Stellar Astrophysics

Lecture 4

Radiative Transfer

* Or, why do we see the spectra we see?

Opacities

* Processes that involve the destruction or creation of photons, the scattering of photons, and/or the transfer of energy between photons and electrons
* Scattering and absorption
* Or, how does the energy generated in the form of light change as it passes from the core to the photosphere of a star?

Opacities

* Bound - Bound
  * Photon absorbed
* Bound - Free
  * Photon absorbed
* Free - Free
  * Photon Absorbed
* Scattering
  * Photon 'bounces'

Atomic processes in a stellar atmosphere

* Matter Radiation interactions
  * Photoexcitation - a photon stimulates the excitation of a bound electron

$$X + h\nu \rightarrow X^*$$

  * Collisional de-excitation - a photon stimulates the excitation of an electron, a passing, free electron takes this energy and the bound electron drops back down. The free electron gains kinetic energy

$$X + h\nu \rightarrow X^* \rightarrow X^* + \frac{1}{2}m_e v^2 \rightarrow X + \frac{1}{2}m_e v'^2$$

* Collisional excitation - a free electron collides with an atom and excites a bound electron which then drops back down and releases a photon, the free electron looses energy

$$X + \frac{1}{2}m_e v^2 = X^* + \frac{1}{2}m_e v'^2 = X + h\nu + \frac{1}{2}m_e v'^2$$

* Where

$$\frac{1}{2}m_e v'^2 = \frac{1}{2}m_e v^2 - h\nu$$

* Photoionization - a photon with sufficient energy strikes an atom and releases an electron

$$X + h\nu = X^+ + \frac{1}{2}m_e v^2$$

* Recombination - A free electron is captured by an ionized atom and a photon is released

$$X^+ + \frac{1}{2}m_e v^2 = X + h\nu$$

* Collisional Ionization - A free electron collides with an atom and removes an electron

$$X + \frac{1}{2}m_e v^2 = X^+ + \frac{1}{2}m_e v'^2 + \frac{1}{2}m_e v''^2$$

There are more

* A collision with a free electron may stimulate a transition, slowing the free electron. This is followed by a second collision de-exciting the atom whereby the second electron gains kinetic energy. This transfers energy to the second electron from the first

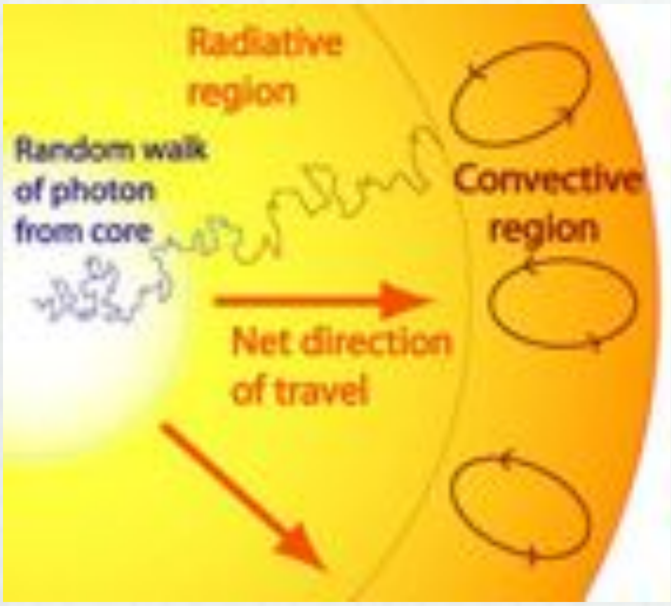
* A photon may excite a bound electron several levels which then jumps down multiple levels and releases multiple photons
* A free electron and photon may interact such that either the photon looses energy and gives it to the electron or vice versa (Compton and inverse Compton scattering)
    * More generally the photon is elastically scattered from an electron (Thomson Scattering)
* A free electron may be decelerated by interacting with an ionized atom causing it to slow down and release energy in the form of photons (Bremsstrahlung radiation)

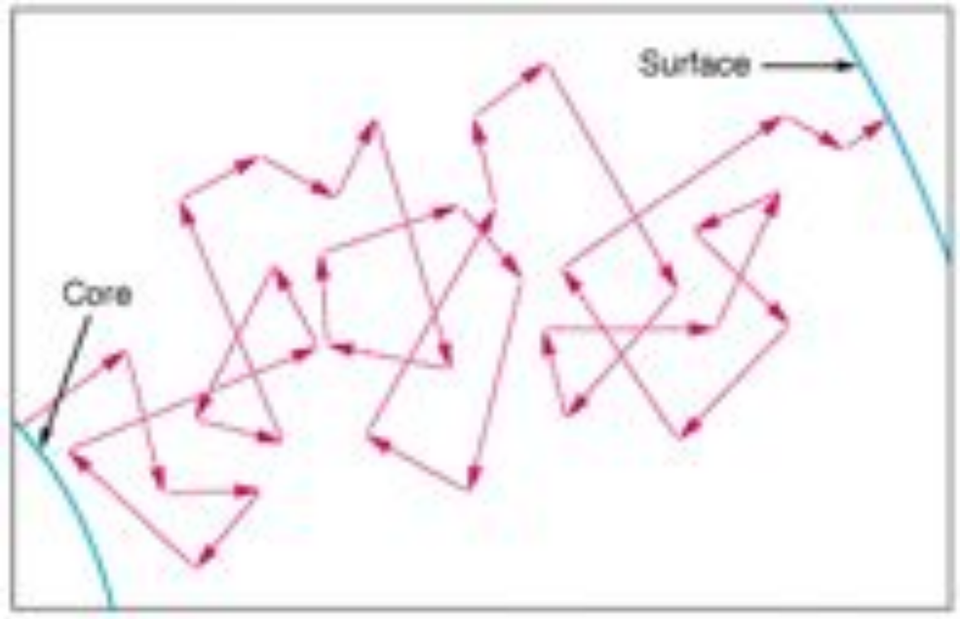
Opacity

- K_ν - cm^2/g for a given photon frequency
- ρK_ν - (number density absorbers/scatterers)x(Opacity) = α_ν = opacity per unit length. Your book uses χ_ν
- $K_\nu = k_\nu + \sigma_\nu$ = absorption plus scattering opacity For stars $\sigma_\nu = \sigma_e = n_e\sigma_t/\rho$ where $\sigma_t = 6.6524 \times 10^{-25}$ cm^2, $\sigma_\nu = .4 cm^2\ g^{-1}$
  - σ_t = Thomson scattering cross section, area within which a photon will scatter elastically off an electron, energy independent, the Thomson cross section has units of inverse area

Random Walks

* What happens to a photon trying to get out of a star?





Stellar Model

* We'll assume a star is a blackbody emitter with a temperature T_1 different than that of the upper atmosphere at T_2
* Photons random walk from the core and then participate in opacity interactions beneath the photosphere
* Opacity degrades photon energy from gamma ray range in core to visible in photosphere
* Momentum is transferred to particles via radiation pressure
* This is what supports the star against gravitational collapse

Random Walk

* As photons diffuse from the core they traverse a total distance d with a series of steps given by path length l
* At each step they scatter or are absorbed and re-emitted and their energy may degrade (does on the large scale)
* The length l is determined by the mean free path or distance they travel before interacting due to opacity. l is given by $1/(n\sigma)$ where σ is the effective cross section for interaction, not the same as opacity, be careful, two conventions are used
* $n\sigma$ and ρK_ν both have units of inverse length

* |d| is the magnitude of the total radial path traveled in terms of little steps of length l

$$\vec{d}\cdot\vec{d} = \vec{l}_1\cdot\vec{l}_1 + \vec{l}_1\cdot\vec{l}_2 + \ldots + \vec{l}_1\cdot\vec{l}_n$$

$$+\vec{l}_2\cdot\vec{l}_1 + \vec{l}_2\cdot\vec{l}_2 + \ldots + \vec{l}_2\cdot\vec{l}_n + \ldots$$

$$+\vec{l}_n\cdot\vec{l}_1 + \vec{l}_n\cdot\vec{l}_2 + \ldots + \vec{l}_n\cdot\vec{l}_n$$

* $$= \sum_{i=1}^{N}\sum_{j=1}^{N}\vec{l}_i\cdot\vec{l}_j = Nl^2 + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} l_i l_j cos(\theta_{ij})$$

* Where the Cosine term's sum approaches zero for N >> 1

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- giving $d = l\sqrt{N} = \tau l$
- or $N = \tau^2$
- where $\tau = \int K_\nu \rho ds = \int n\sigma ds$
- Tau is the optical depth, more on this later
- Generically, to travel d = 10l need 100 steps, d = 100l need 10000 steps, etc.
- Need 10,000 to 100,000 years for a gamma ray photon to escape.

