Quantum Theory I: Homework 8

Due: 21 February 2023

1 Expectation values of spin measurements

Consider an ensemble of particles, each of which is in the state

$$|+\hat{\boldsymbol{n}}\rangle = \cos\left(\frac{\theta}{2}\right)|+\hat{\boldsymbol{z}}\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\hat{\boldsymbol{z}}\rangle .$$

Situations of this type arise in nuclear magnetic resonance (NMR) spectroscopy on spin-1/2 nuclei such as the hydrogen nucleus. NMR spectrometers are well equipped to measure ensemble averages of the components of spin; for large ensemble sizes these will approximate the expectation values $\langle S_x\rangle, \langle S_y\rangle$ and $\langle S_z\rangle$. It is useful to gather these into a *magnetization vector*

$$\mathbf{M} := \frac{ge}{2m_p}\Big(\langle S_x\rangle\,\hat{\mathbf{x}} + \langle S_y\rangle\,\hat{\mathbf{y}} + \langle S_z\rangle\,\hat{\mathbf{z}}\Big)$$

where e is the charge of a proton, m_p its mass and g a g-factor.

a) Consider particles described by the state $|+\hat{\boldsymbol{n}}\rangle$. Determine expressions for $\langle S_x\rangle, \langle S_y\rangle$ and $\langle S_z\rangle$. Substitute these into the magnetization vector defined above. Show that the magnetization is

$$\mathbf{M}_{\text{state }|+\hat{\boldsymbol{n}}\rangle} := \alpha\left(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\right)$$

where α is a constant independent of θ and ϕ. Find an expression for α.

b) Determine the direction, $\hat{\mathbf{n}}$, in which an SG apparatus must be oriented so that any single particle in this state $|+\hat{\boldsymbol{n}}\rangle$ will give a measurement outcome $S_n = +\hbar/2$ with 100% certainty and express $\hat{\mathbf{n}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. Find a simple relationship between $\hat{\mathbf{n}}$ and $\mathbf{M}$.

Actually, in most NMR situations, the spin-1/2 nuclei are not all in the same state. In effect, some are in the state $|+\hat{\boldsymbol{n}}\rangle$ and the rest are in the state $|-\hat{\boldsymbol{n}}\rangle$. Suppose that the fraction of particles in the state $|+\hat{\boldsymbol{n}}\rangle$ is $(1+\varepsilon)/2$ and that of particles in the state $|-\hat{\boldsymbol{n}}\rangle$ is $(1-\varepsilon)/2$ where $0 < \varepsilon < 1$ is a constant that depends on the particular NMR situation. The spectrometer still measures the ensemble averages of the components of spin; for large ensemble sizes these will approximate *suitably weighted* expectation values. For example,

$$\langle S_x\rangle_{\text{entire ensemble}} = \frac{1+\varepsilon}{2}\langle S_x\rangle_{\text{state }|+\hat{\boldsymbol{n}}\rangle} + \frac{1-\varepsilon}{2}\langle S_x\rangle_{\text{state }|-\hat{\boldsymbol{n}}\rangle}$$

where $\langle S_x\rangle_{\text{state }|+\hat{\boldsymbol{n}}\rangle}$ is the expectation value of measurements of S_x for an ensemble of particles each in state $|+\hat{\boldsymbol{n}}\rangle$, etc, ... It follows that the magnetization for the ensemble is

$$\mathbf{M} = \frac{1+\varepsilon}{2}\mathbf{M}_{\text{state }|+\hat{\boldsymbol{n}}\rangle} + \frac{1-\varepsilon}{2}\mathbf{M}_{\text{state }|-\hat{\boldsymbol{n}}\rangle}$$

where $\mathbf{M}_{\text{state }|+\hat{\boldsymbol{n}}\rangle}$ and $\mathbf{M}_{\text{state }|-\hat{\boldsymbol{n}}\rangle}$ are the magntizations for particles in the states $|+\hat{\boldsymbol{n}}\rangle$ and $|-\hat{\boldsymbol{n}}\rangle$ respectively.

c) Show that, for particles described by the state $|-\hat{\boldsymbol{n}}\rangle$,

$$\mathbf{M}_{\text{state }|-\hat{\boldsymbol{n}}\rangle} := -\alpha\left(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\right)$$

where α is the same as in part (a).

d) Most NMR spectrometers are configured so that the *signal strength* that they obtain is proportional to the magnitude of the *transverse magnetization,* i.e. $\sqrt{M_x^2 + M_y^2}$. Determine an expression for the signal strength. Note that typically $\varepsilon \approx 10^{-4}$.

2 Eigenvalues and eigenvectors of a spin observable

Consider the spin observable

$$\hat{S} = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

as represented in the $\{|+\hat{\boldsymbol{z}}\rangle, |-\hat{\boldsymbol{z}}\rangle\}$ basis. Determine the eigenvalues and eigenvectors of $\hat{S}$.

3 Spin-1/2 Observable

A physicist working with spin-1/2 particles claims that a measuring device that he is using corresponds to the observable

$$\hat{A} = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}$$

where this representation is in terms of the $\{|+\hat{\boldsymbol{z}}\rangle, |-\hat{\boldsymbol{z}}\rangle\}$ basis.

a) Verify that $\hat{A}$ satisfies the requirements for an observable.

b) Determine the eigenvalues of $\hat{A}$.

c) Suppose the physicist subjects a spin-1/2 particle to this measuring device. What are the possible outcomes of the measurement?

d) Determine the states associated with each of the two measurement outcomes (i.e. the states such that each gives an outcome with certainty).

e) Determine the direction $\hat{\mathbf{n}}$ associated with the measurement.

4 Observable actions

Consider the observable

$$\hat{B} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

where this representation is in terms of the $\{|+\hat{\boldsymbol{z}}\rangle, |-\hat{\boldsymbol{z}}\rangle\}$ basis. The states $|+\hat{\boldsymbol{y}}\rangle$ and $|-\hat{\boldsymbol{y}}\rangle$ are eigenstates of $\hat{B}$.

a) Describe whether $\hat{B}\,|+\hat{\boldsymbol{y}}\rangle$ is exactly the outcome of a measurement on a particle in the state $|+\hat{\boldsymbol{y}}\rangle$. Explain your answer. Identify how this is might be associated with a measurement *outcome.*

b) Describe whether $\hat{B}\,(|+\hat{\boldsymbol{y}}\rangle + |-\hat{\boldsymbol{y}}\rangle)/\sqrt{2}$ is exactly the outcome of a measurement on a particle in the state $(|+\hat{\boldsymbol{y}}\rangle + |-\hat{\boldsymbol{y}}\rangle)/\sqrt{2}$. Explain your answer. Identify how this is might be associated with a measurement *outcome.*

c) In general does operating with an observable on a state produce a measurement outcome?