Quantum Theory I: Homework 12

Due: 14 March 2023

1 Bloch equations

Consider a spin-1/2 particle with mass m, charge q and g-factor g placed in a constant magnetic field

$$\mathbf{B} = B_0\hat{\mathbf{z}}.$$

Use $\omega_0 := -gqB_0/2m$ to answer the following.

a) Express the Hamiltonian for the particle in terms of ω_0 and $\hat{S}_x, \hat{S}_y$ and $\hat{S}_z$.

b) Using Ehrenfest's theorem, find a set of coupled differential equations for $\langle S_x \rangle, \langle S_y \rangle$ and $\langle S_z \rangle$ which describe the time-evolution of these quantities, i.e. find equations for

$$\frac{d}{dt}\langle S_x \rangle, \frac{d}{dt}\langle S_y \rangle, \frac{d}{dt}\langle S_z \rangle$$

in terms of $\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle$ and relevant constants.

c) Defining the magnetization vector as

$$\mathbf{M} := \frac{gq}{2m}\left(\langle S_x \rangle \hat{\mathbf{x}} + \langle S_y \rangle \hat{\mathbf{y}} + \langle S_z \rangle \hat{\mathbf{z}} \right),$$

show that it satisfies

$$\frac{d}{dt}\mathbf{M} = -\frac{gq}{2m}\mathbf{B} \times \mathbf{M}.$$

In nuclear magnetic resonance these are called the *Bloch equations*, named after Felix Bloch who shared the Nobel Prize for inventing NMR. The Bloch equations have the same form as the equations for a classical magnetic dipole precessing in a constant magnetic field and they allow one to view an ensemble of spin-1/2 particles in a classical way. This makes it easy to picture how to manipulate them. They also allow you to include the effects of relaxation into the description of the dynamics of the ensemble of spins - you merely add appropriate terms to the Bloch equations.

2 Time-dependence and measurements for spin-1/2 particles

Consider a spin-1/2 particle in the initial state $|+\hat{\boldsymbol{x}}\rangle$. This is subsequently subjected to a constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{y}}$. After time t in the field, S_x is measured. Explain your answers for the following.

a) Do the possible outcomes of the measuremennt depend on t?

b) Do the probabilities of the measurement outcomes depend on t?

c) Does the expectation value of S_x depend on t?

Consider a spin-1/2 particle in the initial state $|+\hat{x}\rangle$. This is subsequently subjected to a constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. After time t in the field, S_z is measured. Explain your answers for the following.

d) Do the possible outcomes of the measuremennt depend on t?

e) Do the probabilities of the measurement outcomes depend on t?

f) Does the expectation value of S_z depend on t?

3 Photon passing through a Mach-Zehnder interferometer

A Mach-Zehnder interferometer consists of an arrangement of two beam splitters, B_1 and B_2, two mirrors, M_1 and M_2, and two detectors, D_0 and D_1, as illustrated in Fig. 1. Note that the reflective side of B_1 is down and right and that of B_2 is up and left. Ignore the thickness of the glass in the beam-splitters and assume that they reflect 50% of the beam and transmit 50%.

The orthonormal input states $\{|0\rangle, |1\rangle\}$ are indicated in the figure. The effects of B_1 and B_2 can be described by the operators
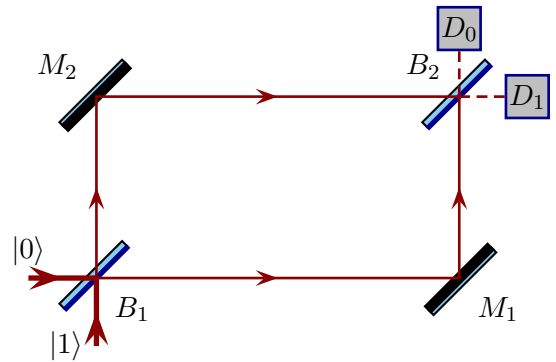

Figure 1: Mach-Zehnder interferometer

$$\hat{U}_{B_1} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \hat{U}_{B_2} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

where the matrices indicate representations in the basis $\{|0\rangle, |1\rangle\}$. Suppose that a single photon is incident on the apparatus in the state $|1\rangle$. Determine the probability with which the photon will arrive in either detector.

4 Mach-Zehnder interferometer and phase shifters

A Mach-Zehnder interferometer consists of an arrangement of two beam splitters, B_1 and B_2, two mirrors, M_1 and M_2, and two detectors as illustrated in Fig. 2. Note that the reflective side of B_1 is down and right and that of B_2 is up and left. Ignore the thickness of the glass in the beamsplitters and assume that they reflect 50% of the beam and transmit 50%.