Solid Angles

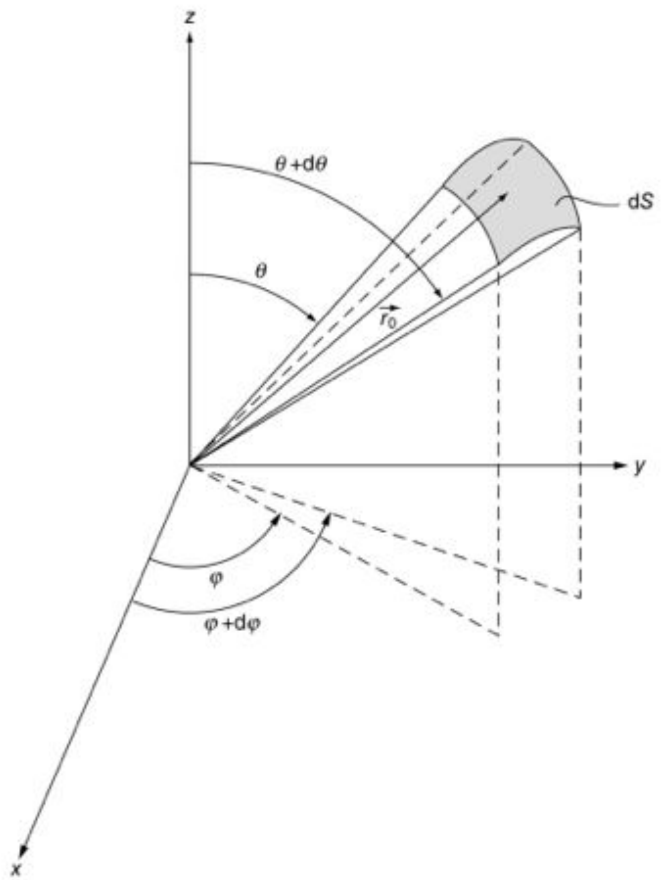
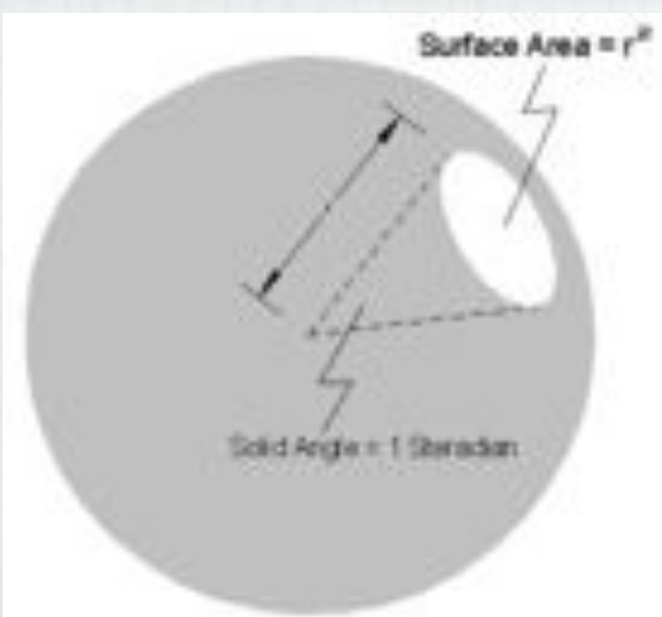
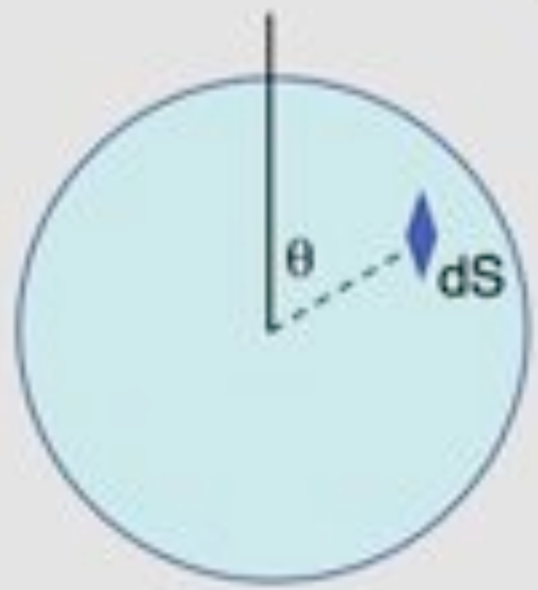


Figure 3.4 Illustration showing the solid angle subtended between the angles θ and $\theta + d\theta$, and φ and $\varphi + d\varphi$.



How does the intensity of radiation change in the presence of emission and / or absorption?

Definition of solid angle and steradian



Sphere radius r - area of a patch dS on the surface is:

$$dS = r d\theta \times r \sin\theta d\phi = r^2 d\Omega$$

dΩ is the solid angle subtended by the area dS at the center of the sphere.

Unit of solid angle is the steradian.

4π steradians cover whole sphere.

Specific Intensity

* I_ν= Energy emanating from surface area dA per unit time in the spectral range ν+dν traveling between solid angles Ω+dΩ
* Units ers/s/Hz/cm^2/steradian

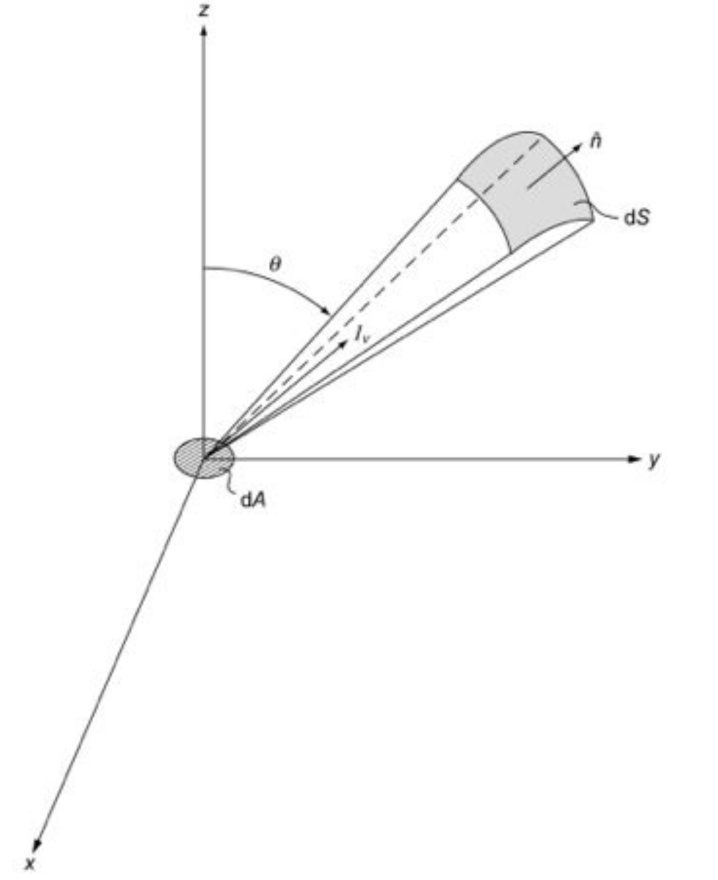
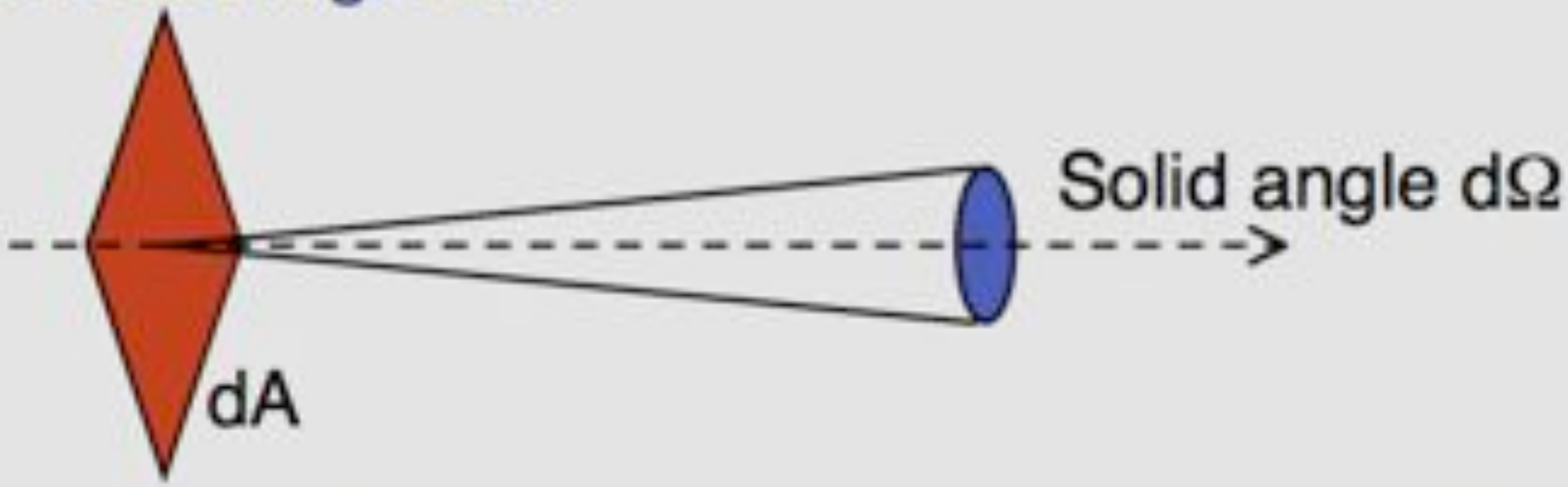


Figure 3.5 Illustration defining the specific intensity I_ν.

Definition of the specific intensity

Construct an area dA normal to a light ray, and consider all the rays that pass through dA whose directions lie within a small solid angle dΩ.



The amount of energy passing through dA and into dΩ in time dt in frequency range dν is:

$$dE = I_\nu dA dt d\nu d\Omega$$

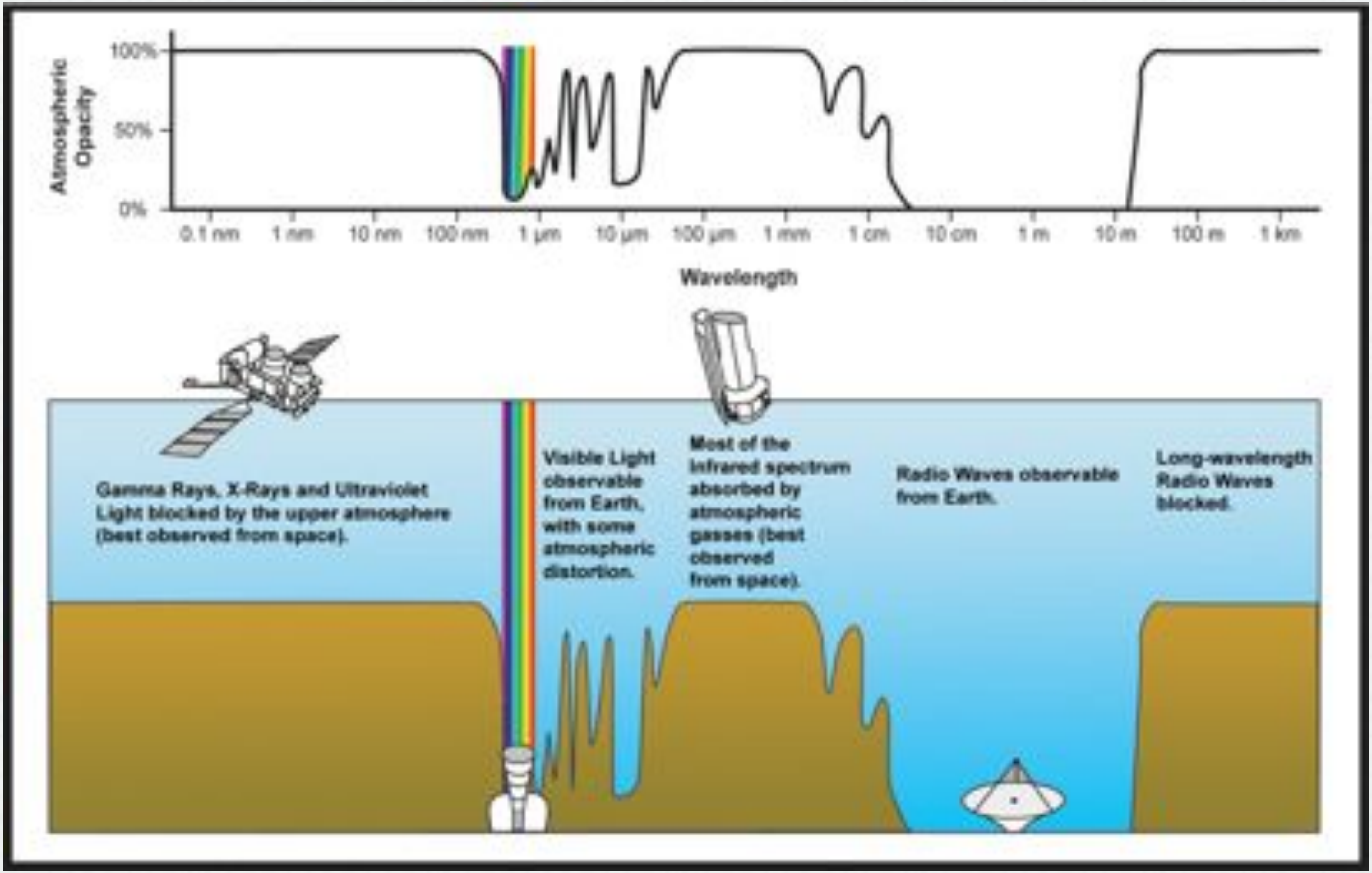
↑

Specific intensity of the radiation.

