Quantum Theory I: Homework 4

Due: 7 February 2023

1 Kets and Stern-Gerlach measurement outcome probabilities

A particle is in the state

$$|\psi\rangle = \frac{1}{2}\,|+\hat{\boldsymbol{z}}\rangle + i\frac{\sqrt{3}}{2}\,|-\hat{\boldsymbol{z}}\rangle$$

a) The particle is subjected to an SG $\hat{\boldsymbol{z}}$ measurement. Determine the probabilities of the two measurement outcomes.

b) The particle is subjected to an SG $\hat{\boldsymbol{x}}$ measurement. Determine the probabilities of the two measurement outcomes.

2 Kets and outputs from Stern-Gerlach measurements

a) A Stern-Gerlach apparatus is oriented in the direction $\hat{\mathbf{n}} = (\hat{\mathbf{x}} - \hat{\boldsymbol{z}})/\sqrt{2}$. Express, as a superposition of $|+\hat{\boldsymbol{z}}\rangle$ and $|-\hat{\boldsymbol{z}}\rangle$, the state which emerges from the measuring apparatus if it gave measurement outcome $S_n = +\hbar/2$. Repeat this for a particle emerging from the device if it yielded $S_n = -\hbar/2$. Demonstrate that the two states are orthogonal.

b) A Stern-Gerlach apparatus is oriented in the direction $\hat{\mathbf{n}} = \left(\hat{\mathbf{x}} + \sqrt{3}\hat{\boldsymbol{y}}\right)/2$. Express, as a superposition of $|+\hat{\boldsymbol{z}}\rangle$ and $|-\hat{\boldsymbol{z}}\rangle$, the state which emerges from the measuring apparatus if it gave measurement outcome $S_n = +\hbar/2$. Repeat this for a particle emerging from the device if it yielded $S_n = -\hbar/2$. Demonstrate that the two states are orthogonal.

Note: In the solutions to these problems the sin and cosine of angles do not need to be converted into numbers except when the angle is a multiple of $\pi/4$. Similarly you only need to convert complex exponentials of angles into complex numbers except when the angle is a multiple of $\pi/2$.

3 Spin-1/2 states and measurement direction

Consider a spin-1/2 system prepared in the state

$$|\Psi\rangle = A\big\{15\,|+\hat{\boldsymbol{z}}\rangle - (12 - 16i)\,|-\hat{\boldsymbol{z}}\rangle\big\}$$

where A is a normalization constant.

a) Apply the normalization condition to determine A.

b) Suppose that you would like to subject this particle to measurement via a Stern-Gerlach apparatus whose magnetic field is oriented in some direction $\hat{\mathbf{n}}$ so that the outcome of the measurement is $S_n = +\hbar/2$ with 100% certainty. Determine values for the spherical coordinate parameters θ, ϕ corresponding to $\hat{\mathbf{n}}$ which will ensure this.

4 Measurements on ensembles of particles

Suppose one ensemble (A) of particles are each in the state

$$|\psi_A\rangle = \frac{1}{\sqrt{2}} |+\hat{\boldsymbol{z}}\rangle + \frac{1}{\sqrt{2}} |-\hat{\boldsymbol{z}}\rangle .$$

Another ensemble (B) of particles are each in the state

$$|\psi_B\rangle = \frac{1}{\sqrt{2}} |+\hat{\boldsymbol{z}}\rangle - \frac{1}{\sqrt{2}} |-\hat{\boldsymbol{z}}\rangle .$$

a) Suppose that all of the particles in each ensemble are subjected to an SG $\hat{\boldsymbol{z}}$ measurement. Determine the probability of each measurement outcome and the mean $\langle S_z \rangle$ for each ensemble. Can this measurement be used to distinguish between the two ensembles?

b) Suppose that all of the particles in each ensemble are subjected to an SG $\hat{\boldsymbol{x}}$ measurement. Determine the probability of each measurement outcome and the mean $\langle S_x \rangle$ for each ensemble. Can this measurement be used to distinguish between the two ensembles?

c) Nuclear magnetic resonance (NMR) deals with spin-1/2 particles and an NMR spectrometer can measure quantities such as $\langle S_x \rangle$. At room temperature the states of the spin-1/2 particles in a sample are not known with certainty. However, the probabilities of the states are known. A simple example of this would be that the fraction of particles in state $|\psi_A\rangle$ is $(1 + \epsilon)/2$ and the fraction in state $|\psi_B\rangle$ is $(1 - \epsilon)/2$ where $\epsilon > 0$ is usually very small. Determine $\langle S_z \rangle$ and $\langle S_x \rangle$ for this entire sample.