues

Let $L^2 F = \lambda F$ and $L_z F = \mu F$ $\quad [L^2, L_z] = 0$

$L_\pm = L_x \pm i L_y$

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y]$$

$$= i\hbar L_y \pm i(-i\hbar) L_x = i\hbar L_y \pm \hbar L_x$$

$$= \hbar(i L_y \pm L_x) = \pm\hbar(L_x \pm i L_y)$$

$$[L_z, L_\pm] = \pm\hbar L_\pm \rightarrow L_z L_\pm = L_\pm L_z \pm \hbar L_\pm$$

and $[L^2, L_\pm] = 0 \rightarrow L^2 L_\pm - L_\pm L^2 = 0$

Claim $\rightarrow$ F eigenfunction of L^2, L_z so is $[L_\pm F]$

#1. $L^2(L_\pm F) = L_\pm L^2 F = L_\pm \lambda F = \lambda[L_\pm F]$

#2. $L_z(L_\pm F) = (L_\pm L_z \pm \hbar L_\pm) F = (L_\pm \mu \pm \hbar L_\pm) F$

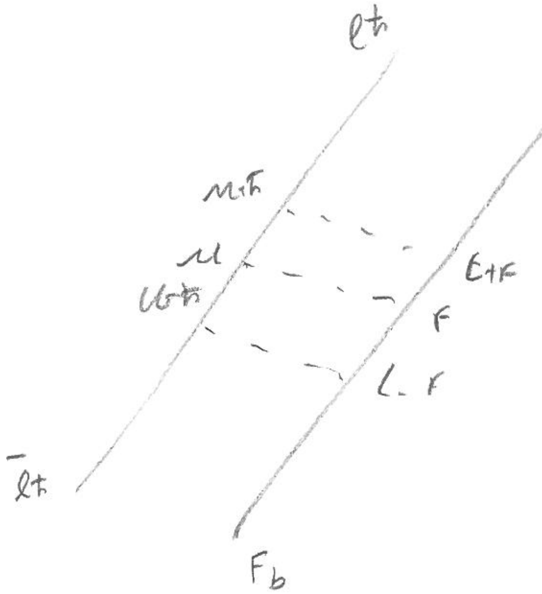
$$= (\mu \pm \hbar) L_\pm F$$

So $L^2(L_\pm f) = \lambda (L_\pm f)$

$L_z (L_\pm f) = (\mu \pm \hbar) L_\pm f$

$L_z f \to \mu \quad L_z(L_\pm f) = \mu \pm \hbar \quad \uparrow\hbar$ or $\downarrow\hbar$

$L_+ f_T = 0 \quad \uparrow$ end



Let

$L_z f_T = \hbar \ell f_T \qquad L^2 f_T = \lambda f_T$

Now $\quad L_\pm L_\mp = (L_x \pm i L_y)(L_x \mp i L_y) = L_x^2 + L_y^2 \mp i\underbrace{(L_x L_y - L_y L_x)}_{i\hbar L_z}$

$= L^2 - L_z^2 \mp (i)^2 \hbar L_z = L^2 - L_z^2 \pm \hbar L_z$

$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$

So $L^2 F_T = (L_- L_+ + L_z^2 + \hbar L_z) F_T$

Pick to kill F_T

$= (\hbar^2 l^2 + \hbar^2 l) F_T = \hbar^2 l(l+1) F_T$

$\lambda = \hbar^2 l(l+1) \rightarrow$ Eigenvalue L^2!!

$L^2 F_b = (L_+ L_- + L_z^2 - \hbar L_z) F_b = \hbar^2 \bar{l}^2 - \hbar^2 \bar{l}$

to kill F_b

$= \hbar^2 \bar{l}(\bar{l}-1) F_b = \lambda$

So $l(l+1) = \bar{l}(\bar{l}-1)$

either $\bar{l} = l+1$ No $\bar{l}$ bottom eigenvalue piece
l top

or $\bar{l} = -l$ ✓

$L^2 F_l^m \rightarrow \hbar^2 (l)(l+1) F_l^m$
$l = 0, 1/2, 1, 3/2 \ldots\ldots$

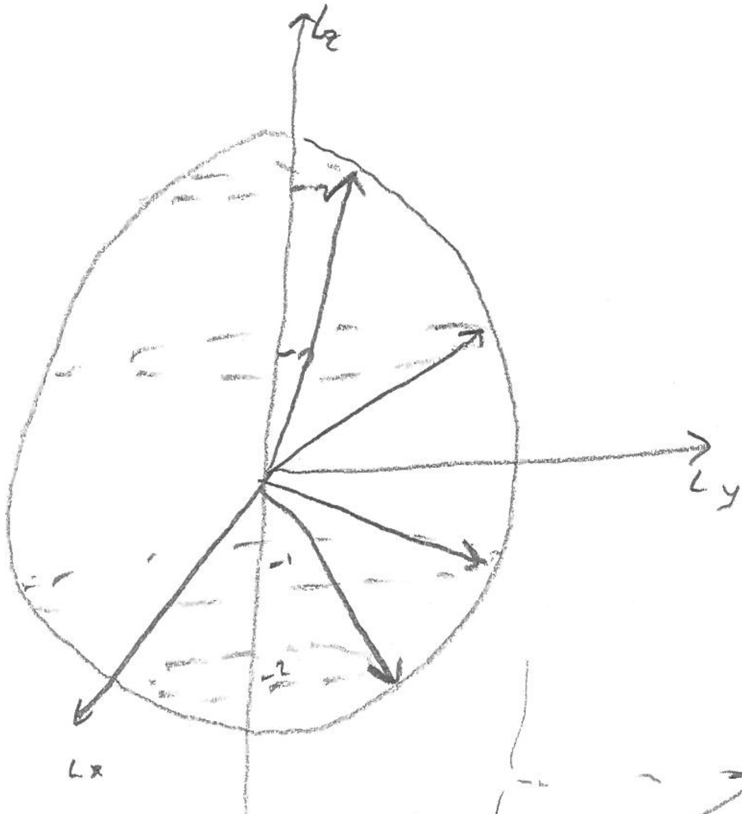
$L_z F_l^m = m \hbar F_l^m$

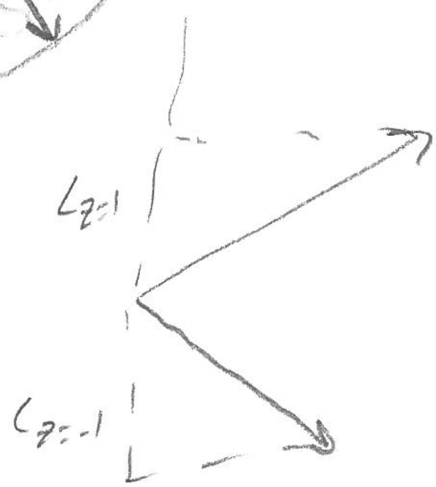
$m_l = -l, -l+1, \ldots 0, l-1, l$

$2l+1$ m_l

E_x

can't know
L_x and L_z
or L_z and L_y
or L_x and L_y





$|\hat{L}| = \sqrt{|\hat{L}^2|}$

$= \hbar\sqrt{\ell(\ell+1)}$

Now $F_\ell^m = Y_\ell^m$