LTE

* Assumes the mean free path of photon interactions is much smaller than scale on which the temperature changes
* Sometimes assume the source function equals a blackbody spectrum, more stringently the specific intensity is a blackbody
* Fails when radiative processes become important and density is low - NLTE
* In LTE Saha, Boltzman, and Maxwell Boltzman hold
* Fails in upper photosphere and above
* For now we'll pretend like things reduce to blackbodies at different temperatures

Side Note



Atmospheric Opacity
100%
50%
0%
0.1 nm
1 nm
10 nm
100 nm
1 µm
10 µm
100 µm
1 mm
1 cm
10 cm
1 m
10 m
100 m
1 km
Wavelength
Gamma Rays, X-Rays and Ultraviolet Light blocked by the upper atmosphere (best observed from space).
Visible Light observable from Earth, with some atmospheric distortion.
Most of the infrared spectrum absorbed by atmospheric gasses (best observed from space).
Radio Waves observable from Earth.
Long-wavelength Radio Waves blocked.

How does specific intensity change along a ray

If there is no emission or absorption, specific intensity is just **constant** along the path of a light ray. Consider any two points along a ray, and construct areas dA_1 and dA_2 normal to the ray at those points. How much energy is carried by those rays that pass through **both** dA_1 and dA_2?

$$dE_1 = I_{\nu 1} dA_1 dt d\nu_1 d\Omega_1$$
$$dE_2 = I_{\nu 2} dA_2 dt d\nu_2 d\Omega_2$$

where $d\Omega_1$ is the solid angle subtended by dA_2 at dA_1 etc

The same photons pass through both dA_1 and dA_2, without change in their frequency. Conservation of energy gives:

$$dE_1 = dE_2 \quad \text{- equal energy}$$

$$d\nu_1 = d\nu_2 \quad \text{- same frequency interval}$$

Using definition of solid angle, if dA_1 is separated from dA_2 by distance r:

$$d\Omega_1 = \frac{dA_2}{r^2}, \; d\Omega_2 = \frac{dA_1}{r^2}$$

Substitute:

$$I_{\nu 1} dA_1 dt d\nu_1 d\Omega_1 = I_{\nu 2} dA_2 dt d\nu_2 d\Omega_2 \qquad dE_1=dE_2$$

$$I_{\nu 1} dA_1 dt d\nu_1 \frac{dA_2}{r^2} = I_{\nu 2} dA_2 dt d\nu_2 \frac{dA_1}{r^2} \qquad d\nu_1=d\nu_2$$

$$I_{\nu 1} = I_{\nu 2}$$

Conclude: specific intensity remains the same as radiation propagates through free space.

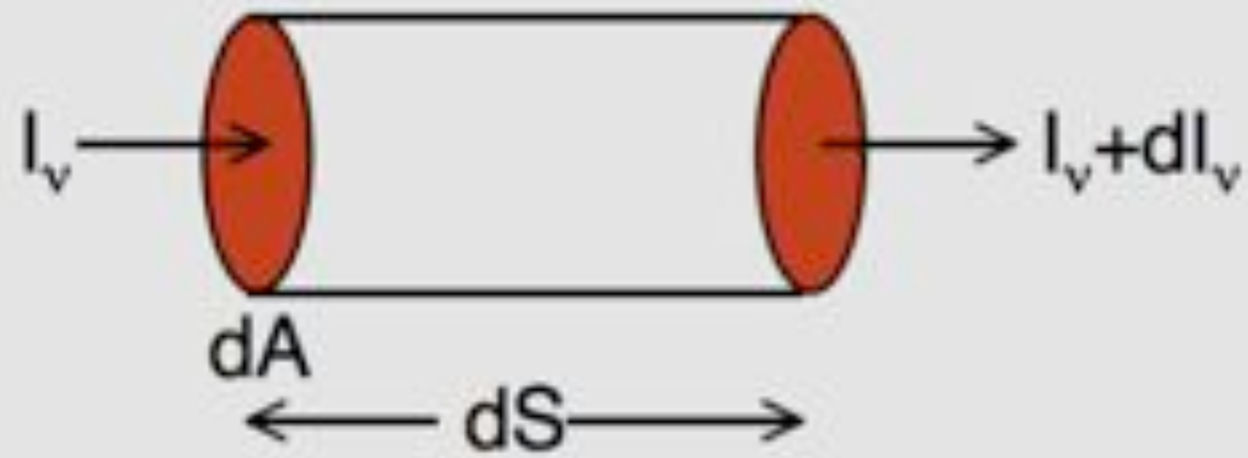
Justifies use of alternative term `brightness' - e.g. brightness of the disk of a star remains same no matter the distance - **flux** goes down but this is compensated by the light coming from a smaller area.

If we measure the distance along a ray by variable s, can express result equivalently in differential form:

$$\frac{dI_\nu}{ds} = 0$$

Emission

If the radiation travels through a medium which is itself emitting radiation, that will add to the energy:



Spontaneous **emission coefficient** is the amount of energy emitted per unit time, per unit solid angle, per unit frequency interval, and per unit volume:

$$dE = j_\nu dV d\Omega dt d\nu$$

In going a distance ds, beam of cross-section dA travels through a volume dV = dA x ds.

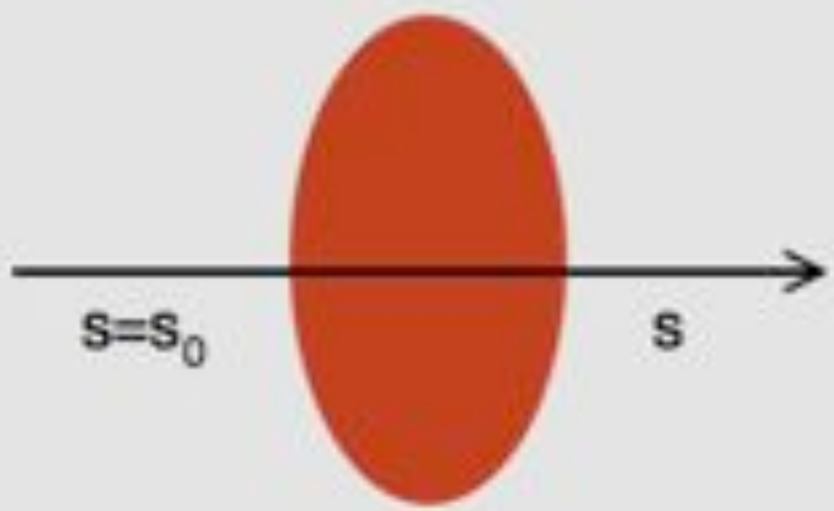
Change (increase) in specific intensity is therefore:

$$dI_\nu = j_\nu ds$$

Equation of radiative transfer for pure emission becomes:

$$\frac{dI_\nu}{ds} = j_\nu$$

If we know what j_ν is, can integrate this equation to find the change in specific intensity as radiation propagates through the gas:

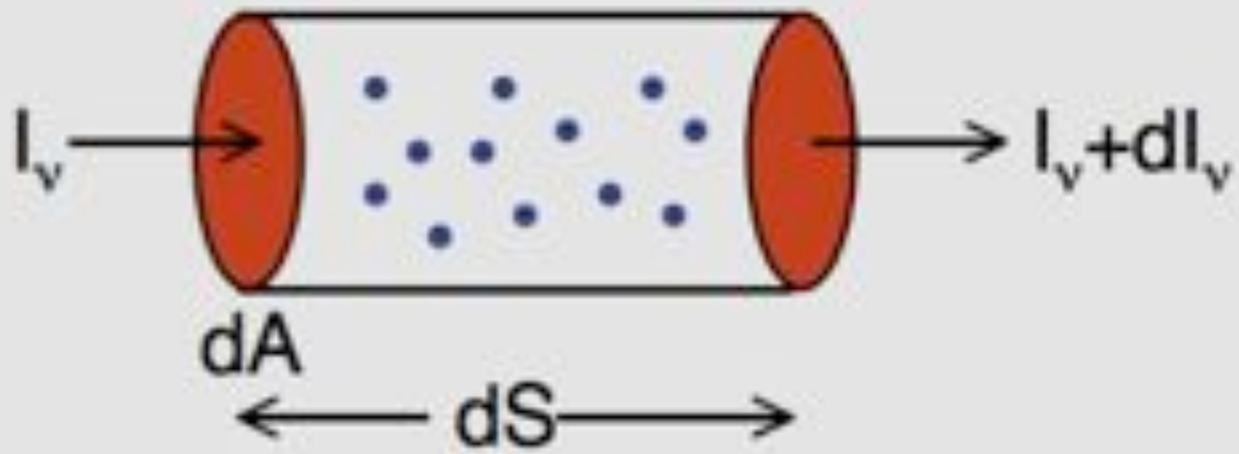


$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')ds'$$

i.e. add up the contributions to the emission all along the path.