Phase shifters are inserted into the upper and lower beams. Classically these slow the light waves that pass through them, thus shifting the phase of the light in each beam. Denote the shift in the upper beam by φ_1 and the lower beam by φ_0.
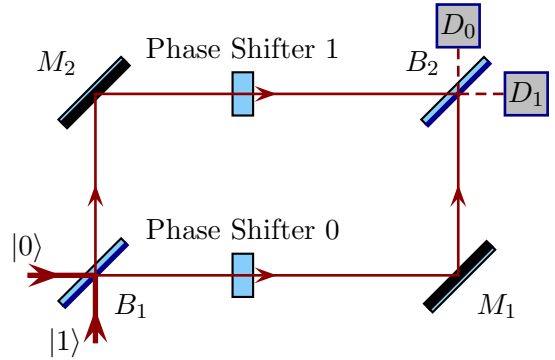

Figure 2: Mach-Zehnder interferometer with phase shifter.

In terms of the kets describing the passage of light this amounts to:

$$|0\rangle \to e^{i\varphi_0} |0\rangle$$
$$|1\rangle \to e^{i\varphi_1} |1\rangle .$$

a) Determine the unitary operator which represents the action of the phase shifter.

b) Suppose that B_2 is removed and a photon is incident in the beam represented by $|0\rangle$. Determine the probability with which it arrives at detector D_0. Does this depend on either of the phase shifters (i.e. φ_0 or φ_1)? Could you reasonably say that, if the photon arrives at D_0, then only one trajectory through the interferometer was involved in the process? If so, which trajectory?

c) With B_2 in place, consider a photon incident in the beam represented by $|0\rangle$. Determine the probability with which it arrives at detector D_0. Does this depend on either of the phase shifters (i.e. φ_0 or φ_1)? Does it depend on *both* of the phase shifters? Could you reasonably say that, if the photon arrives at D_0, then *only one* trajectory through the interferometer was involved in this process? If so, which trajectory?

d) If the particle were known to be traveling in the upper path prior to B_2, determine the probabilities of detection at either detector. Does this depend on either of the phase shifters? What if it were known to be traveling in the lower path?

In the following, consider the special case where $\varphi_0 = \pi/2$ and $\varphi_1 = -\pi/2$ and the incident photon is in state $|0\rangle$.

e) With B_2 removed, determine the probabilities with which the photon will arrive in either detector. In the absence B_2 do the phase shifters affect the outcome of the experiment?

f) With B_2 present, determine the probabilities with which the photon will arrive in either detector. In the presence B_2 do the phase shifters affect the outcome of the experiment?

g) Does the absence or presence of B_2 affect whether the phase shifters are relevant or not for the outcome of the experiment?

h) If B_2 were inserted after the photon has passed through either phase shifter then experiments indicate that the outcomes are the same as though B_2 has been present from the time that the photon entered B_1. Does it appear possible that in this case the absence or presence of B_2 should affect whether the phase shifters are relevant or not for the outcome of the experiment?

The last scenario is called a *delayed choice experiment.* Such experiments have been performed.

5 Multiple Mach-Zehnder interferometers: sending messages

Consider two Mach-Zehnder interferometers arranged so that a photon that passes through the first is applied to the second. The beams are redirected between the two interferometers by two mirrors (the paths taken between the two interferometers are indicated by dashed lines).
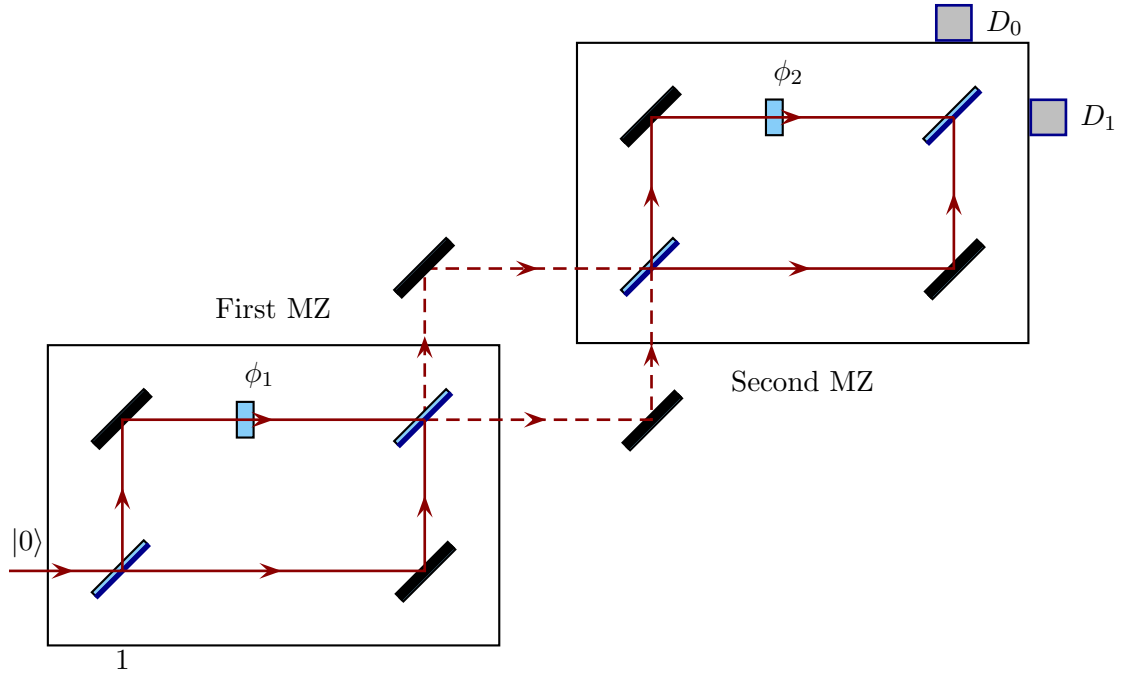
For each interferometer the phase shifter performs

