Quantum Theory I: Homework 7

Due: 17 February 2023

1 Measurement operators

Consider a spin-1/2 particle subjected to an SG $\hat{\boldsymbol{y}}$ measurement. The measurement operators associated with the measurement outcomes are

$$S_y = +\frac{\hbar}{2} \leftrightarrow \left|+\hat{\boldsymbol{y}}\right\rangle \left\langle +\hat{\boldsymbol{y}}\right| = \hat{P}_{+y}$$
$$S_y = -\frac{\hbar}{2} \leftrightarrow \left|-\hat{\boldsymbol{y}}\right\rangle \left\langle -\hat{\boldsymbol{y}}\right| = \hat{P}_{-y}$$

a) Determine the matrix representation of $\left|+\hat{\boldsymbol{y}}\right\rangle \left\langle +\hat{\boldsymbol{y}}\right|$ in the $\{\left|+\hat{\boldsymbol{z}}\right\rangle, \left|-\hat{\boldsymbol{z}}\right\rangle\}$ basis.

b) Determine the matrix representation of $\left|-\hat{\boldsymbol{y}}\right\rangle \left\langle -\hat{\boldsymbol{y}}\right|$ in the $\{\left|+\hat{\boldsymbol{z}}\right\rangle, \left|-\hat{\boldsymbol{z}}\right\rangle\}$ basis.

c) Suppose that the spin-1/2 particle is initially in the state $\left|+\hat{\boldsymbol{x}}\right\rangle$. Using row and column vector representations of bra and ket vectors and the measurement operators above, determine $\Pr(S_y = \pm\hbar/2)$.

d) Determine the matrix representation of $\hat{S}_y$ in the $\{\left|+\hat{\boldsymbol{z}}\right\rangle, \left|-\hat{\boldsymbol{z}}\right\rangle\}$ basis and use this to determine the expectation value $\langle S_y \rangle$ for an ensemble of particles in the state $\left|+\hat{\boldsymbol{x}}\right\rangle$.

2 Matrix multiplication and the adjoint operation

Consider the matrices

$$A = \begin{pmatrix} 1 & -3i \\ i & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Determine the matrix products AB and BA. Are these equal?

b) Determine $A^\dagger$ and $B^\dagger$. Determine whether each matrix is Hermitian.

c) Verify, by explicitly evaluating either side that

$$(AB)^\dagger = B^\dagger A^\dagger.$$

3 Observables for spin-1/2

Consider the observable corresponding to the x component of spin

$$\hat{S}_x := \frac{\hbar}{2}\Big(\left|+\hat{\boldsymbol{x}}\right\rangle \left\langle +\hat{\boldsymbol{x}}\right| - \left|-\hat{\boldsymbol{x}}\right\rangle \left\langle -\hat{\boldsymbol{x}}\right|\Big).$$

a) Find the matrix representation of $\hat{S}_x$ in the $\{\left|+\hat{\boldsymbol{z}}\right\rangle, \left|-\hat{\boldsymbol{z}}\right\rangle\}$ basis.

b) Verify that $\hat{S}_x$ is Hermitian.

The observable corresponding to the component of spin along the direction $\hat{\mathbf{n}}$ is

$$\hat{S}_n := \frac{\hbar}{2}\Big(|+\hat{\boldsymbol{n}}\rangle\langle+\hat{\boldsymbol{n}}| - |-\hat{\boldsymbol{n}}\rangle\langle-\hat{\boldsymbol{n}}|\Big).$$

c) Using the standard representations

$$|+\hat{\boldsymbol{n}}\rangle = \cos\left(\frac{\theta}{2}\right)|+\hat{\boldsymbol{z}}\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\hat{\boldsymbol{z}}\rangle$$
$$|-\hat{\boldsymbol{n}}\rangle = \sin\left(\frac{\theta}{2}\right)|+\hat{\boldsymbol{z}}\rangle - e^{i\phi}\cos\left(\frac{\theta}{2}\right)|-\hat{\boldsymbol{z}}\rangle$$

find the matrix representation of $\hat{S}_n$ in the $\{|+\hat{\boldsymbol{z}}\rangle, |-\hat{\boldsymbol{z}}\rangle\}$ basis.

d) Verify that

$$\hat{S}_n = \cos\phi\sin\theta\,\hat{S}_x + \sin\phi\sin\theta\,\hat{S}_y + \cos\theta\,\hat{S}_z.$$

This is reminiscent of $\hat{\mathbf{n}} = \cos\phi\sin\theta\hat{\mathbf{x}} + \sin\phi\cos\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$. In this sense one can regard the spin as a vector quantity.