Absorption

If the radiation travels through a medium which absorbs (or scatters) radiation, the energy in the beam will be reduced:



- Mass density of absorbers/scatterers per unit volume = ρ
- Each absorber/scatterer has cross sectional area K_ν (cm^2/g)
- $\rho K_\nu dA ds$ is essentially the amount of stuff per until length photons can expect to encounter multiplied by the differential surface area through which they pass

* Fraction of area absorbed = fraction of area blocked
* or cm2g-1 of stuff times g/cm3
* Minus because this is a loss term

$$\frac{dI_\nu}{I_\nu} = -\frac{\rho K_\nu I_\nu dA ds}{dA} = -\rho K_\nu ds$$

$$dI_\nu = -\rho K_\nu I_\nu ds = -\alpha_\nu I_\nu ds$$

Notice, I use big K instead of little k, big K is absorption plus scattering or cm^2/g

don't forget α is in units of 1/cm or opacity per unit length

Equation of radiative transfer for pure absorption. Rearrange previous equation:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

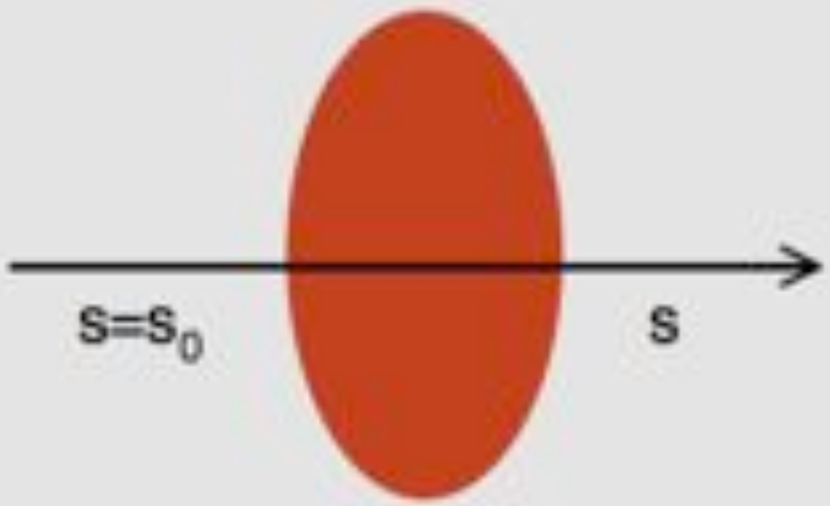
Different from emission because depends on how much radiation we already have.

Integrate to find how radiation changes along path:

$$\int_{s_0}^{s} \frac{dI_\nu}{I_\nu} = -\int_{s_0}^{s} \alpha_\nu(s')ds'$$

$$\left[\ln I_\nu\right]_{s_0}^{s} = -\int_{s_0}^{s} \alpha_\nu(s')ds'$$

$$I_\nu(s) = I_\nu(s_0) e^{-\int_{s_0}^{s} \alpha_\nu(s')ds'}$$



e.g. if the absorption coefficient is a constant (example, a uniform density gas of ionized hydrogen):

$$I_\nu(\Delta s) = I_0 e^{-\alpha_\nu \Delta s}$$

Specific intensity after distance Δs

Initial intensity

Radiation exponentially absorbed with distance

Radiative transfer equation with both absorption and emission:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

absorption

emission

Optical depth

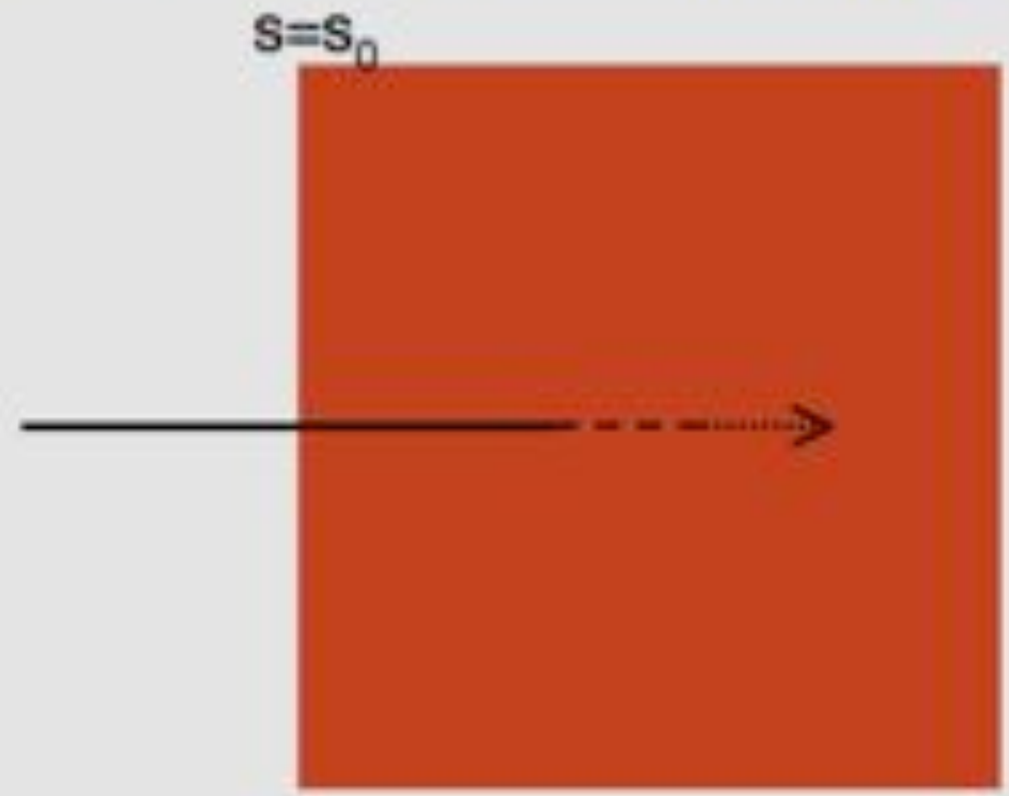
Look again at general solution for pure absorption:

$$I_\nu(s) = I_\nu(s_0)e^{-\int_{s_0}^{s} \alpha_\nu(s')ds'}$$

Imagine radiation traveling into a cloud of absorbing gas, exponential defines a scale over which radiation is attenuated.

When: $\int_{s_0}^{s} \alpha_\nu(s')ds' = 1$

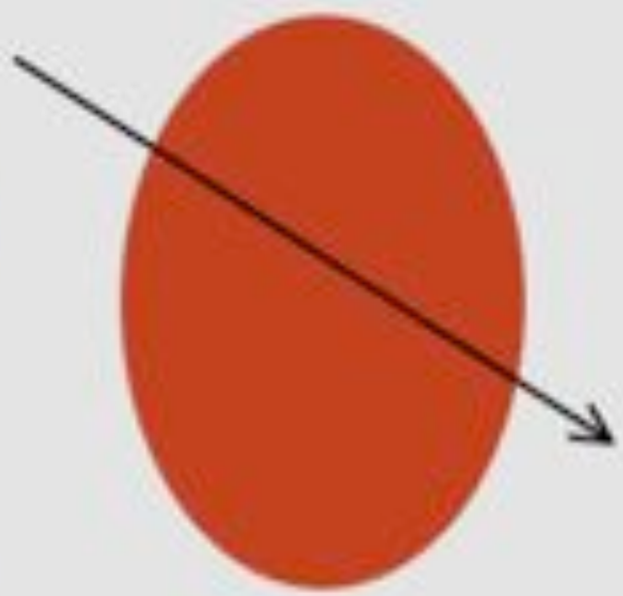
…intensity will be reduced to 1/e of its original value.



Define **optical depth** τ as:

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')ds'$$

or equivalently $d\tau_\nu = \alpha_\nu ds$



A medium is **optically thick** at a frequency ν if the optical depth for a typical path through the medium satisfies:

$$\tau_\nu \geq 1$$

Medium is said to be **optically thin** if instead:

$$\tau_\nu < 1$$

Interpretation: an optically thin medium is one which a typical photon of frequency ν can pass through without being absorbed.

General Equation of Time Independent Radiative

* Divide by α $$\frac{1}{-\alpha_\nu}\frac{dI_\nu}{ds} = I_\nu - \frac{j_\nu}{K_\nu}$$
* Define the source function $$S_\nu = \frac{j_\nu}{K_\nu}$$

  essentially the ratio of emission to absorption

* Recall $$-\alpha_\nu ds = -K_\nu \rho ds = d\tau_\nu$$

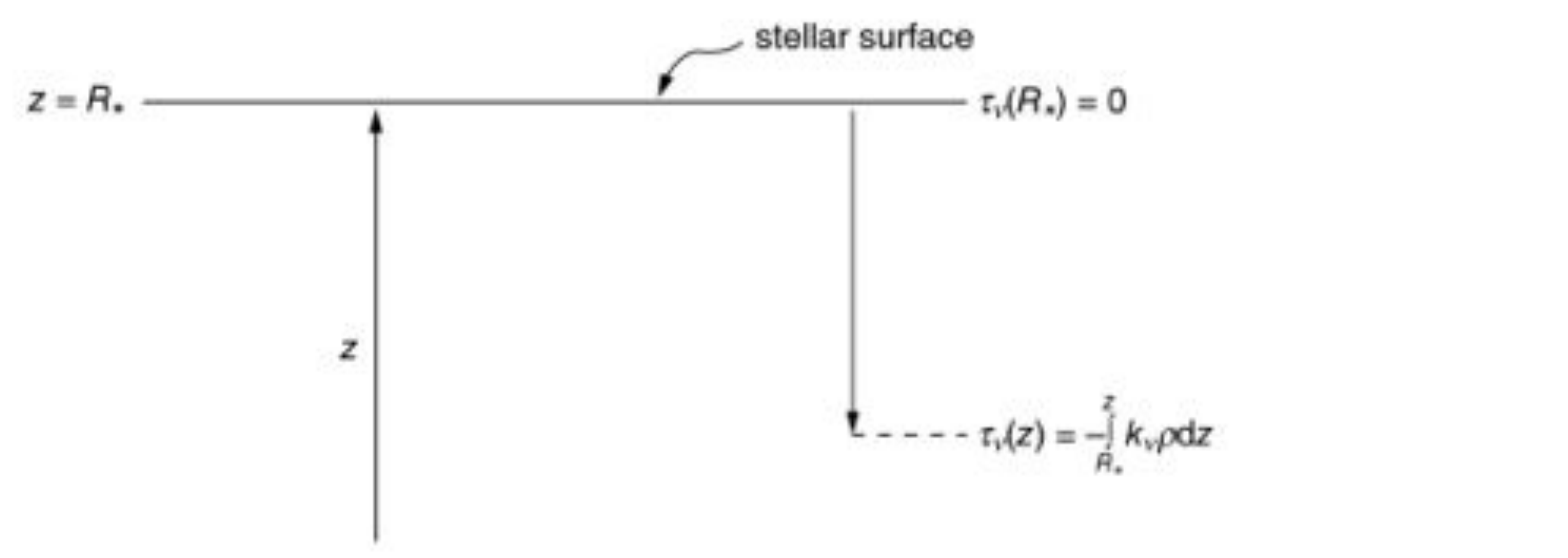


Figure 3.10 Illustration of the concept of optical depth with respect to the stellar surface. The integration to obtain $\tau_\nu(z)$ is performed from the surface to the depth z.