$$\begin{aligned} |0\rangle &\rightarrow |0\rangle \\ |1\rangle &\rightarrow e^{i\phi_i} |1\rangle . \end{aligned} \tag{1}$$

a) Suppose that both of the phase shifters are absent and that all of the beam splitters are absent. Determine which detector (after the second MZ) will register a photon in the state $|0\rangle$ (before the first MZ). Repeat this with the state $|1\rangle$. Are these consistent with the usual physical interpretation of the states?

b) Now suppose that the beam splitters and phase shifters are in place. Suppose that a photon is initially in the state $|0\rangle$. Determine the state after the last beam splitter. Use this to show that the probabilities with which this will arrive at the detectors are

$$\Pr(D_0) = \cos^2\left(\frac{\phi_1 + \phi_2}{2}\right)$$

$$\Pr(D_1) = \sin^2\left(\frac{\phi_1 + \phi_2}{2}\right).$$

Repeat this for a photon initially in the state $|1\rangle$.

c) Suppose that a photon is initially in the state $|0\rangle$ and that $\phi_1 = 0$ and $\phi_2 = 0$. Determine the probability with which a photon will arrive at either of the detectors. Repeat this for a photon initially in the state $|1\rangle$. Based on final photon detection, is it possible to say which of the two states was initially used? If so how?

d) Suppose that a photon is initially in the state $|0\rangle$ and that $\phi_1 = \pi/2$ and $\phi_2 = 0$. Determine the probability with which a photon will arrive at either of the detectors. Repeat this for a photon initially in the state $|1\rangle$. Based on final photon detection, is it possible to say which of the two states was initially used? If so how?

e) Suppose that a photon is initially in the state $|0\rangle$ and that $\phi_1 = 0$ and $\phi_2 = -\pi/2$. Determine the probability with which a photon will arrive at either of the detectors. Repeat this for a photon initially in the state $|1\rangle$. Based on final photon detection, is it possible to say which of the two states was initially used? If so how?

f) Suppose that a photon is initially in the state $|0\rangle$ and that $\phi_1 = \pi/2$ and $\phi_2 = -\pi/2$. Determine the probability with which a photon will arrive at either of the detectors. Repeat this for a photon initially in the state $|1\rangle$. Based on final photon detection, is it possible to say which of the two states was initially used? If so how?

This combination of Mach-Zehnder interferometers forms the conceptual basis for some quantum cryptography schemes. Such schemes allow for the detection of eavesdropping, which would occur between the two interferometers.

g) Suppose that one person had control of the first MZ and another the second MZ. Each randomly chooses the phase, so the first chooses 0 or $\pi/2$ and the other 0 or $-\pi/2$. The first then sends a photon either using $|0\rangle$ or $|1\rangle$. Whenever either both phases are zero or both are non-zero, can the second person learn which photon state the first person used? Can you see how they could communicate?

h) Suppose that a photon is initially in the state $|0\rangle$ and that $\phi_1 = \pi/2$ and $\phi = -\pi/2$. Determine the probability with which the photon will emerge in D_0. Now determine the state of the photon immediately after the second beam splitter of the first interferometer. Suppose that a photon detector is placed in each path immediately after the second beam splitter of the first interferometer and that the photon is detected in the upper detector (usually called D_0). This photon is then retransmitted to the second interferometer, i.e. entering in the state $|0\rangle$. Determine the probability with which it emerges in each of the final detectors. What is strikingly different about this compared to the case where no intermediate detection took place. Is it possible for someone using the second interferometer to learn of the presence of intermediate detection? If so how?