Source Function

* Emissivity is the absorption opacity times the Planck function + scattering opacity times the average intensity (to be defined later)

$$j_\nu = k_\nu B_\nu + \sigma_\nu J_\nu$$

* The source function is then

$$\frac{k_\nu B_\nu + \sigma_\nu J_\nu}{k_\nu + \sigma_\nu}$$

* Essentially the attenuated thermal contribution plus the scattered radiation

A more physical demonstration of the equation of radiative transfer

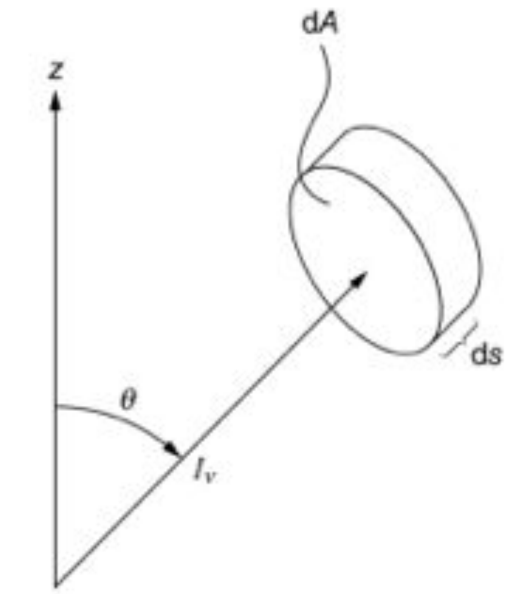


Figure 3.18 Illustration of a beam of radiation defined by the specific intensity I_ν arriving perpendicular to the surface dA of a mass element of volume dAds.

Energy removed passing through a slab with absorption/scattering

$$dE_\nu = -K_\nu \rho I_\nu d\Omega d\nu dt dA ds$$

Energy added passing through a slab with emission

$$dE_\nu = j_\nu \rho d\Omega d\nu dt dA ds$$

Variation of specific intensity is therefore

$$dI_\nu d\Omega d\nu dt dA = j_\nu \rho d\Omega d\nu dt dA ds - K_\nu \rho I_\nu d\Omega d\nu dt dA ds$$

Which leads to

$$\frac{\mu}{\rho}\frac{dI_\nu}{dz} = -K_\nu I_\nu + j_\nu$$

* Where dz =υds
* Exactly (almost) what I got before
* Divide though by -Kν
* υ? $$\mu\frac{dI_\nu(\tau_\nu,\mu)}{d\tau_\nu} = I_\nu(\tau_\nu,\mu) - S_\nu(\tau_\nu)$$
* Let's do 3.7

Attenuation Depends on Path travelled

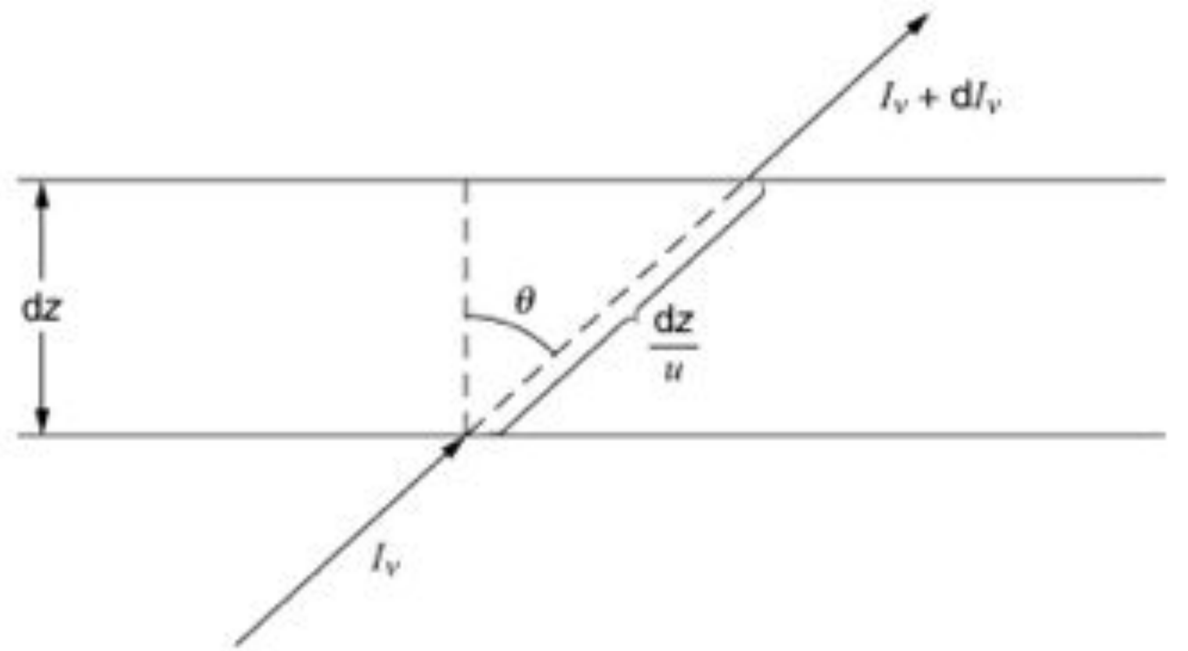


Figure 3.8 Illustration of a beam of photons characterized by I_ν that crosses a slab of plasma of thickness dz at an angle θ and emerges as $I_\nu + dI_\nu$. The distance travelled through the slab is dz/u.

Path traversed is $dz/\cos(\theta)$

define $\mu = \cos(\theta)$

Plane Parallel Approximation

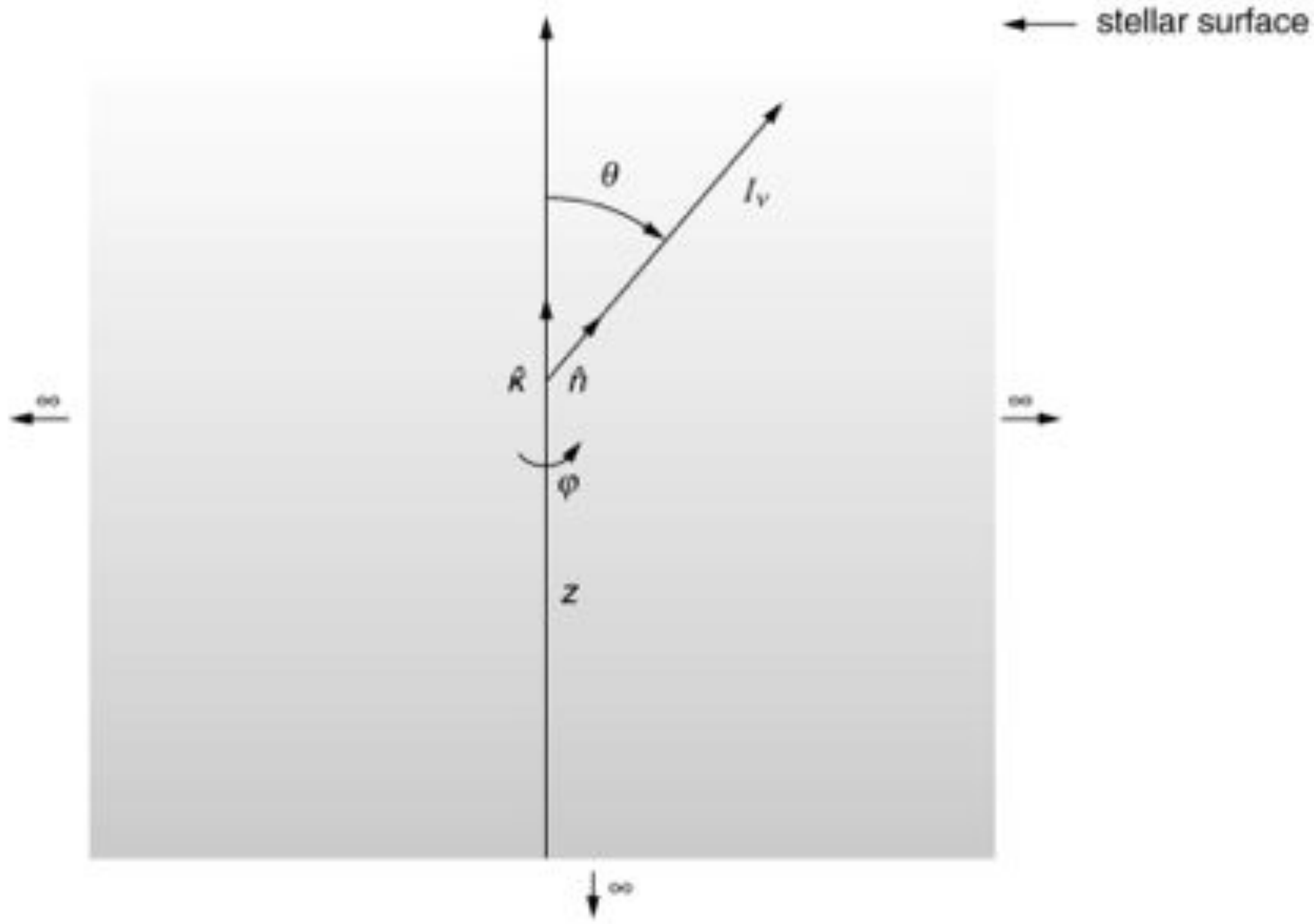


Figure 3.6 Diagram describing the plane-parallel approximation. The dimensions of the atmosphere in this approximation are infinite in all directions except $+z$ that ends at the stellar surface. In this approximation, the specific intensity is independent of φ. The grey scale represents the growing density in the atmosphere.

Optical Depth Home Work Hints

- We see stuff from an optical depth of approximately 2/3rds, we'll show this later
- Neglecting angular dependence and integrating in from the surface where the optical depth is zero to depth z gives

$$\tau(z) = \int_0^z K_\nu \rho(z')dz'$$

Remember, optical depth is 0 or greater, you either integrate inwards from 0 or add a negative sign to the integral above and integrate from the surface to a depth z which is less than the surface radius

Formal Equation

$$\mu \frac{dI_\nu(\tau_\nu, \mu)}{d\tau_\nu} = I_\nu(\tau_\nu, \mu) - S_\nu(\tau_\nu)$$

* The source function is generally the product of absorption opacity times blackbody radiation plus scattering opacity times a radiation field term divided by total opacity
* In practice we will approximate the source function as blackbody radiation only

Ok, different books use different conventions

* Some define optical depth with a minus, some without
* This just changes the direction you integrate
* Basically we are looking at the ingoing and outgoing radiation for some slab which is contributing due to a source function plus some incident radiation from above or below

Solution of Equation

* Multiply by integrating factor
  * integration factor = $e^{-\frac{\tau_\nu}{u}}$
  * First two terms combine

$$\frac{d(I_\nu(\tau_\nu, u)e^{-\frac{\tau_\nu}{u}})}{d\tau_\nu} = -\frac{S_\nu(\tau_\nu)e^{-\frac{\tau_\nu}{u}}}{u}$$

Now integrate

* for $u > 0$ outgoing radiation
    * τ_1 closer to stellar surface, τ_2 farther in the star
* for $u < 0$ ingoing radiation
    * τ_1 farther from stellar surface, τ_2 is at stellar surface and = 0
* Basically τ_1 is where radiation under consideration is being examined, τ_2 is at the start of the slab being considered

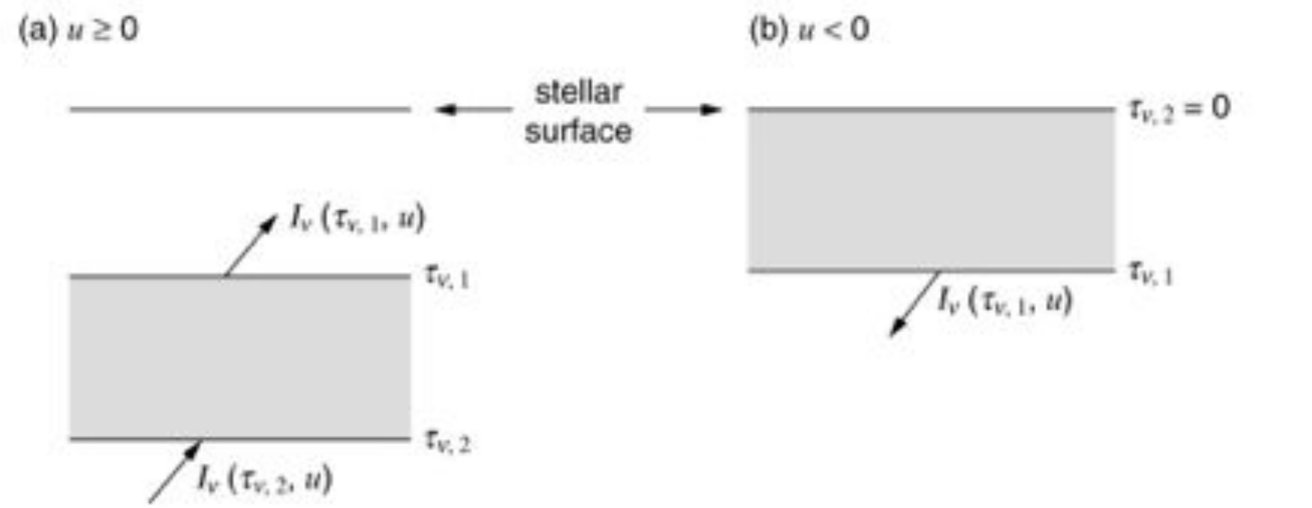


Figure 3.11 Illustrative explanation of the general solution for the radiative-transfer equation for the outgoing (a) and ingoing (b) directions. The shaded areas represent the portion of the star that contributes to the intensity at depth $\tau_{\nu,1}$.

Solution

* Intensity of radiation emerging from optical depth one is now

$$I_\nu(\tau_{\nu,1}, u) = I_\nu(\tau_{\nu,2}, u) e^{\frac{\tau_{\nu,1} - \tau_{\nu,2}}{u}} + \int_{\tau_{\nu,1}}^{\tau_{\nu,2}} S_\nu(t) e^{\frac{\tau_{\nu,1} - t}{u}} \frac{dt}{u}$$

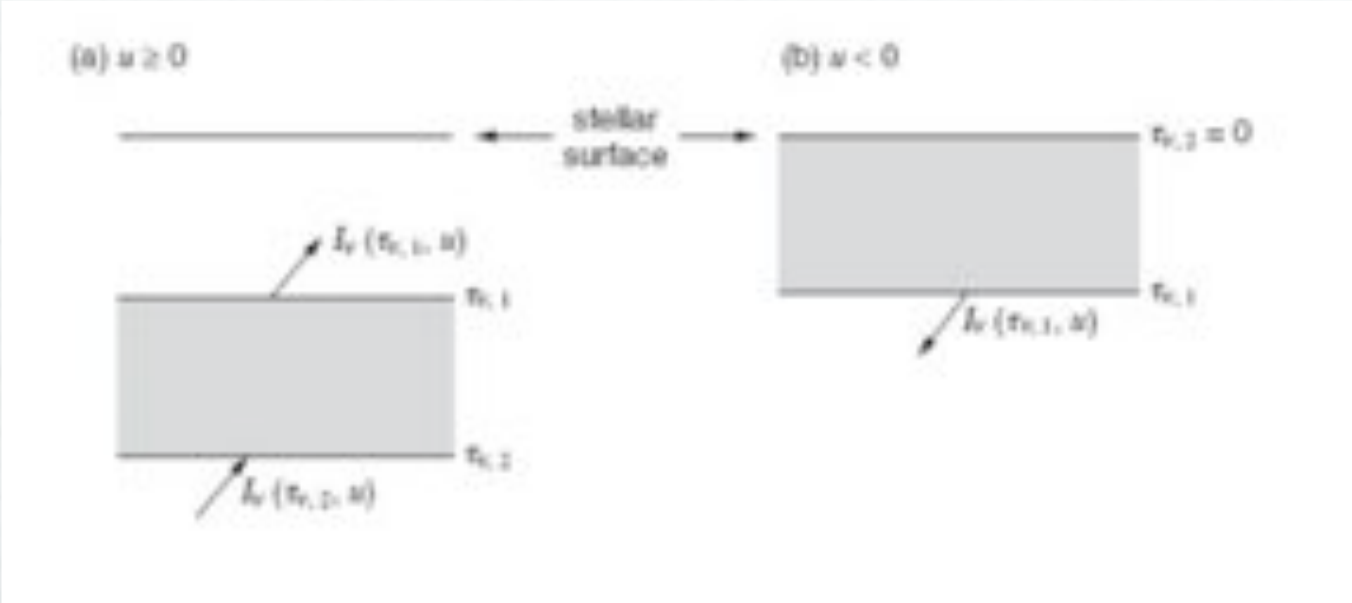
* Where the exponential term is negative or positive
    * Emission or absorption added to source function
* and t replaces τ as the dummy variable for integration
* The first term on the right is the attenuated initial specific intensity and the second is the attenuated source function within the slab
* Notice this still needs to be integrated over all angles

Specific intensity

* You integrate from where you measure to where it originated therefore $\tau_1 - \tau_2$ may be positive or negative.
* If you are going outwards in a star the term will be negative as specific intensity is decreasing, if inwards it will be positive
* We will be approximating the source and intensity as blackbodies with different temperatures at different radii and a star goes from hotter to cooler so you should expect the intensity to increase when going inwards and decrease when going outwards

Specific Intensity

* Integrate from optical depth one to optical depth two
* two cases
  * μ > 0 outgoing $\tau_1 = \tau_\nu \quad \tau_2 = \infty$
  * μ < 0 ingoing $\tau_1 = \tau_\nu \quad \tau_2 = 0$



Solutions? Let's see

Huh?

* Some insight
* For radiation emerging from the surface of a star, neglecting angular dependence,and assuming a source function that depends only on frequency specific intensity becomes

$$I_\nu(\tau_\nu) = S_\nu(1 - e^{-\tau_\nu})$$

* We assume a semi-infinite plane parallel slab
* For a very high optical depth $I_\nu = S_\nu$ often taken as $B_\nu(T)$ only
* For low optical depth we can do a first order expansion of the exponential and $I_\nu = S_\nu\tau_\nu = S_\nu\alpha_\nu L = j_\nu L$, this is I from infinite optical depth
* For optically thick the intensity is blackbody radiation
* For optically thin the intensity is just the emissivity times the path length

Moments of the radiation field

* First moment, average intensity, same units as specific intensity

$$J_\nu(\nu) = \frac{1}{4\pi}\oint I_\nu(z,u)d\Omega = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} I_\nu(z,u)sin\theta d\theta$$

* Let u = cosθ

$$J_\nu(\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(z,u)du$$

* Typically broken into outgoing and ingoing parts

$$J_\nu = \frac{1}{2}[\int_0^{1} I_\nu(z,u)d\nu + \int_{-1}^{0} I_\nu(z,u)d\nu]$$

From here on

* If a quantity lacks a frequency subscript I am referring to the frequency integrated quantity

Notational inconvenience

* I used K for opacity
* K is also the common name for the second moment of specific intensity
* From here on please note, K is still opacity
* I will denote the second moment which is entirely distinct from opacity by $K_{2\nu}$ and K_2 for the frequency dependent and frequency integrated second moment

Generic Moments

$$M_\nu(z,n) = \frac{1}{2}\int_{-1}^{1} I_\nu(z,u)u^n du$$

$$M_\nu(z,0) = J_\nu(z)$$

$$M_\nu(z,1) = H_\nu(z) = \frac{F_\nu(z)}{4\pi}$$ Eddington Flux

$$M_\nu(z,2) = K_{2\nu}(z)$$

- These seem quite odd to care about
- Physically
  - J is related to the energy density
  - H is related to the monochromatic flux
  - K_2 is related to radiation pressure

Energy Density

$$dE_\nu = I_\nu cos(\theta) dA d\Omega d\nu dt$$

$$dV = dAcos(\theta)cdt$$

$$dE_\nu = \frac{I_\nu dV d\nu d\Omega}{c}$$

$$U_\nu = \frac{1}{c} \oint I_\nu d\Omega = \frac{4\pi}{c} J_\nu$$

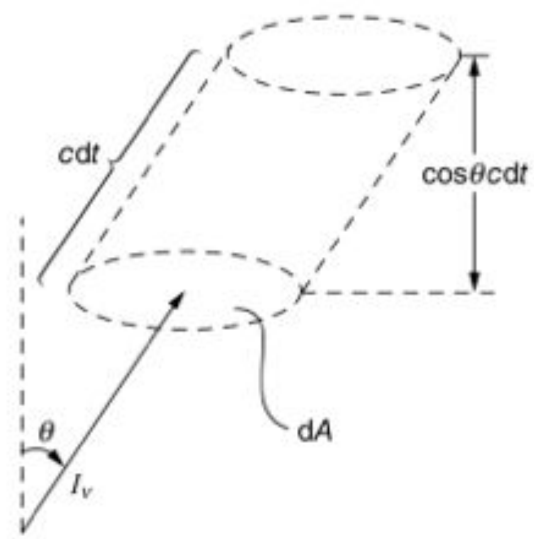


Figure 3.7 Illustration showing a beam of radiation crossing a surface dA. During the time interval dt, the volume dV = dAcosθcdt is filled with radiation from the specific intensity in the given direction that crosses the surface dA.

energy density is proportional to the average intensity

Monochromatic Flux

* Measurable, ergs/s/Hz/cm^2
* Solid angle integrated projection of specific intensity along the z direction
* This one describes whether there is a net flow of energy or not

$$\oint I_\nu(z,u) cos(\theta) d\Omega$$

$$F_\nu(z = surface) = 2\pi \int_0^1 I_\nu(z,u) u du = \pi B_\nu(T_{eff})$$

$$F = \int_0^\infty F_\nu d\nu = \pi\sigma T_{eff}^4$$

Integrated monochromatic flux depends on effective temperature of star, not local temperature.

Monochromatic Flux

* There is a temperature gradient in stars
* Generally the flux is greater for outgoing radiation as the temperature increases with depth so there is a net transport outwards
* The book is kind of vague but when we talk of monochromatic flux we generally speak of what is emerging and this is dependent on the effective temperature at the surface
* When we speak of blackbodies we are generally referring to the blackbody at the local temperature

Radiative Equilibrium

* Integrate the equation of radiative transfer over all frequencies and solid angles

$$\frac{1}{\rho}\frac{d(\int_0^\infty \oint \mu I_\nu d_\nu d\Omega)}{dz} + \int_0^\infty \oint [-j_\nu + K_\nu I_\nu] d\nu d\Omega = 0$$

The first integral is the frequency independent flux and is constant with depth and is therefore zero

* Scattering does not destroy or create photons therefore

$$j_\nu = j_\nu \frac{K_\nu}{K_\nu} = K_\nu S_\nu = \frac{K_\nu k_\nu B_\nu}{K_\nu} = k_\nu B_\nu$$

After Some Math

$$\int_0^\infty k_\nu (J_\nu - B_\nu) d\nu = 0$$

* Only absorption opacities remain
* Says quantity absorbed equals quantity emitted
* The Planck function is local
* The average intensity is not
* Local equilibrium depends on non-local influences

Radiative Transfer at large optical depths

* At large optical depths the mean free path is small therefore the radiative field depends primarily on local conditions
* Make the ansatz that source function can be written as a taylor series expansion of the blackbody function at depth t expanded about depth τ_ν

$$S_\nu(t) = \sum_{n=0}^{\infty} \frac{(t-\tau_\nu)^n}{n!} \frac{d^n B_\nu}{d\tau_\nu^n}|_{\tau_\nu}$$

Which leads to

$$I_\nu(\tau_\nu, u) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n B_\nu}{d\tau_\nu^n}|_{\tau_\nu} \int_{\tau_\nu}^{\infty} (t - \tau_\nu)^n e^{\frac{\tau_\nu - t}{u}} \frac{dt}{u}$$

using $$x = \frac{t - \tau_\nu}{u}$$

and $$\int_0^\infty x^n e^{-x} dx = n!$$

Results in

$$I_\nu(\tau_\nu, u) = \sum_{n=0}^{\infty} u^n \frac{d^n B_\nu}{d\tau_\nu^n}|_{\tau_\nu}$$

The resulting moments are then

$$I_\nu(\tau_\nu, u) = B_\nu + u\frac{dB_\nu}{d\tau_\nu} + u^2\frac{dB_\nu^2}{d\tau_\nu^2} + ...$$

Alternating even odd functions
Only odd or even terms remain in the moments

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau_\nu, u)du = B_\nu + \frac{1}{3}\frac{d^2B_\nu}{d\tau_\nu^2} + ...$$

$$H_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau_\nu, u)udu = \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} + \frac{1}{5}\frac{d^3B_\nu}{d\tau_\nu^3} + ...$$

$$K_{2\nu}(\tau_\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau_\nu, u)u^2du = \frac{B_\nu}{3} + \frac{1}{5}\frac{d^2B_\nu}{d\tau^2} + \ldots$$

Successive derivatives vary as

$$\frac{\frac{d^{n+2}B_\nu}{d\tau_\nu^{n+2}}}{\frac{d^n B_\nu}{d\tau_\nu^n}} \sim \frac{1}{\tau_\nu^2}$$

- For an optical depth of 2 this means the next surviving term is 1/4th
- For an optical depth of 10 the ratio is 1/100 or a correction of one percent
- Truncate all expansions after 1st derivative
- Optional - show this for homework extra credit

Approximate moments

$$I_\nu \sim B_\nu + u\frac{dB_\nu}{d\tau_\nu}$$

$$J_\nu \sim B_\nu$$

$$H_\nu \sim \frac{1}{3}\frac{dB_\nu}{d\tau_\nu}$$

$$K_{2\nu}(\tau_\nu) \sim \frac{B_\nu}{3}$$

The relative value of the anisotropic term in the specific intensity to the isotropic term is

$$\frac{\frac{dB_\nu}{d\tau_\nu}}{B_\nu} \sim \frac{3H_\nu}{B_\nu} \sim \frac{3H}{B} = 3(\frac{T_{eff}}{T})^4$$

$$\frac{T_{eff}^4}{T^4} << 1$$

* Due to the fact that the temperature interior in a star is much more than it's effective temperature
* Essentially this says that the radiation field deep in the star is essentially isotropic
* It is the anisotropic term that causes the slow outward transport of radiation

* Recalling that we can relate the true flux to the Eddington flux via $F_\nu=4\pi H_\nu$
* We may re-write the flux as dependent on the star's temperature gradient

$$H_\nu \sim \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} = -\frac{1}{3k_\nu\rho}\frac{dB_\nu}{dz} = -\frac{1}{3k_\nu\rho}\frac{dB_\nu}{dT}\frac{dT}{dz}$$

Physical insight?

* 1 - low transport of monochromatic flux at frequencies where opacity is high
* 2 - Flux is proportional to temperature gradient
  * low gradient, low flux
  * high gradient, high flux

Rosseland and other mean opacities

* Free-Free - Thomson
* Bound-Bound - Boltzmann
* Bound-Free - Saha
* Temperature dependent, pressure dependent, turbulent velocity dependent, composition dependent
* HARD!!!

This is the opacity for ONE line transition

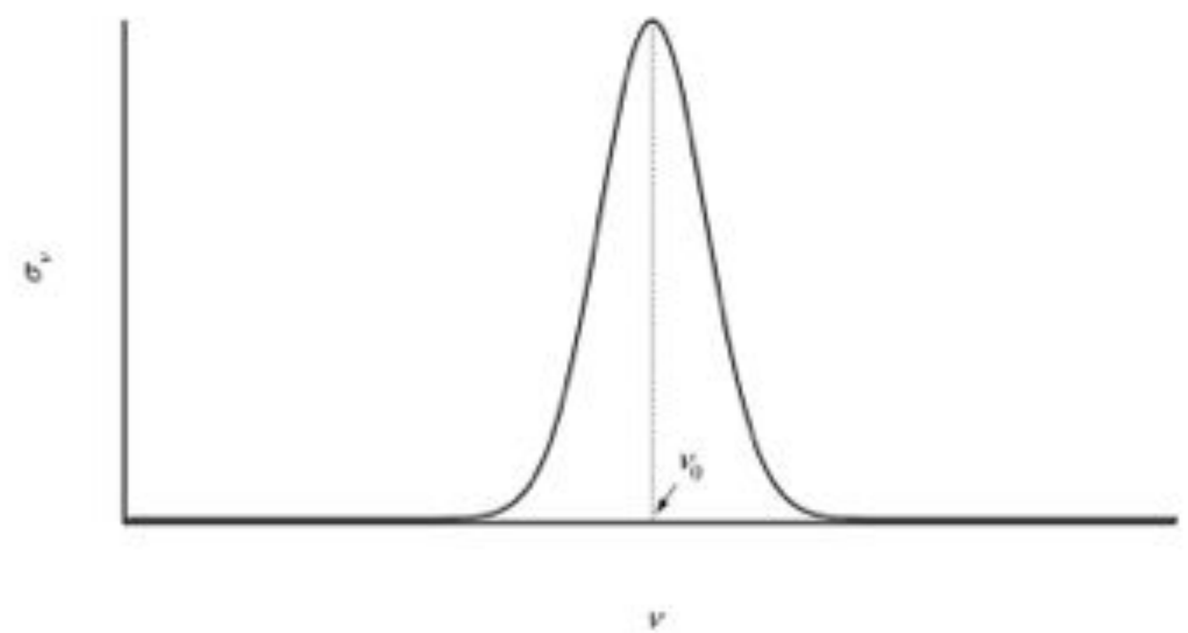


Figure 3.2 Schematic frequency dependence of the cross section per absorber (i.e. per atom in the initial energy state of the transition considered) due to an atomic line (bound–bound transition). The natural frequency of the transition is represented by ν_0 (see Eq. (1.47) in Section 1.6). The width of the atomic lines is due to three broadening mechanisms (natural, Doppler and pressure broadening) that will be discussed in Section 4.3.

Enter the Rosseland mean opacity

* Defined by the definition of the frequency integrated Eddington Flux

$$H = -\frac{1}{3k_R\rho}\frac{dB}{dT}\frac{dT}{dZ}$$

$$H = \int_0^\infty H_\nu d\nu$$

$$H_\nu \sim \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} = -\frac{1}{3k_\nu\rho}\frac{dB_\nu}{dz} = -\frac{1}{3k_\nu\rho}\frac{dB_\nu}{dT}\frac{dT}{dz}$$

$$H = -\frac{1}{3\rho}\frac{dT}{dz}\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{d\tau}d\nu = -\frac{1}{3\kappa_r\rho}\frac{dB}{dT}\frac{dT}{dz}$$

$$\frac{1}{k_R} = \frac{\int_0^\infty [\frac{1}{k_\nu}\frac{dB_\nu}{dT}d\nu]}{\frac{dB}{dT}}$$

$$B = \frac{\sigma T^4}{\pi} \quad \frac{dB}{dT} = \frac{d(\frac{\sigma T^4}{\pi})}{dT} = \frac{4\sigma T^3}{\pi}$$

$$\frac{1}{k_R} = \frac{\pi}{4\sigma T^3}\int_0^\infty \frac{1}{k_\nu}\frac{dB_\nu}{dt}d\nu$$

Result

* Depends on temperature and density
* Generally people use tables of previously tabulated opacity
* Third dimension omitted
* log(R)=-3 here
* T_6 is units of 10^6 kelvin

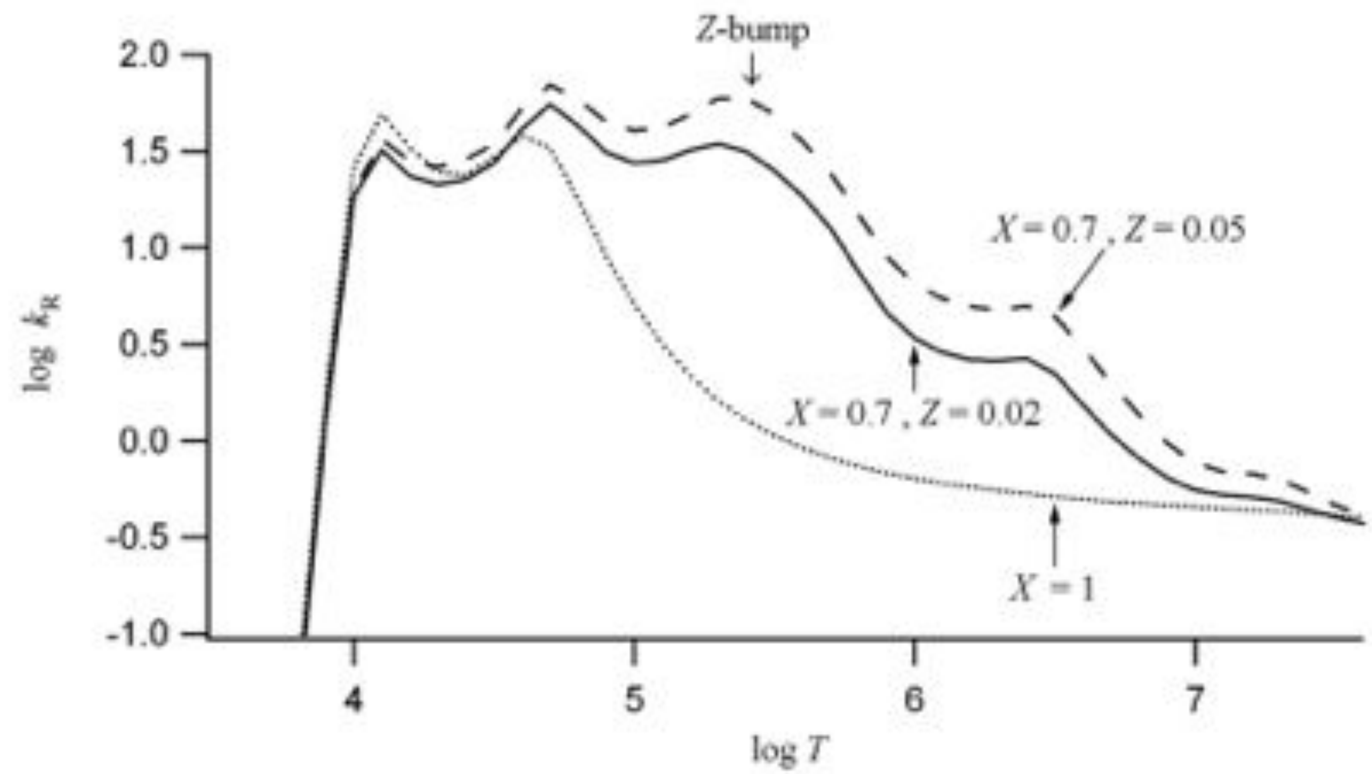


Figure 3.17 Rosseland mean opacity as a function of temperature from the Opacity Project data. The densities used are those for log R = –3. Three curves are shown for different abundances that are defined in the figure. Also identified in the figure is the position of the Z-bump.

$$log(R) = log(\frac{\rho}{T_6^3})$$

Radiative Acceleration

* There exists a net outward flux of radiation from stars
* Photons possess momentum = E/c
* This momentum is transferred to matter providing an outward acceleration to it
* This provides the pressure support for stars but also sets an upper limit on stellar masses and temperatures

* The energy matter absorbs is

$$dE_\nu = K_\nu \rho I_\nu d\Omega d\nu dt dA ds$$

* The outward momentum transferred to matter is (p=E/c for a photon)

$$dp_\nu = \frac{u dE_\nu}{c} = \frac{u}{c} K_\nu \rho I_\nu d\Omega d\nu dt dA ds$$

* Recalling the definition for the Eddington flux and integrating over all solid angles and frequencies results in a net momentum transfer of

$$dp = \frac{4\pi}{c} \rho dt dA ds \int_0^\infty K_\nu H_\nu d\nu$$

Radiative Acceleration

* momentum is dp/dt

$$\frac{dp}{dt} = F_{rad} = \frac{4\pi}{c}\rho dA ds \int_0^\infty K_\nu H_\nu d\nu = \rho dA ds g_{rad}$$

* Where

$$g_{rad} = \frac{4\pi}{c}\int_0^\infty K_\nu H_\nu d\nu$$

* Which modifies our pressure profile to become

$$\frac{dP}{dr} = -\rho[g - g_{rad}]$$

Radiative Pressure

- using $H_\nu \sim \frac{dK_{2\nu}}{d\tau_\nu}$ and $-\alpha_\nu ds = -K_\nu \rho ds = d\tau_\nu$

$$\frac{dP_{\nu(rad)}}{ds} = -\rho\frac{4\pi}{c}\int_0^\infty K_\nu H_\nu d\nu = -\rho\frac{4\pi}{c}\int_0^\infty K_\nu \frac{dK_{2\nu}}{d\tau_\nu} d\nu$$

$$\frac{dP_{\nu(rad}}{ds} = -\rho\frac{4\pi}{c}\int_0^\infty K_\nu \frac{dK_{2\nu}}{-K_\nu \rho ds} d\nu = \frac{4\pi}{c}\frac{d}{ds}\int_0^\infty K_{2\nu} d\nu$$

- Gives $K_{2\nu} = \frac{c}{4\pi} P_\nu$

* At large optical depths the frequency integrated radiative pressure is

$$P = \frac{4\pi}{c}\int_0^\infty K_{2\nu} d\nu = \frac{4\pi}{c}\int_0^\infty \frac{B_\nu}{3} d\nu = \frac{4\sigma T^4}{3c}$$

* We need to add this to the total pressure when at large optical depths

* Just remember, gas pressure opposes gravitational

Radiative Pressure

* Let's do a heuristic derivation for the Eddington Luminosity recalling P = E/c for a photon
* Flux is energy per second per unit area
* Flux*Thomson cross section/(c) is momentum transfer per second or roughly dp/dt or a force
* Let's balance with gravitational attraction and derive the Eddington luminosity

$$L_{\rm Edd} = \frac{4\pi G M m_{\rm p} c}{\sigma_{\rm T}}$$
$$\cong 1.26 \times 10^{31} \left(\frac{M}{M_\odot}\right) {\rm W} = 3.2 \times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot$$

Review

* Opacity $K_\nu = k_\nu + \sigma_\nu$ = absorption plus scattering opacity cm^2/g
    * ρK_ν - (number density absorbers/scatterers)x(Opacity) = α_ν = opacity per unit length. Your book uses χ_ν
* Optical Depth, a measure of path integrated opacity, always 0 or greater, increases inwardly

$$-\alpha_\nu ds = -K_\nu \rho ds = d\tau_\nu \quad \tau(z) = \int_0^z K_\nu \rho(z')dz'$$

* Source Function $S_\nu = \frac{j_\nu}{K_\nu}$ ratio of emission to absorption

$$\frac{k_\nu B_\nu + \sigma_\nu J_\nu}{k_\nu + \sigma_\nu}$$

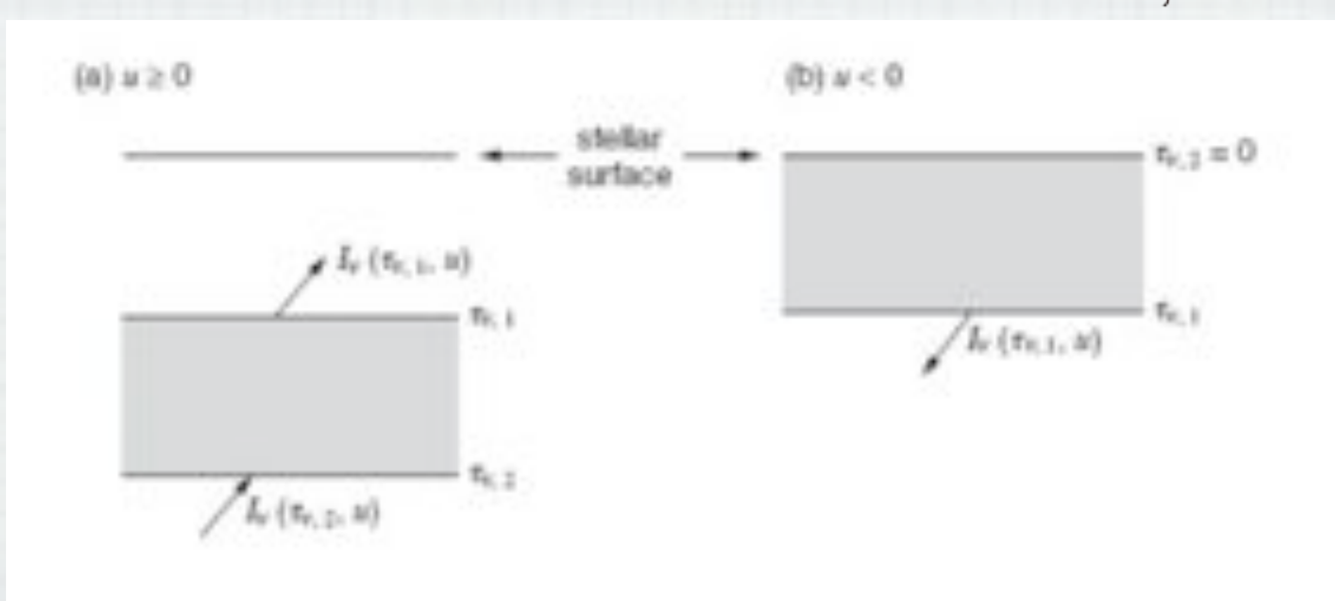
Review

* Equation for specific intensity

$$\frac{\mu}{\rho}\frac{dI_\nu}{dz} = -K_\nu I_\nu + j_\nu$$

$$\mu\frac{dI_\nu(\tau_\nu,\mu)}{d\tau_\nu} = I_\nu(\tau_\nu,\mu) - S_\nu(\tau_\nu)$$

$$I_\nu(\tau_{\nu,1},u) = I_\nu(\tau_{\nu,2},u)e^{\frac{\tau_{\nu,1}-\tau_{\nu,2}}{u}} + \int_{\tau_{\nu,1}}^{\tau_{\nu,2}} S_\nu(t)e^{\frac{\tau_{\nu,1}-t}{u}}\frac{dt}{u}$$



For large optical depths

$$I_\nu(\tau_\nu, u) = B_\nu + u\frac{dB_\nu}{d\tau_\nu} + u^2\frac{dB_\nu^2}{d\tau_\nu^2} + ...$$

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau_\nu, u)du = B_\nu + \frac{1}{3}\frac{d^2B_\nu}{d\tau_\nu^2} + ...$$

$$H_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau_\nu, u)udu = \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} + \frac{1}{5}\frac{d^3B_\nu}{d\tau_\nu^3} + ...$$

$$K_{2\nu}(\tau_\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau_\nu, u)u^2du = \frac{B_\nu}{3} + \frac{1}{5}\frac{d^2B_\nu}{d\tau^2} + \ldots$$

Review

* Moments
  * Zeroth J - average specific intensity
  * 1st H Eddington Flux
  * 2nd K_2 integral
* At large optical depth
  * $H_\nu = \frac{dK_{2\nu}}{d\tau_\nu}, \; J_\nu = 3K_{2\nu}$

$$J_\nu = \frac{c}{4\pi} U_\nu$$

$$H_\nu = \frac{1}{4\pi} F_\nu$$

$$K_{2\nu} = \frac{c}{4\pi} P_\nu$$

Homework

* Use equations 3.31 and 3.97
* 3.6 - assume u =1, integrate from 0 to d1 and then d1 to d2
* 3.8 define Y = I - S so dY = dI (S = constant)
* Skip 3.7, 3.10, and 3.11